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On the representation of integers by certain binary cubic and biquadratic forms

by

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1. Let $f_n(x, y)$ denote any irreducible form with rational integral coefficients of degree $n > 2$. Some sixty years ago Axel Thue [18] proved that the diophantine equation

$$(1) \qquad f_n(x, y) = m,$$

where m is a given rational integer $\neq 0$, has only a finite number of solutions in rational integers x and y. However, the method used does not help in finding the values of x and y in question. It is only possible to obtain an upper bound for the *number* of solutions. Recently Alan Baker [4] has shown that bounds can also be given for the *magnitude* of the solutions. However, the bounds for $|x|$ and $|y|$ are extremely big, even for small values of n.

A quite different method of investigating (1) has been developed by Th. Skolem in a series of papers. See for example [23], [24], [25] and [26]. The method works in case $f_n(x, -1)$ has at least one pair of conjugate complex zeros, and it can be considered as a generalization of a method used by T. Nagell and B. Delone [18] for $n = 3$. The application of Dirichlet's unit theorem plays an important role. Otherwise Skolem's method is a p-adic one, which is also of a non-effective character. But it can be used to obtain an upper bound for the number of solutions, and in all numerical examples hitherto treated these bounds are low, such that it in many cases has been possible to attain a complete solution.

As to the case where $f_n(x, -1)$ has only real zeros Skolem himself proposed a manner of proceeding to deal with the simplest case where $n = 3$. I have carried out this proposal in [10]. In 1939 C. Chabauty gave an outline of a proof of the proposition that it was possible to use Skolem's method also in the remaining cases. A complete treatment appeared in 1941 [6]. However, it seems to be too complicated to get a low upper bound for the number of solutions by Chabautay's method, even for small values of n.

The purpose of this paper is to show that the cubic forms with positive discriminant can be treated in a way, which is simpler than the manner of proceeding proposed by Th. Skolem. In addition we solve some cases with $n = 4$, where $f_4(x, -1)$ has only real zeros.

2. It is well-known that any cubic form $f = f_3(x, y)$ has two covariants:

$$H = H(x, y) = -\frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2\right)$$

and

$$Q = Q(x, y) = \frac{\partial f}{\partial x}\cdot\frac{\partial H}{\partial y} - \frac{\partial f}{\partial y}\cdot\frac{\partial H}{\partial x},$$

satisfying the identity

(2) $$Q^2 = 4H^3 - 27Df^2,$$

D denoting the discriminant of the form. See [8], pp. 132–135. In case $f_3(x, y) = 1$, we obtain from (2)

(3) $$Q^2 = 4H^3 - 27D.$$

It was shown by Mordell [17] that the diophantine equation

(4) $$v^2 = 4u^3 - g_2 u - g_3,$$

where g_2 and g_3 are given rational integers, has at most a finite number of rational integral solutions (u, v), when its right-hand side has no squared factor in u. He proved that to every integral solution (u, v) of (4) there corresponded a binary *quartic* with invariants g_2 and g_3 which represented unity, and conversely.

In (3) we have $g_2 = 0$ and $g_3 = 27D$. The problem of solving $f_3(x, y) = 1$ is now transposed into the problem of finding all representations of 1 by certain binary biquadratic forms having these invariants. Since such a form $f_4(x, y)$ has negative discriminant $D_1 = 2^8(g_2^3 - 27g_3^2) = -2^8\cdot 3^6\cdot D^2$, the corresponding equation $f_4(x, -1) = 0$ will have two real roots η and η' and two complex roots η'' and η'''. In $Q(\eta)$ we have two independent units, and the only roots of unity are ± 1. Hence, the method of Skolem can be used.

In an earlier paper [16] I have given the complete solution of the equation

(5) $$y^2 - k = x^3$$

in case $k = -7$ and $k = -15$, using the p-adic method in the treatment of the occuring binary biquadratic forms. By the way we got the complete solution of

(6) $$x^3 - 6xy^2 + 2y^3 = 1$$

and

(7) $$x^3 - 6x^2y + 2y^3 = 1,$$

where the cubic forms on the left-hand side of (6) and of (7) have positive discriminants. The equation (6) has the two solutions $(x, y) = (1, 0)$ and $(x, y) = (1, 3)$, while the equation (7) has the only solution $(x, y) = (1, 0)$.

However, I did not emphasize that we could start with an arbitrary cubic form with positive discriminant.

In my paper [10] I gave the complete solution of the equation

(8) $$x^3 - 3xy^2 - y^3 = 1, \quad D = 81.$$

Corresponding to (2) we find

$$(x^3 + 6x^2y + 3xy^2 - y^3)^2 = 4(x^2 + xy + y^2)^3 - 3(x^3 - 3xy^2 - y^3)^2.$$

In order to solve (8) in rational integers (x, y) we must solve in rational integers (u, v) the equation

$$v^2 = 4u^3 - 3,$$

which leads to the problem of solving

$$x^4 - 6x^2y^2 - 4xy^3 - 3y^4 = 1$$

in rational integers (x, y). I do not enter into this here since (8) is already completely solved.

It may even happen that our problem can be solved by considering only one exponential equation with one unknown exponent. We shall illustrate this by an example. Using $f_3(x, y) = x^3 - 6xy^2 + 2y^3$ as in (6), we obtain the following identity, corresponding to (3):

$$(x^3 + 12x^2y + 6xy^2 - 6y^3)^2 + 7(x^3 - 6xy^2 + 2y^3)^2 = 8(x^2 + xy + 2y^2)^3.$$

The equation

(9) $$x^3 + 12x^2y + 6xy^2 - 6y^3 = 1, \quad D = 4\cdot 21^3$$

leads to the equation

(10) $$1 + 7u^2 = 8v^3.$$

In an earlier paper [11], pp. 50–52, I have proved that $(u, v) = (1, 1)$, $(3, 2)$ and $(39, 11)$ are the only solutions of (10) in rational integers (u, v). Combining (9) with $x^2 + xy + 2y^2 = v$, we find that (9) has exactly the following three solutions $(x, y) = (1, 0), (x, y) = (1, -1)$ and $(x, y) = (1, 2)$. In the proof use was made of Skolem's method and only one exponential equation occured. The cause of this is the reducibility of $8v^3 - 1$ in Q.

3. In this section we are going to treat some cases with $n = 4$, assuming that the corresponding quartic field $Q(\eta)$ has a real quadratic subfield, i.e. that the reducing cubic equation has a rational root. See for instance T. Nagell [19], p. 349.

We need some well-known results from the theory of covariants and invariants of biquadratic forms. Referring to the textbook [8], pp. 138–147, we write

$$f_4(x, y) = a_0x^4+4a_1x^3y+6a_2x^2y^2+4a_3xy^3+a_4y^4.$$

The form $f = f_4(x, y)$ has the two covariants

$$H = H(x, y) = -\frac{1}{144}\left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2\right)$$

and

$$Q = Q(x, y) = \frac{1}{8}\left(\frac{\partial f}{\partial x}\cdot\frac{\partial H}{\partial y}-\frac{\partial f}{\partial y}\cdot\frac{\partial H}{\partial x}\right),$$

satisfying the identity

$$(11)\qquad 4H^3-g_2Hf^2-g_3f^3 = Q^2,$$

where the two invariants g_2 and g_3 are given by

$$(12)\qquad g_2 = a_0a_4-4a_1a_3+3a_2^2,\qquad g_3 = a_0a_2a_4+2a_1a_2a_3-a_2^3-a_0a_3^2-a_1^2a_4.$$

The discriminant D of the form has the value

$$(13)\qquad D = 256(g_2^3-27g_3^2).$$

Putting

$$(14)\qquad H(x, y) = b_0x^4+b_1x^3y+b_2x^2y^2+b_3xy^3+b_4y^4,$$

we have

$$(14')\qquad \begin{gathered} b_0 = a_1^2-a_0a_2,\quad b_1 = 2(a_1a_2-a_0a_3),\quad b_2 = 3a_2^2-2a_1a_3-a_0a_4,\\ b_3 = 2(a_2a_3-a_1a_4),\quad b_4 = a_3^2-a_2a_4.\end{gathered}$$

Denoting by e_1, e_2 and e_3 the roots of the reducing cubic

$$(15)\qquad 4z^3-g_2z-g_3 = 0,$$

then the following formulas are valid

$$(16)\qquad H-e_if = \varphi_i^2(x, y),\qquad i = 1, 2, 3.$$

The three binary quadratics $\varphi_i(x, y) = \varphi_i$ are the "irrational" covariants, and $Q = 2\varphi_1\varphi_2\varphi_3$.

Let e_1 be a rational root of (15). Then we may write

$$(17)\qquad H-e_1f = k\psi_1^2(x, y),$$

and therefore $(H-e_2f)(H-e_3f)$ in the form

$$(18)\qquad H^2+e_1Hf+(e_1^2-\tfrac{1}{4}g_2)f^2 = k\psi_2^2(x, y),$$

where $\psi_i(x, y) = \psi_i = p_ix^2+q_ixy+r_iy^2$ $(i = 1, 2)$ with rational integer coefficients and rational k.

Replacing $H-e_1f$ by $k\psi_1^2$ in (18) we obtain

$$k^2\psi_1^4+3e_1kf\psi_1^2+f^2(3e_1^2-\tfrac{1}{4}g_2) = k\psi_2^2.$$

In case

$$(19)\qquad f_4(x, y) = 1$$

we get

$$k^2\psi_1^4+3e_1k\psi_1^2+(3e_1^2-\tfrac{1}{4}g_2) = k\psi_2^2,$$

or

$$(20)\qquad Au^4+Bu^2+C = Dv^2,$$

A, B, C, D denoting rational integers.

Since $f_4(t, -1) = 0$ has four real roots, the reducing cubic (15) has three real roots. This fact implies that $B^2-4AC > 0$. However, in an earlier paper [15], p. 8, I have shown how it is possible to apply Skolem's method to solve (20) by passing to an appropriate algebraic field of degree 8, then getting two exponential equations with two unknown exponents.

Assuming $g_3 = 0$ V. Krechmar [9] proved that $f_4(x, y) = 1$ had at most twenty solutions in rational integers x, y. This case is included in our considerations above, because then $z = 0$ is a rational root of (15). Besides, the following identity is easily verified

$$(21)\qquad 4a_0b_0f_4(x, y) = 4b_0(a_0x^2+2a_1xy+a_2y^2)^2-a_0(b_1y^2+4b_0xy)^2.$$

Hence, $Q(\sqrt{a_0b_0})$ is a quadratic subfield of $Q(\eta)$. Here we have $a_0b_0 = a_0(a_1^2-a_0a_2) > 0$. In (20) is then $B = 0$.

We shall give two examples, chosen for the purpose of avoiding laborious calculations.

Example 1.

$$(22)\qquad f_4(x, y) = (x^2-6y^2)^2-8(xy-2y^2)^2 = 1.$$

We find $g_2 = \frac{1}{3}\cdot 7\cdot 16$, $g_3 = -\frac{1}{27}\cdot 17\cdot 16$, $D = 2^{13}$ and $e_1 = \frac{4}{3}$.

These values yield further

$$H-e_1f = 2(x^2-4xy+6y^2)^2,$$

and corresponding to (20) we obtain

$$(23)\qquad u^4+2u^2-1 = 2v^2,$$

where

$$u = x^2-4xy+6y^2\quad\text{and}\quad v = x^4-8x^3y+12x^2y^2+16xy^3-28y^4.$$

However, in [15] I have proved that the only solutions of (23) in rational positive integers (u, v) are $(u, v) = (1, 1)$ and $(u, v) = (3, 7)$. The only solutions of (22) are then $x = \pm 1$, $y = 0$ and $x = 3, y = 1, x = -3, y = -1$.

Example 2.

(24) $$f_4(x, y) = (x^2+2xy-y^2)^2-8x^2y^2 = 1.$$

Here we have $g_2 = 8, g_3 = 0, D = 2^{17}$ and $H = 2(x^2+y^2)^2$. We note the following identity

$$(x^4-6x^2y^2+y^4+4x^3y-4xy^3)^2+(x^4-6x^2y^2+y^4-4x^3y+4xy^3)^2 = 2(x^2+y^2)^4,$$

corresponding to equation (20) in the form

(25) $$2u^4-1 = v^2.$$

The only solutions in natural numbers (u, v) of (25) are $(u, v) = (1, 1)$ and $(u, v) = (13, 239)$. See [12], pp. 9–12. The equation (24) has the eight solutions $(\pm 1, 0), (0, \pm 1), (3, 2), (-3, -2), (2, -3)$ and $(-2, 3)$.

4. In the following we wish to add some considerations concerning quartic forms with negative discriminant, with the further property that the corresponding field $Q(\eta)$ has a real quadratic subfield. Here we have $r = 2$. As in the preceeding section we may obtain the solution of (19) by solving (20), where now $B^2-4AC < 0$.

In my paper [11] I have shown how it is possible to deal with (20) by discussing only one exponential equation with one unknown exponent. Besides [11] contains several general theorems on the solvability of equations of the type (20).

Example.

(26) $$f_4(x, y) = (x^2+xy+y^2)^2-3(xy-y^2)^2 = 1.$$

We find $g_2 = -6, g_3 = -\frac{7}{2}, e_1 = -\frac{1}{2}$,

$$H-e_1f = \tfrac{3}{4}(x^2-2xy-2y^2)^2$$

and

(27) $$u^4-2u^2+4 = 3v^2,$$

where

$$u = x^2-2xy-2y^2, \qquad v = x^4+2x^3y+6x^2y^2-4xy^3+4y^4.$$

In [11], Theorem III, p. 4, I have proved that the only solutions in natural numbers u, v of (27) are given by

$$(u, v) = (1, 1), (13, 97) \text{ and } (2, 2).$$

The only solutions of (26) are then $(x, y) = (\pm 1, 0), (3, -1)$ and $(-3, 1)$.

V. D. Podsypanin [21] has given another method for solving the same problem by working in a totally complex quartic field. As an application he treats the equation $x^4+4xy^3-y^4 = 1$, proving that the only integral solutions are $(x, y) = (\pm 1, 0), (1, 4)$ and $(-1, -4)$.

As to more general investigations we quote the following two theorems:

Let a, b, c, d be integers in Q; a, b, c positive, $d \neq 0$, a and c squarefree; d^2+4acb^2 not a perfect square. For fixed a, b, c, d and for every pair of natural numbers b_1, b_2 satisfying $b_1b_2 = b$, the equations

$$ab_1^2x^4-cb_2^2y^4-dx^2y^2 = \pm 1,$$
$$a^3b_1^2x^4-c^3b_2^2y^4+acdx^2y^2 = \pm 1$$

are insoluble in natural numbers x, y, except possibly one, which has exactly one solution.

See Ljunggren [13] and Podsypanin [20].

Let p and q be integers in Q. The diophantine equation

$$f_4(x, y) = x^4+px^3y+qx^2y^2+pxy^3+y^4 = 1,$$

where $f_4(t, -1)$ has exactly two real zeros, and where $q \neq 2|p|-3$ has at most 8 solutions in rational integers x and y, including the four trivial solutions $x = \pm 1, y = 0$ and $x = 0, y = \pm 1$. If $q = 2|p|-3$ the equation has at most 10 solutions.

In the proof use is made of the fact that the algebraic integer

$$(x+y\eta)^2(x+y\eta'')(x+y\eta''')$$

is a unit with relative norm 1 in $Z[\eta]$. See [14] and [14'].

5. Special diophantine equations of the form $f_4(x, y) = 1$ has been investigated by the Russian mathematician E. T. Avanesov. See the papers [1] and [3]. In [2] he obtained a result concerning (1) in case $n = 3$, $m = 1$, where $f_3(x_1-1) = 0$ has three real roots. His procedure is exactly that of Skolem, mentioned in Section 1. See [23], pp. 58–61. However, Skolem gave no details. One ends with systems of four exponential equations with four unknown exponents. In my treatment in [10] I showed that one could handle the problem using only two exponential equations with two unknown exponents. I did not discuss which of these methods would give a minimum of laboriouse calculations. By use of the method in [5] V. I. Baulin gave the complete solution of $f_3(x, y) = 1$, where the occuring cubic form had discriminant 49.

There are at most fifteen solutions of (1) if $n = 3$ and the discriminant of the form is positive and great compared with m. See papers [22] and [7].

Added in proof. If the rank of (1) over Q does not exceed 1, m is without cubic factors, and $(D, m) = 1$, then for $|D|m^2 > 6^{144}$, the equation (1) has at most three solutions in rational integers in the case $D \neq s^2$, $s \in Z$, and nine solutions in the case $D = s^2$ (see V. A. Demjanenko, *On the representation of numbers by a binary cubic irreducible form* (Russian), Mat. Zametki 7 (1970), pp. 87–97, English translation, Math. Notes 7 (1–2) (1970)).

References

[1] E. T. Avanesov, *Solution of a problem on figurate numbers* (Russian), Acta Arith. 12 (1967), pp. 409–420.
[2] — *The representation of numbers by binary cubic forms of positive discriminant* (Russian), Acta Arith. 14 (1968), pp. 13–25.
[3] — *On the question of a certain theorem of Skolem* (Russian), Izv. Akad. Nauk Armjan. SSR, Ser. Mat. 3 (1968), no. 2, pp. 160–165.
[4] A. Baker, *Contributions to the theory of Diophantine equations*, Phil. Trans. Roy. Soc. London, Ser. A, 263 (1967/68), pp. 171–208.
[5] V. I. Baulin, *On an indeterminate equation of the third degree with least positive discriminant* (Russian), Tul'sk. Gos. Ped. Inst. U'cen. Zap. Fiz. Mat. Nauk 7 (1960), pp. 138–170.
[6] C. Chabauty, *Démonstration nouvelle d'un théorème de Thue et Mahler sur les formes binaires*, Bull. Sci. Math. (2), 65 (1941), pp. 112–130.
[7] B. N. Delaunay (Delone) and D. K. Faddeev, *Theory of irrationalities of the third degree*, Acad. Sci. URSS. Trav. Inst. Math. Stekloff, 11 (1940); Translations of Mathematical Monograph, v. 10 (1964).
[8] R. Fricke, *Lehrbuch der Algebra*, Bd. I, Braunschweig 1924.
[9] V. Krechmar, *On the superior bound of the number of representations of an integer by binary forms of the fourth degree*, Bull. Acad. Sci. URSS, Ser. Math. (1939), pp. 289–302.
[10] W. Ljunggren, *Einige Bemerkungen über die Darstellung ganzer Zahlen durch binäre kubische Formen mit positiver Diskriminante*, Acta Math. 75 (1942), pp. 1–21.
[11] — *Einige Sätze über unbestimmte Gleichungen von der Form $Ax^4+Bx^2+C = Dy^2$*, Vid.-Akad. Skrifter I, No. 9, (1942).
[12] — *Zur Theorie der Gleichung $x^2+1 = Dy^4$*, Vid. Akad. Avhdl. I, No. 5, (1942).
[13] — *A theorem of diophantine equations of the fourth degree*, Vid. Akad. Avhdl. I, No. 9, (1943).
[14] — *On the representation of integers by binary biquadratic forms of a special class* (Norwegian), Norsk mat. tidsskrift, 26 (1944), pp. 51–59.
[14'] — *Diophantine equations. A p-adic approach*, Lecture notes prepared by R. R. Laxton 1968. University of Nottingham 1970.
[15] — *Proof of a theorem of de Jonquieres* (Norwegian), Norsk mat. tidsskrift, 26 (1944), pp. 3–8.
[16] — *On the diophantine equation $y^2-k = x^3$*, Acta Arith. 8 (1963), pp. 451–463.
[17] L. J. Mordell, *Diophantine equations*, Pure and applied mathematics, vol. 30, London and New York 1969.
[18] T. Nagell, *L'analyse indéterminée du degré superieur*, Mem. des Sci. Math. 39 (1929).
[19] — *Sur quelques questions dans la theorie des corps biquadratiques*, Arkiv för Mat. 26 (1956), pp. 347–376.
[20] V. D. Podsypanin, *On the equation $ax^4+bx^2y^2-cy^4 = 1$* (Russian), Mat. Sbornik, N. S. 18 (60), (1946), pp. 105–114.
[21] — *On the representation of integers by binary biquadratic forms* (Russian), Mat. Sbornik, N. S. 42 (84), (1957), pp. 523–532.
[22] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abh. Preuss. Akad. Wiss., Phys.-math. Kl. nr. 1 (1929).
[23] Th. Skolem, *Einige Sätze über gewisse Reihenentwicklungen und exponentiale Beziehungen mit Anwendung auf diophantische Gleichungen*, Skr. Norske Vid. Akad., Oslo, I, 1933, No. 6, pp. 1–61.
[24] Th. Skolem, *Ein Verfahren zur Behandlung gewisser exponentialer Gleichungen und diophantischer Gleichungen*, 8de Skand. Mat. Kongress, Stockholm 1934, pp. 163–188.
[25] — *Einige Sätze über p-adische Potenzreihen mit Anwendung auf gewisse exponentielle Gleichungen*, Math. Ann. 11 (1935), pp. 399–424.
[26] — *Anwendung exponentieller Kongruenzen zum Beweis der Unlösbarkeit gewisser diophantischer Gleichungen*, Avh. Norske Vid. Akad., Oslo, I, 1937, No. 12, pp. 1–16.

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