A combinatorial problem connected with differential equations II

by

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1. Let us call a sequence admissible if it has no immediate repetition and contains no subsequence of the form $a, b, a, b, a$ with $a \neq b$. Let $N(n)$ be the greatest length (that is, greatest number of terms) of an admissible sequence formed from $n$ distinct elements.

The problem of estimating $N(n)$ has been investigated in [1] and it has been proved there that

$$5n - C < N(n) = O(n \log n).$$

($N(n)$ was denoted by $N_{4}(n)$. The aim of this paper is to improve the above result in both directions. We prove

**Theorem 1.** We have

$$N(n) = \Theta \left( \frac{n \log n}{\log \log n} \right).$$

**Theorem 2.** We have

$$\lim_{n \to \infty} \frac{N(n)}{n} \geq 8.$$

**Theorem 3.** For positive integers $l, m$ the following inequality holds

$$N(lm + 1) \geq 6lm - m - 5l + 2.$$

Theorem 3, found in collaboration with J. H. Conway, gives in general a weaker bound for $N(n)$ than that which can be obtained from the proof of Theorem 2. It is included as useful for small values of $n$. In particular, it implies

**Corollary.** We have $N(n) \geq 5n - 8$ and the equality sign is excluded for odd $n \geq 13$ and even $n \geq 18$.

It is interesting to note that $N(n) = 5n - 8$ for $n = 4, \ldots, 10$ (cf. [2]).
Let \( M(n) \) be the maximum length of a sequence formed from the integers 1, 2, \ldots, \( n \) with the following property: for some \( r \) (0 \( \leq r \leq n \)) there exists an admissible sequence of which the given sequence is a section, and the integers 1, 2, \ldots, \( r \) occur before this section and the integers \( r+1, \ldots, n \) occur after it.

**Lemma 1.** \( M(n) \leq 5n \).

**Proof.** We can write the given sequence as
\[
\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_a, \mathcal{B}_s,
\]
where the elements of each \( \mathcal{A}_i \) are from 1, \ldots, \( r \) and the elements of each \( \mathcal{B}_i \) are from \( r+1, \ldots, n \), and \( \mathcal{A}_1 \) or \( \mathcal{B}_s \) may be empty but the others are not.

If we remove the \( \mathcal{B}_s \)'s and eliminate any immediate repetitions we get a sequence formed from 1, \ldots, \( r \) of length \( \sum L(\mathcal{A}_i) - s \), where \( L(\mathcal{A}_i) \) is the length of \( \mathcal{A}_i \). Since this sequence is admissible when preceded by a sequence containing 1, \ldots, \( r \), it contains no \( a, a', a, a' \). Hence by Theorem 1 of [1]
\[
\sum_{i=1}^{s} L(\mathcal{A}_i) \leq s + (2r-1).
\]

Similarly
\[
\sum_{i=1}^{s} L(\mathcal{B}_i) \leq s + (2n-2r-1),
\]
whence
\[
M(n) \leq 2s + 2n - 2.
\]

It remains to estimate \( s \). We select one element \( \alpha \) from each \( \mathcal{A}_i \) and one element \( \beta \) from each \( \mathcal{B}_i \). The elements \( \alpha \) selected from consecutive \( \mathcal{A}_i \) may be equal and we can enumerate all the selected elements as
\[
(\star) \quad \alpha_1, \beta_1^{(1)}, \alpha_2, \beta_2^{(1)}, \ldots, \alpha_i, \beta_i^{(1)}, \alpha_{i+1}, \beta_{i+1}^{(1)}, \ldots, \alpha_s, \beta_s^{(1)}, \ldots, \alpha_k, \beta_k^{(o)},
\]
where possibly \( \alpha_i \) or \( \beta_k^{(o)} \) may be missing. We have
\[
s = \sum_{i=1}^{h} v_i.
\]

If \( \beta_i^{(o)} = \beta_j^{(o)} = \beta \) for \( i < k \) or \( i = k, j < l < v_k \), then the sequence (\( \star \)) contains the subsequence \( \beta, \alpha_i, \beta, \alpha_j \), which is impossible. Therefore, the elements \( \beta_i^{(o)} \) with \( j < v_i \) are distinct and
\[
\sum_{i=1}^{h} (v_i - 1) \leq n - r.
\]

Also the sequence \( a_1, a_2, \ldots, a_h \) forms part of an admissible sequence when preceded by 1, \ldots, \( r \), whence
\[
h \leq 2r - 1.
\]

Finally, we have
\[
s \leq (n-r) + h \leq n+r-1,
\]
whence
\[
M(n) \leq 2(n+r-1) + 2n - 2.
\]

By symmetry this implies
\[
M(n) \leq 5n - 4.
\]

This proves Lemma 1.

We now consider any admissible sequence \( \mathcal{F} \) of length \( N(n) \) formed from 1, \ldots, \( n \), and construct a partition of \( \mathcal{F} \) depending on an arbitrary integer \( m \) with 1 \( \leq m \leq n \). First we take the minimal left-hand section \( \mathcal{B} \) of \( \mathcal{F} \) with \( m \) distinct terms, then the minimal right-hand section \( \mathcal{V} \) of \( \mathcal{F} \) including all the elements of \( \mathcal{F} \) not appearing in \( \mathcal{B} \). We now write
\[
\mathcal{F} = (\mathcal{B}, \mathcal{V}, \mathcal{V}').
\]

Let for a given set \( A, \mathcal{G} \) be its complement, \( |A| \) its cardinality and for a given sequence \( \mathcal{A}, \mathcal{A}' \) be the set of its elements. We put
\[
m' = |\mathcal{V}'|,
\]
\[
m_1 = |\mathcal{G}^{\mathcal{V}} \cap \mathcal{V}'|,
\]
\[
m' = |\mathcal{G}^{\mathcal{V}} \cap \mathcal{V}^{\mathcal{V}}|,
\]
\[
m_4 = |\mathcal{G} \cap \mathcal{V} \cap \mathcal{V}'|,
\]
\[
m_4 = |\mathcal{G} \cap \mathcal{V} \cap \mathcal{V}'|.
\]

Then
\[
m = m_1 + m_2 + m_3 + m_4,
\]
\[
m' = m_1' + m_2' + m_3' + m_4',
\]
\[
n = m_1 + m_1' + m_2 + m_2' + m_3 + m_3' + m_4.
\]

We note that of the \( m_1 + m_1' + m_2 \) distinct elements of \( \mathcal{F} \), \( m_2 \) occur also to the left in \( \mathcal{F} \), and \( m_1' \) occur also to the right in \( \mathcal{F} \), and \( m_4 \) occur in both \( \mathcal{F} \) and \( \mathcal{F}' \).

**Lemma 2.** \( N(n) < N(m_1) + N(m_2) + 13n \).

**Proof.** By Lemma 1 we have
\[
L(\mathcal{F}) < 5(m_1 + m_1' + m_2) \leq 5n.
\]
Now enumerate the terms of $\mathcal{V}$, picking out explicitly those that have occurred already in $\mathcal{V}$ or $\mathcal{V}'$, the number of such terms (distinct) being $m'_i + m_1 + m_2 = m' - m'$. Write

$$\mathcal{V} = (a_1, \mathcal{S}_1^{(1)}, a_2, \ldots, a_k, \mathcal{S}_1^{(n)}, a_2, \ldots, a_k, \mathcal{S}_h^{(1)}, \ldots, \mathcal{S}_h^{(n)})$$

where $a_1, \ldots, a_k$ are terms just mentioned (not necessarily distinct) and the $\mathcal{S}_i^{(1)}$ are formed from the $m'$ distinct terms of $\mathcal{V}$ which do not occur in $\mathcal{V}$ or $\mathcal{V}'$ ($a_i$ may be missing and $\mathcal{S}_i^{(1)}$ may be empty). By the arguments used in the proof of Lemma 1

$$h \leq 2(m' - m'_1) - 1,$$

$$\sum_{i=1}^{h} (v_i - 1) \leq m'_1.$$

If we remove from $\mathcal{V}$ the $a_i$'s and eliminate any immediate repetitions we get an admissible sequence formed from $m'_1$ distinct integers of length $\sum \sum L(\mathcal{S}_i^{(j)}) - r$, where $r$ is the number of immediate repetitions. However, $r$ does not exceed $h$ since (cf. the proof of Lemma 1)

$$\mathcal{S}_1^{(h)} \cap \mathcal{S}_1^{(h)} = \emptyset,$$

if $v_i > 1$ and either $i < h$ or $i = h$, $j \leq l < v_i$. Hence

$$\sum \sum_{i=1}^{h} L(\mathcal{S}_i^{(j)}) \leq N(m'_1) + h$$

and

$$L(\mathcal{V}) \leq N(m'_1) + h + \sum_{i=1}^{h} v_i \leq N(m'_1) + m'_i + 2h \leq N(m'_1) + m'_i + 4(m' - m'_1) \leq N(m'_1) + 4n.$$

Similarly

$$L(\mathcal{V}) \leq N(m_1) + 4n$$

and on addition we obtain the result.

**Lemma 3.** $N(n) \leq N(m) + N(n-m) + (n-m) + 4(m-m_1).$

**Proof.** We set

$$\mathcal{V} = (\mathcal{V}, a_1, \mathcal{S}_1^{(1)}, a_2, \ldots, a_k, \mathcal{S}_h^{(1)})$$

where the $a_i$ are terms that have occurred in $\mathcal{V}$, the $\mathcal{S}_i^{(1)}$ do not contain such terms, $a_i$ may be missing and $\mathcal{S}_i^{(1)}$ may be empty. Since the number of distinct terms available for the $a_i$ is $m - m_1$, we have

$$k \leq 2(m - m_1) - 1.$$
Proof. Put \( F(u) = nL(u) \), where \( L(u) = \frac{\log u}{\log \log u} \). We note that

\[
L'(u) = \frac{1}{\log \log u} - \frac{1}{u \log u},
\]

and that this is a decreasing function and is greater than \( \frac{1}{2u \log \log u} \).

The first result is easy:

\[
F(n) - F(n-h) = nL(n) - (n-h)L(n-h) = n(L(n) - L(n-h)) + hL(n-h) > hL(n-h) > hL(\frac{1}{3}n) > \frac{1}{2h} \frac{\log n}{\log \log n}.
\]

For the second result, using part of the preceding chain of inequalities, we have

\[
F(n) - F(n-h) - F(h) > hL(n-h) - hL(h) = h \int_{n-h}^{n} L(t) \, dt > \frac{1}{3h} \int_{n-h}^{n} \frac{dt}{\log \log t} > \frac{1}{3h} \frac{\log n}{\log \log n} \int_{n-h}^{n} \frac{dt}{t},
\]

whence the result.

Proof of Theorem 1. We suppose that \( N(n) < AF(m) \) for \( m < n \), where \( A \) is a suitable large constant, and prove that then this also holds for \( n = m \). We take

\[
h = \left[ \frac{n \log \log n}{\log n} \right]
\]

in Lemma 4. It suffices to prove that

\[ AF(n-h) + 13n < AF(n) \quad \text{and} \quad AF(n-h) + AF(h) + 5h < AF(n). \]

By Lemma 5, the former holds if

\[
\frac{1}{2} hA \frac{\log n}{\log \log n} > 13n
\]

and this is so if \( A \) is a sufficiently large constant. Also the second inequality holds if

\[
\frac{1}{2} hA \frac{\log (n-h)/h}{\log \log n} > 5h.
\]

Now

\[
\log \frac{n-h}{h} > \log \frac{n}{2h} > \log \frac{\log n}{3 \log \log n} \rightarrow \frac{1}{2} \log \log n.
\]

Hence the condition is again satisfied if \( A \) is a sufficiently large constant.

3. Proof of Theorem 2. Consider a sequence \( \mathcal{A} \) formed from \( m^2 \) distinct terms, typified by the following example

\[
1, 2, 3; 3, 2, 1; 4, 5, 6; 6, 5, 4; 7, 8, 9; 9, 8, 7; 7, 1, 4; 1, 4, 7; 8, 5, 2; 2, 5, 8; 9, 6, 3; 3, 6, 9.
\]

In general

\[
\mathcal{A} = \{ B_1, B_2, \ldots, B_m, C_m, C_{m+1}, C_{m+1}, \ldots, C_{2m}, B_{2m}\},
\]

where

\[
B_k = \{(k-1)m+1, \ldots, km\}, \quad C_k = \{(km, \ldots, (k-1)m+1) \quad (1 \leq k \leq m); \quad B_k = (k-m, \ldots, k+m^2-2m), \quad C_k = (k+m^2-2m, \ldots, k-m) \quad (m < k \leq 2m).
\]

\( \mathcal{A} \) contains no subsequence \( a, b, c, d, a, b, c, d \). It contains some immediate repetitions, but they will disappear later.

The first appearances of all the integers are in the blocks \( B_1, B_2, \ldots, B_m \) and their last appearances are in the blocks \( B_{m+1}, B_{m+2}, \ldots, B_{2m} \). We shall expand each of these blocks.

For each block we look for a new set \( U_k \) of \( l \) integers \( u^{(1)}, \ldots, u^{(l)} \), where \( l > m+1 \). Thus there are \( 2ml \) new integers, and the total number of integers

\[
n = m^2 + 2ml.
\]

For each set \( U_k \) \( (1 \leq k \leq m) \) we take an admissible sequence \( \mathcal{S}_k \) of length \( N(l) \) formed from the elements of \( U_k \) and arranged so that the last appearance of \( u^{(i)} \) occurs before the last appearance of \( u^{(j)} \) for \( i < j \). We replace the last appearance of \( u^{(i)} \) by \( u^{(j)} \), \( (k-1)m+j \), \( u^{(j)} \), \( (k-1)m+j \) for \( j = 1, 2, \ldots, m \). Thus if \( n = 3 \) and \( l = 4 \) we can take

\[
\mathcal{S}_1 = \{ u_1, u_1, u_1, u_1, u_2, u_2, u_2, u_3, u_3, u_3, u_4, u_4 \}
\]

and this becomes

\[
\mathcal{S}_1 = \{ u_1, u_1, u_1, u_1, u_2, u_2, u_2, 1, u_1, 1, u_1, u_2, u_2, 2, u_2, u_2, 3, u_2, 3, u_2 \},
\]

where the superscripts over \( u \)'s are omitted. \( \mathcal{S}_k \) replaces the block 1, 2, 3. Note that the last term is now not 3, so the immediate repetition of 3 in
disappears and in general the same holds for the repetition of \( mk \) 
\((1 \leq k \leq m)\).

For each set \( U_k \) \((m < k \leq 2m)\) we take similarly an admissible 
sequence \( \mathcal{S}_k \) of length \( N(l) \) formed from the elements of \( U_k \) 
and arranged so that the first appearance of \( \mathcal{S}_k(i) \) occurs before the first appearance 
of \( \mathcal{S}_j(i) \) for \( i < j \). We replace the first appearance of \( \mathcal{S}_j(i) \) by \((j-2) m + 
(k-m), \mathcal{S}_j(j-2)m + (k-m), \mathcal{S}_j(k) \) for \( j = 2, 3, \ldots, m+1 \).

The number of terms of the expanded block \( \mathcal{S}_n \) is 
\[ N(l) + 3m. \]

Of these terms, \( m \) were already present in \( \mathcal{R}_n \). So the length of the sequence 
\[ \mathcal{S}_{11}, \mathcal{S}_{12}, \ldots, \mathcal{S}_{m}, \mathcal{S}_{m+1}, \mathcal{S}_{m+2}, \ldots, \mathcal{S}_{2m} \]
is \( 4m^2 + 2m(N(l) + 2m) \). If the sequence \( \mathcal{S} \) obtained from the above by 
cancelling the central term \( m^2 - m + 1 \) (in the example) is admissible, we get 
\[ N(m^2 + 2ml) \geq 8m^2 + 2mlN(l) - 1. \]
Since \( N(m^2 + n_k) \geq N(m^2 + n_2), \ N(m)/n \) tends to a limit (finite or 
infinite). Choose \( l = m+1 \). If 
\[ \lambda = \lim_{n \to \infty} \frac{N(m)}{n} < \infty, \]
then \( N(l) > (\lambda - \varepsilon) l \) and 
\[ N(m^2 + 2ml) \geq \frac{8m^2 + 2ml(\lambda - \varepsilon) - 1}{m^2 + 2ml} \]
for \( m > m_0(\varepsilon) \). Making \( m \to \infty \) we get 
\[ \lambda \geq \frac{8 + 2\lambda}{3}, \text{ whence } \lambda \geq 8. \]

In order to prove that \( \mathcal{S} \) is admissible consider two distinct elements 
a and b. If \( a \in U_k \), \( b \in U_k \), \( \mathcal{S} \) contains no subsequence \( a, b, a, b, a, b \) in view of 
the same property of \( \mathcal{S}_k \). If \( a \in U_k \), \( b \in U_j \) with \( k \neq j \), or \( b \leq m^2 \)
and b does not occur in \( \mathcal{S}_k \), \( \mathcal{S} \) contains no subsequence \( a, b, a, b, a \) if \( a \in U_k \)
and \( b \) occurs in \( \mathcal{S}_k \), then the maximal subsequence of \( \mathcal{S} \) formed from 
a and b is 
\[ a, \ldots, a, b, a, b, b, a, b, b \quad \text{if } k \leq m, \]
\[ b, b, b, a, b, a, b, a, b, \ldots, a \quad \text{if } k > m, \]
with one b missing if \( b = m^2 - m + 1 \).

Finally, if \( a < b \leq m^2 \) the maximal subsequence of \( \mathcal{S} \) formed from 
a and b is 
\[ a, a, b, b, a, a, b, b, b \quad \text{if } \left\lfloor \frac{a}{m} \right\rfloor \geq \frac{b}{m} \text{ and } \left\lfloor \frac{a}{m} \right\rfloor \geq \frac{b}{m}, \]
\[ a, a, a, b, b, a, a, b, b \quad \text{if } \left\lfloor \frac{a}{m} \right\rfloor < \frac{b}{m} \text{ and } \left\lfloor \frac{a}{m} \right\rfloor < \frac{b}{m}, \]
\[ a, a, a, b, a, b, b, a, a \quad \text{if } \left\lfloor \frac{a}{m} \right\rfloor < \frac{b}{m} \text{ and } \left\lfloor \frac{a}{m} \right\rfloor = \frac{b}{m}, \]
\[ a, a, a, b, a, b, b, a, b \quad \text{if } \left\lfloor \frac{a}{m} \right\rfloor = \frac{b}{m} \text{ and } \left\lfloor \frac{a}{m} \right\rfloor < \frac{b}{m}. \]

with one letter missing if \( a = m^2 - m + 1 \) or \( b = m^2 - m + 1 \). None of 
the above sequences contains either a, b, a, b, a or b, a, b, a, b, which 
completes the proof.

4. Proof of Theorem 3. We take \( l \) pairwise disjoint sets of \( m-1 \) 
integers \( C_i = \{c_i, c_{i+1}, \ldots, c_{i+1} \} \), where 
\[ c_i = (j-1)(m-1) + i \quad (1 \leq i < m, 1 \leq j \leq l), \]
say, and \( l+1 \) other integers \( x_k = l(m-1) + k \quad (1 \leq k \leq l+1) \). Set 
\[ \mathcal{A}_1 = (a_1, c_1, x_1, \ldots, x_{l+1}, c_{l+1}), \]
\[ \mathcal{A}_k = (a_k, c_{k-1}, c_k, c_{k-1}, c_k, x_k, c_{k-1}, c_k, x_k, \ldots, x_k, c_{k-1}, c_k), \]
\[ \mathcal{A}_{l+1} = (x_{l+1}, c_{l+1}, x_{l+1}, c_{l+1}, x_{l+1}), \]
and form the sequence \( \mathcal{S} \)
\[ \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \ldots, \mathcal{S}_l, \mathcal{S}_{l+1}, \mathcal{S}_{l+2}, \ldots. \]
The number of distinct terms in \( \mathcal{S} \) is 
\[ n = l(m-1) + l + 1 = lm + 1 \]
and the length of \( \mathcal{S} \)
\[ N = \sum_{k=1}^{l+1} L(\mathcal{A}_k) + \sum_{j=1}^{l} L(\mathcal{S}_j) \]
\[ = 2(2m-1) + (l-1)(5m-4) + l(m-1) = 6l - 2m - 3l + 2. \]

It remains to prove that \( \mathcal{S} \) is admissible. Clearly it contains no 
immediate repetitions. Consider two elements \( a < b \). If \( a = x_k, b = x_k \)}
Density inequalities for a restricted sum
of sets of lattice points

by

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§ 1. Introduction. Let \( Q \) be the set of all 2-dimensional lattice points \((x, y)\) such that \( x \) and \( y \) are nonnegative integers and either \( x \) or \( y \) is positive. Addition and subtraction of elements of \( Q \) will be done componentwise.

Let a set \( B \) of positive integers be a basis of order \( k \) for the positive integers. Then clearly any subset of \( Q \) containing all points \((b, 0)\) and \((0, b)\) with \( b \in B \) is a basis for \( Q \), and is of order no more than \( 2k \), if addition of subsets of \( Q \) is done as in [2]. For example, Schnirelmann has shown ([3], p. 680) that the set consisting of 1 and all positive primes \( 4t + 3 \) is a basis for the positive integers. Therefore the set \( P \) consisting of 1, \( i \), and the Gaussian primes \( p + qi \) where \((p, q) \in Q \) is a basis for the set of all Gaussian integers \( a + bi \) where \((a, b) \in Q \).

However, it might be of interest to know whether these Gaussian integers can be written as sums of elements of \( P \) in some less trivial way than as sums of elements on the axes. More specifically, we might ask which subsets \( A \) of \( Q \) have the property that each point \((x, y)\) of \( Q \) can be written as a sum of no more than \( k \) elements of \( A \), and in such a way that no two of its summands are on different axes. This question leads us to make the following definition of sums of sets in \( Q \). These restricted sum sets are not only smaller than the sum sets used in [1] and [2], but this addition of sets is not, in general, associative. In particular, we cannot assume \( kA + A = (k+1)A \).

§ 2. Definitions and notation. For any \( k \) subsets \( A_1, \ldots, A_k \) of \( Q \) let \( A_1 + \ldots + A_k \) be the set of all \( a_1 + \ldots + a_k \) in \( Q \) such that \( 1 \) each \( a_i \in A_i \cup \{(0, 0)\} \), and \( 2 \) \( i \) if two of the summands have the forms \( a_i = (a, 0) \) and \( a_j = (0, b) \) then one of them is \((0, 0)\). If \( A_1 = \ldots = A_k = A \) we write \( kA \) instead of \( A + \ldots + A \).

For any \( p \) and \( q \) in \( Q \), \( p < q \) if and only if \( q - p \in Q \). Let \( L_q = \{ p \in Q : p \leq q \} \). We will also use the definitions and notation of [2], except that any subset of \( Q \) of \( A \) the density of \( A \), as defined in [2], will be denoted by \( d(A) \).

References