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The number of k-free divisors of an integer

by

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1. Introduction. Let k be a fixed integer $\geqslant 2$. A divisor $d > 0$ of the positive integer n is called *k-free* if d is not divisible by kth power of any integer > 1. Let $\tau_{(k)}(n)$ denote the number of k-free divisors of n. It has been recently shown by the first author (cf. [10], Theorem 3.1) that for $x \geqslant 2$,

$$(1.1)\qquad \sum_{n\leqslant x}\tau_{(k)}(n) = \frac{x}{\zeta(k)}\left(\log x + 2\gamma - 1 - \frac{k\zeta'(k)}{\zeta(k)}\right) + \Delta_k(x),$$

where $\Delta_k(x) = O(x^{1/3})$ or $O(x^{\alpha})$ according as $k = 3$ or $k \geqslant 4$; α being the number which appears in the Dirichlet divisor problem, viz.,

$$(1.2)\qquad \sum_{n\leqslant x}\tau(n) = x(\log x + 2\gamma - 1) + O(x^{\alpha}),$$

where $\tau(n)$ is the number of divisors of n.

It is known that $\frac{1}{4} < \alpha < \frac{1}{3}$ (cf. [3], p. 272). The best known result is obtained by Yin Wen-lin [13], who proved that the error term in (1.2) is $O(x^{\frac{13}{40}+\varepsilon})$, where $\varepsilon > 0$. There is a conjecture that $\alpha = \frac{1}{4}+\varepsilon$. In the formula (1.1), $\zeta(s)$ denotes the Riemann zeta function, $\zeta'(s)$ its derivative and γ is Euler's constant.

The case $k = 2$ was originally considered in 1874 by Mertens [5], who proved that $\Delta_2(x) = O(x^{1/2}\log x)$ and an alternative proof has been given by Cohen [1]. Gioia and Vaidya [2] have improved this result to $\Delta_2(x) = O(x^{1/2})$. Recently, Saffari Bahman [6] has also shown that $\Delta_2(x) = O(x^{1/2})$ and $\Delta_3(x) = O(x^{1/3})$. These results have already been obtained by Otto Hölder [4] in 1932, who also proved that $\Delta_k(x) = O(x^{33/100})$ for $k \geqslant 4$. However, as early as in 1924, Zyoiti Suetuna [9] has obtained a better order estimate for $\Delta_2(x)$. He proved that $\Delta_2(x) = O(x^{1/2}\exp\{-A\sqrt{\log x \log\log x}\})$, where A is a positive constant. Further, he stated that if the Riemann hypothesis is true, then $\Delta_2(x) = O(x^{\theta})$, where $\theta < \frac{367}{768}$. Recently, Saffari Bahman [7] obtained some results in case $k \geq 4$ and he [8] stated estimates for $\Delta_k(x)$ on the assumption of the Riemann hypothesis in cases $k = 2$ and 3.

The object of the present paper is to further improve the order estimates of $\Delta_k(x)$ for $k \geq 2$. In fact, we prove that

$$\Delta_2(x) = O\big(x^{1/2}\delta(x)\big) \quad \text{and} \quad \Delta_3(x) = O\big(x^{1/3}\delta(x)\big),$$

where $\delta(x) = \exp\{-A\log^{3/5}x(\log\log x)^{-1/5}\}$, A being a positive constant. We also show on the assumption of the Riemann hypothesis that

$$\Delta_2(x) = O\big(x^{\frac{2-a}{5-4a}}w(x)\big) \quad \text{and} \quad \Delta_3(x) = O\big(x^{\frac{2-a}{7-6a}}w(x)\big),$$

where a is given by (1.2) and $w(x) = \exp\{A\log x(\log\log x)^{-1}\}$, A being a positive constant. Further, we remark that the result $\Delta_k(x) = O(x^a)$ for $k \geq 4$, mentioned in (1.1) above, can not be improved further even on the basis of the Riemann hypothesis.

2. Preliminaries. In this section we prove some lemmas which are needed in our discussion. Throughout the following x denotes a real variable ≥ 3. We need the following best known estimate concerning the Möbius function $\mu(n)$ obtained by Arnold Walfisz [12]:

Lemma 2.1 (cf. [12]; Satz 3, p. 191).

$$M(x) = \sum_{n\leq x}\mu(n) = O\big(x\delta(x)\big) \tag{2.1}$$

where

$$\delta(x) = \exp\{-A\log^{3/5}x(\log\log x)^{-1/5}\}, \tag{2.2}$$

A being a positive constant.

Lemma 2.2. *For any $s > 1$,*

$$\sum_{n\leq x}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O\left(\frac{\delta(x)}{x^{s-1}}\right). \tag{2.3}$$

Proof. From

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

we have

$$\sum_{n\leq x}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} - \sum_{n>x}\frac{\mu(n)}{n^s}.$$

Putting $f(n) = 1/n^s$, it can be easily shown that

$$f(n+1)-f(n) = O\left(\frac{1}{n^{s+1}}\right).$$

Therefore by partial summation and (2.1),

$$\sum_{n>x}\mu(n)f(n) = -M(x)f([x]+1) - \sum_{n>x}M(n)[f(n+1)-f(n)]$$
$$= O\left(\frac{\delta(x)}{x^{s-1}}\right) + O\left(\sum_{n>x}\frac{\delta(n)}{n^s}\right).$$

Since $\delta(x)$ is monotonic decreasing and $\sum_{n>x}\frac{1}{n^s} = O\left(\frac{1}{x^{s-1}}\right)$,

$$\sum_{n>x}\mu(n)f(n) = O\left(\frac{\delta(x)}{x^{s-1}}\right) + O\left(\delta(x)\sum_{n>x}\frac{1}{n^s}\right) = O\left(\frac{\delta(x)}{x^{s-1}}\right),$$

which proves the lemma.

Lemma 2.3. *For $s > 1$,*

$$\sum_{n\leq x}\frac{\mu(n)\log n}{n^s} = \frac{\zeta'(s)}{\zeta^2(s)} + O\left(\frac{\delta(x)\log x}{x^{s-1}}\right). \tag{2.4}$$

Proof. From

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

we have

$$\sum_{n\leq x}\frac{\mu(n)\log n}{n^s} = \frac{\zeta'(s)}{\zeta^2(s)} - \sum_{n>x}\frac{\mu(n)\log n}{n^s}.$$

Putting $g(n) = \frac{\log n}{n^s}$, it can be easily shown that

$$g(n+1)-g(n) = O\left(\frac{\log n}{n^{s+1}}\right).$$

Therefore by partial summation and (2.1),

$$\sum_{n>x}\mu(n)g(n) = -M(x)g([x]+1) - \sum_{n>x}M(n)[g(n+1)-g(n)]$$
$$= O\left(\frac{\delta(x)\log x}{x^{s-1}}\right) + O\left(\sum_{n>x}\frac{\delta(n)\log n}{n^s}\right)$$
$$= O\left(\frac{\delta(x)\log x}{x^{s-1}}\right) + O\left(\delta(x)\sum_{n>x}\frac{\log n}{n^s}\right) = O\left(\frac{\delta(x)\log x}{x^{s-1}}\right),$$

which proves the lemma.

Lemma 2.4 (cf. [11], Theorem 14–26 (A), p. 316). *If the Riemann hypothesis is true, then*

$$M(x) = \sum_{n\leqslant x}\mu(n) = O\big(x^{1/2}w(x)\big) \tag{2.5}$$

where

$$w(x) = \exp\{A\log x(\log\log x)^{-1}\}, \tag{2.6}$$

A being a positive constant.

Lemma 2.5. *If the Riemann hypothesis is true, then for any* $s > 1$,

$$\sum_{n\leqslant x}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O\big(x^{\frac{1}{2}-s}w(x)\big). \tag{2.7}$$

Proof. Following the same argument adopted in Lemma 2.2, we get this lemma. We have only to replace $\delta(x)$ in Lemma 2.2 by $x^{-1/2}w(x)$.

Lemma 2.6. *If the Riemann hypothesis is true, then for any* $s > 1$,

$$\sum_{n\leqslant x}\frac{\mu(n)\log n}{n^s} = \frac{\zeta'(s)}{\zeta^2(s)} + O\big(x^{\frac{1}{2}-s}w(x)\log x\big). \tag{2.8}$$

Proof. This follows by adopting the same argument in Lemma 2.3 and replacing $\delta(x)$ by $x^{-1/2}w(x)$.

3. Main results. In this section we prove the following:

Theorem 3.1. *For* $x \geqslant 3$,

$$\sum_{n\leqslant x}\tau_{(k)}(n) = \frac{x}{\zeta(k)}\left(\log x+2\gamma-1-\frac{k\zeta'(k)}{\zeta(k)}\right) + \Delta_k(x), \tag{3.1}$$

where $\Delta_k(x) = O\big(x^{1/k}\delta(x)\big)$ *or* $O(x^a)$, *according as* $k = 2, 3$ *or* $k \geqslant 4$; $\delta(x)$ *being given by* (2.2) *and* a *is given by* (1.2).

Proof. It has been proved (cf. [10], Theorem 2.2) that

$$\tau_{(k)}(n) = \sum_{d^k\delta=n}\mu(d)\tau(\delta).$$

Hence

$$\sum_{n\leqslant x}\tau_{(k)}(n) = \sum_{n\leqslant x}\sum_{d^k\delta=n}\mu(d)\tau(\delta) = \sum_{d^k\delta\leqslant x}\mu(d)\tau(\delta), \tag{3.2}$$

the summation on the right being taken over all ordered pairs (d, δ) such that $d^k\delta \leqslant x$.

Let $z = x^{1/k}$. Further, let $0 < \varrho = \varrho(x) < 1$, where the function $\varrho(x)$ will be suitably chosen later.

From (3.2), we have

$$\sum_{n\leqslant x}\tau_{(k)}(n) = \sum_{n^kr\leqslant x}\mu(n)\tau(r).$$

If $n^kr \leqslant x$, then both $n > \varrho z$ and $r > \varrho^{-k}$ can not simultaneously hold good, and so we have

$$\sum_{n\leqslant x}\tau_{(k)}(n) = \sum_{\substack{n^kr\leqslant x\\ n\leqslant \varrho z}}\mu(n)\tau(r) + \sum_{\substack{n^kr\leqslant x\\ r\leqslant\varrho^{-k}}}\mu(n)\tau(r) - \sum_{\substack{n\leqslant\varrho z\\ r\leqslant\varrho^{-k}}}\mu(n)\tau(r) \tag{3.3}$$
$$= S_1+S_2-S_3, \text{ say}.$$

Now, by (1.2),

$$S_1 = \sum_{\substack{n^kr\leqslant x\\ n\leqslant\varrho z}}\mu(n)\tau(r) = \sum_{n\leqslant\varrho z}\mu(n)\sum_{r\leqslant x/n^k}\tau(r) \tag{3.4}$$
$$= \sum_{n\leqslant\varrho z}\mu(n)\left\{\frac{x}{n^k}\log\frac{x}{n^k} + \frac{x}{n^k}(2\gamma-1) + O\left(\frac{x^a}{n^{ka}}\right)\right\}$$
$$= (x\log x+2\gamma-1)\sum_{n\leqslant\varrho z}\frac{\mu(n)}{n^k} - kx\sum_{n\leqslant\varrho z}\frac{\mu(n)\log n}{n^k} + E_k(x),$$

where

$$E_k(x) = O\left(x^a\sum_{n\leqslant\varrho z}\frac{1}{n^{ka}}\right).$$

If $k = 2$ or 3, then since $\frac{1}{4} < a < \frac{1}{3}$, we have $ka < 1$, so that

$$E_k(x) = O\big(x^a(\varrho z)^{1-ka}\big) = O(\varrho^{1-ka}z);$$

and if $k \geqslant 4$, then $ka > 1$, so that $E_k(x) = O(x^a)$. Hence

$$E_k(x) = O(\varrho^{1-ka}z) \quad \text{or} \quad O(x^a) \tag{3.5}$$

according as $k = 2, 3$ or $k \geqslant 4$.

Now, by (3.4), (2.3) and (2.4),

$$S_1 = x(\log x+2\gamma-1)\left\{\frac{1}{\zeta(k)} + O\left(\frac{\delta(\varrho z)}{(\varrho z)^{k-1}}\right)\right\} - \tag{3.6}$$
$$-kx\left\{\frac{\zeta'(k)}{\zeta^2(k)} + O\left(\frac{\delta(\varrho z)\log(\varrho z)}{(\varrho z)^{k-1}}\right)\right\} + E_k(x)$$
$$= \frac{x}{\zeta(k)}\left(\log x+2\gamma-1-\frac{k\zeta'(k)}{\zeta(k)}\right) + O\big(\varrho^{1-k}z\log z\,\delta(\varrho z)\big) + E_k(x).$$

We have,

$$S_2 = \sum_{\substack{n^k r \leqslant x \\ r \leqslant \varrho^{-k}}} \mu(n)\tau(r) = \sum_{r \leqslant \varrho^{-k}} \tau(r) \sum_{n \leqslant \left(\frac{x}{r}\right)^{1/k}} \mu(n) = \sum_{r \leqslant \varrho^{-k}} \tau(r) M\left\{\left(\frac{x}{r}\right)^{1/k}\right\}$$

$$= O\left(x^{1/k} \sum_{r \leqslant \varrho^{-k}} \tau(r) r^{-1/k} \delta\left(\frac{x^{1/k}}{r^{1/k}}\right)\right),$$

by (2.1). Since $\delta(x)$ is monotonic decreasing and $\frac{x^{1/k}}{r^{1/k}} > \varrho z$, we have $\delta\left(\frac{x^{1/k}}{r^{1/k}}\right) < \delta(\varrho z)$. Also,

$$\sum_{r \leqslant \varrho^{-k}} \tau(r) r^{-1/k} = \sum_{r \leqslant \varrho^{-k}} \sum_{d\delta = r} d^{-1/k} \delta^{-1/k} = \sum_{d \leqslant \delta \varrho^{-k}} d^{-1/k} \delta^{-1/k}$$

$$= \sum_{d \leqslant \varrho^{-k}} d^{-1/k} \sum_{\delta \leqslant \varrho^{-k}/d} \delta^{-1/k} = O\left(\sum_{d \leqslant \varrho^{-k}} d^{-1/k} \left(\frac{\varrho^{-k}}{d}\right)^{1-1/k}\right)$$

$$= O\left(\varrho^{1-k} \sum_{d \leqslant \varrho^{-k}} \frac{1}{d}\right) = O\left(\varrho^{1-k} \log(\varrho^{-k})\right).$$

Hence

(3.7) $$S_2 = O\left(\varrho^{1-k} z \log\left(\frac{1}{\varrho}\right) \delta(\varrho z)\right).$$

Also, we have

(3.8) $$S_3 = \sum_{\substack{n \leqslant \varrho z \\ r \leqslant \varrho^{-k}}} \mu(n)\tau(r) = \sum_{r \leqslant \varrho^{-k}} \tau(r) M(\varrho z)$$

$$= O\left(\varrho z \delta(\varrho z) \varrho^{-k} \log(\varrho^{-k})\right) = O\left(\varrho^{1-k} z \log\left(\frac{1}{\varrho}\right) \delta(\varrho z)\right).$$

Hence, by (3.3), (3.6), (3.7) and (3.8),

(3.9) $$\sum_{n \leqslant x} \tau_{(k)}(n) = \frac{x}{\zeta(k)}\left(\log x + 2\gamma - 1 - \frac{k\zeta'(k)}{\zeta(k)}\right) + O\left(\varrho^{1-k} z \log z \delta(\varrho z)\right) +$$
$$+ O\left(\varrho^{1-k} z \log\left(\frac{1}{\varrho}\right) \delta(\varrho z)\right) + E_k(x).$$

Now, we choose

(3.10) $$\varrho = \varrho(x) = \{\delta(x^{1/2k})\}^{1/k},$$

and write

(3.11) $$f(x) = \log^{3/5}(x^{1/2k})\{\log\log(x^{1/2k})\}^{-1/5} = \left(\frac{1}{2k}\right)^{3/5} U^{3/5}(V - \log 2k)^{-1/5},$$

where $U = \log x$ and $V = \log\log x$.

(3.12) For $V \geqslant 2\log 2k$, that is, $U \geqslant 4k^2$, $x \geqslant \exp(4k^2)$, we have

$$V^{-1/5} \leqslant (V - \log 2k)^{-1/5} \leqslant (V/2)^{-1/5}$$

and therefore

(3.13) $$\tfrac{1}{2} k^{-3/5} U^{3/5} V^{-1/5} \leqslant f(x) \leqslant k^{-3/5} U^{3/5} V^{-1/5}.$$

(3.14) We assume without loss of generality that the constant A in (2.2) is less than 1.

By (3.10), (2.2) and (3.11), we have

(3.15) $$\varrho = \exp\left\{-\frac{A}{k} f(x)\right\}.$$

By (3.12), we have

$$k^{-8/5} U^{3/5} V^{-1/5} \leqslant \frac{U}{2k}.$$

Hence, by (3.13), (3.14), (3.15) and the above,

$$\varrho \geqslant \exp(-Ak^{-8/5} U^{3/5} V^{-1/5}) \geqslant \exp(-k^{-8/5} U^{3/5} V^{-1/5})$$
$$\geqslant \exp\left(-\frac{U}{2k}\right) = \exp\left(-\frac{\log x}{2k}\right),$$

so that $\varrho \geqslant x^{-1/2k}$.

Hence

(3.16) $$\log\left(\frac{1}{\varrho}\right) \leqslant \log(\sqrt{z}) = O(\log x) \quad \text{and} \quad \varrho z \geqslant x^{1/2k}.$$

Since $\delta(x)$ is monotonic decreasing,

$$\delta(\varrho z) \leqslant \delta(x^{1/2k}) = \varrho^k,$$

so that by (3.13) and (3.15), we have

(3.17) $$\varrho^{1-k} \delta(\varrho z) \leqslant \varrho \leqslant \exp\left\{-\frac{A}{2} k^{-8/5} U^{3/5} V^{-1/5}\right\}.$$

Hence, by (3.16) and (3.17), the first and second O-terms of (3.9) are

$$O\left(x^{1/k} \exp\left\{-\frac{A}{2} k^{-8/5} U^{3/5} V^{-1/5}\right\} \log x\right).$$

Hence, if $\Delta_k(x)$ denotes the error term in the asymptotic formula (3.9), then we have

(3.18) $$\Delta_k(x) = O\left(x^{1/k} \exp\left\{-\frac{A}{2} k^{-8/5} U^{3/5} V^{-1/5}\right\} \log x\right) + E_k(x).$$

Case $k = 2$ or 3. In this case, we have $0 < 1-k\alpha < 1$, since $\frac{1}{4} < \alpha < \frac{1}{3}$. By (3.15) and (3.13), we have

$$\varrho^{1-k\alpha} = \exp\left\{-\frac{A(1-k\alpha)}{k} f(x)\right\} \leqslant \exp\left\{-\frac{A(1-k\alpha)}{2} k^{-8/5} U^{3/5} V^{-1/5}\right\},$$

so that by (3.5),

$$E_k(x) = O\left(x^{1/k} \exp\left\{-\frac{A(1-k\alpha)}{2} k^{-8/5} U^{3/5} V^{-1/5}\right\}\right).$$

Again, since $0 < 1-k\alpha < 1$, the first O-term in (3.18) is also of the above order of $E_k(x)$. Hence

$$\Delta_k(x) = O\left(x^{1/k} \exp\{-B \log^{3/5} x (\log\log x)^{-1/5}\}\right), \tag{3.19}$$

where B is a positive constant.

Hence Theorem 3.1 follows in this case.

Case $k \geqslant 4$. In this case, by (3.5), $E_k(x) = O(x^{\alpha})$ and the first O-term in (3.18) is $O(x^{1/k})$. Hence $\Delta_k(x) = O(x^{\alpha})$.

Hence Theorem 3.1 is completely proved.

THEOREM 3.2. *If the Riemann hypothesis is true, then for $x \geqslant 3$,*

$$\sum_{n \leqslant x} \tau_{(k)}(n) = \frac{x}{\zeta(k)}\left(\log x + 2\gamma - 1 - \frac{k\zeta'(k)}{\zeta(k)}\right) + \Delta_k(x), \tag{3.20}$$

where

$$\Delta_k(x) = O\left(x^{\frac{2-\alpha}{1+2k(1-\alpha)}} w(x)\right) \quad \text{or} \quad O(x^{\alpha}),$$

according as $k = 2, 3$ or $k \geqslant 4$; where α is given by (1.2) *and $w(x)$ is given by* (2.6).

Proof: Following the same procedure adopted in Theorem 3.1 and making use of (2.7) and (2.8) instead of (2.3) and (2.4), we get that

$$\Delta_k(x) = O\left(\varrho^{\frac{1}{2}-k} z^{\frac{1}{2}} \log z\, w(\varrho z)\right) + O\left(\varrho^{\frac{1}{2}-k} z^{\frac{1}{2}} \log\left(\frac{1}{\varrho}\right) w(\varrho z)\right) + E_k(x). \tag{3.21}$$

Case $k = 2$ or 3. In this case, choosing

$$\varrho = z^{-\frac{1}{1+2k(1-\alpha)}},$$

we see that $0 < \varrho < 1$, $1/\varrho < z$, so that $\log(1/\varrho) < \log z$, and

$$\varrho^{\frac{1}{2}-k} z^{\frac{1}{2}} = \varrho^{1-k\alpha} z = x^{\frac{2-\alpha}{1+2k(1-\alpha)}}.$$

Since $w(x)$ is monotonic increasing, we have $w(\varrho z) < w(z)$.

Hence, by (3.21) and the above, we have

$$\Delta_k(x) = O\left(x^{\frac{2-\alpha}{1+2k(1-\alpha)}} w(x^{\frac{1}{k}}) \log x\right) = O\left(x^{\frac{2-\alpha}{1+2k(1-\alpha)}} w(x)\right).$$

Case $k \geqslant 4$. We have $w(x) = O(x^{\varepsilon})$ and $\log z = O(x^{\varepsilon})$ for every $\varepsilon > 0$. We assume without loss of generality that $\varepsilon < 1$. Hence, by (3.21), we have

$$\Delta_k(x) = O\left(\varrho^{\frac{1}{2}-k+\varepsilon} z^{\frac{1}{2}+2\varepsilon}\right) + O\left(\varrho^{\frac{1}{2}-k+\varepsilon} z^{\frac{1}{2}+\varepsilon} \log\left(\frac{1}{\varrho}\right)\right) + O(x^{\alpha}). \tag{3.22}$$

Now, choosing

$$\varrho = z^{-\frac{2k\alpha-1+4\varepsilon}{2k-1-2\varepsilon}},$$

we see that $0 < \varrho < 1$, $1/\varrho < z$, so that $\log(1/\varrho) < \log z = O(z^{\varepsilon})$ and

$$\varrho^{\frac{1}{2}-k+\varepsilon} z^{\frac{1}{2}+2\varepsilon} = x^{\alpha}.$$

Hence, by (3.22), $\Delta_k(x) = O(x^{\alpha})$.

Thus Theorem 3.2 is proved.

Remark. In case $k \geqslant 4$, we may choose the function $\varrho = \varrho(x)$, which tends to zero as $x \to \infty$, more rapidly than the function chosen above. In such a case, although the first and second O-terms in (3.22) are $O(x^{\beta})$ where $\beta < \alpha$, but because of the third O-term in (3.22), we again get $\Delta_k(x) = O(x^{\alpha})$. Hence we cannot improve the result that $\Delta_k(x) = O(x^{\alpha})$ for $k \geqslant 4$, even on the assumption of the Riemann hypothesis.

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Some properties of Davenport–Schinzel sequences

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1. Introduction. Davenport and Schinzel [1] introduced the following problem on sequences. Suppose that $Z_n = \{1, 2, \ldots, n\}$ and that $V_n(d)$ denotes the set of all sequences $a, b, a, b, a, \ldots$ of length d, where a and b are distinct elements of Z_n. One considers all sequences made up of elements from Z_n such that no two adjacent elements are equal and no subsequence is an element of $V_n(j)$ for $j > d$. If $N_d(n)$ denotes the maximal length of any such sequence, we call any sequence of length $N_d(n)$ a *Davenport–Schinzel sequence* (or a DS sequence.) The problem is to determine all DS sequences and, in particular, to determine $N_d(n)$. Of course, it will be sufficient to determine all *normal* DS sequences; i.e., DS sequences in which the elements appear in order from left to right.

Davenport and Schinzel consider $N_d(n)$ for fixed d and obtain the results $N_1(n) = 1$, $N_2(n) = n$, $N_3(n) = 2n-1$. Two proofs of the result for $N_3(n)$ are given, the second being based on Mrs. Turán's observation that, in a DS sequence of length $N_3(n)$, there is some element which occurs exactly one time. They remark that $1, 2, 1, 3, 1, \ldots, 1, n, 1$ and $1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1$ are both DS sequences for $d = 3$. Finally, they obtain bounds on $N_d(n)$ for fixed d, including the result $N_4(n) \geqslant 5n-C$, where C is a constant.

The authors [2] have proved that $N_d(2) = d$ and, for $d > 3$, $N_d(3) = 3d-4$ or $3d-5$, depending upon whether d is even or odd. It is also shown that, for $n = 3$, a DS sequence is unique.

The object of the present paper is to prove

$$N_d(4) = 6d-14 \quad (d \text{ odd}, \ d > 4), \tag{1.1}$$

$$N_d(4) = 6d-13 \quad (d \text{ even}, \ d > 4), \tag{1.2}$$

$$\binom{n}{2} d - C \leqslant N_d(n) \quad (d > n), \tag{1.3}$$

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