91. *О некоторых функциональных уравнениях, являющихся следствием уравнений типа Римана*, ИАН СССР, сер. матем. 24 (1960), 469–474.
92. *О теореме Минковского для линейных форм и теоремах переноса*, в книге *Касселс В. С., Введение в теорию диофантовых приближений*, Москва 1961, 202–209.
93. *Элементарные методы в аналитической теории чисел*, Москва 1962, 1–272 (совместно с Ю. В. Линником).
94. *Об одной ошибочной работе Н. И. Гаврилова*, УМН 17 (1962), вып. 1 (103), 265–267 (совместно с Ю. В. Линником, Н. Г. Чудаковым и В. Я. Якубовичем).
95. *Оценка остаточного члена в предельной теореме для рекуррентных событий*, Теория вероятн. и ее примен. 9 (1964), 327–331.
96. *О нулях аналитической функции с заданной арифметикой коэффициентов и представлении чисел*, Acta Arith. 11 (1965), 97–114.
97. *О приближении многочленами со специально выбранными коэффициентами*, УМН 21 (1966), вып. 3 (129), 225–229.
98. *150 лет со дня рождения К. Вейерштрасса*, Вестн. АН СССР, 1966, № 2, 184–185.
99. *Вычеты и их приложения*, Москва 1966, 1–112.
100. *Об оценке мнимых частей корней многочленов с ограниченными производными их логарифмов на действительной оси*, Матем. сб. 71 (113) (1966), 289–296.
101. *The distribution of fractional quantities*, Journ. Math. Anal. Appl. 15 (1966), 65–82.
102. *Об одной теореме единственности*, Матем. зам. 1 (1967), 321–324.
103. *О функциях, принимающих целые значения*, Матем. зам. 1 (1967), 509–513.
104. *Теория чисел в Московском университете за советский период*, Вестн. МГУ (1), № 5 (1967), 7–11.
105. *Sur quelques questions concernant les fonctions entières à valeurs entières*, Compt. Rend. Acad. Sci. (Paris), 264 (1967), 932–934.
106. *О некоторых комбинаторных свойствах (0,1)-матриц*, Матем. сб. 75 (117) (1968), 3.
107. *Sur les nombres qui ont des propriétés additives et multiplicatives donnée*, Acta Arith. 13 (1968), 259–265.
108. *К седьмой проблеме Гильберта*, в сборнике *Проблемы Гильберта*, Москва 1969, 121–127.

ACTA ARITHMETICA
XVII (1971)

Weak forms of Mann's density theorem extended to sets of lattice points

by

BETTY KVARDA GARRISON (San Diego, Calif.)

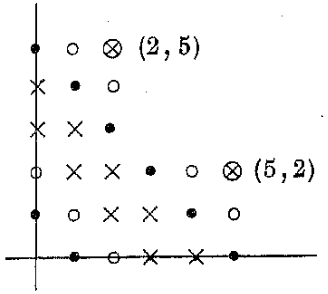
§ 1. Introduction. Let Q_n be the set of all nonzero n-dimensional lattice points with nonnegative integer coordinates. We will use the usual componentwise addition and subtraction of elements of Q_n, and the usual partial ordering: For any x and y in Q_n, $x<y$ if $y-x$ is in Q_n. If S is any subset of Q_n and F is any finite subset of Q_n then $S(F)$ will denote the number of elements in $S\cap F$. For any x in Q_n let $\mathrm{L}x=\{y\in Q_n\colon y\leqslant x\}$. If A and B are subsets of Q_n, $A+B$ will denote the set of all elements of the form $a, b, a+b$, where $a\in A$, $b\in B$, while $A-B$ is the set of all elements of A which are not in B.

A fundamental subset of Q_n or, briefly, a fundamental set, is defined to be any finite nonempty subset R of Q_n such that $x\in R$ implies $\mathrm{L}x\subseteq R$. For any subset A of Q_n Müller [8] has defined the density of A to be the glb $A(R)/Q_n(R)$, taken over all fundamental sets R. For $n=1$ this is clearly the Schnirelmann density of A.

With this family of fundamental sets and definition of density, several results have been obtained for subsets of Q_n which are analogous to well-known theorems of additive number theory for sets of positive integers. (See [2], [3], [5], [6], [8], [9].) In this note we will discuss the extension of the famous theorem of Mann [7] to Q_n. Using the notation given above, an n-dimensional analogue to Mann's theorem may be stated as follows.

(I) Let A and B be subsets of Q_n, let $C=A+B$, and let R be any fundamental subset of Q_n. Then either $C(R)=Q_n(R)$ or there exists a fundamental set $W\subseteq R$ such that no maximal element of W is in C and $C(R)/Q_n(R)\geqslant[A(W)+B(W)]/Q_n(W)$.

The statement (I) is false for $n>1$, as is shown by the following example for Q_2. (For $n>2$ this example may be embedded in Q_n). Let the fundamental set $R=\mathrm{L}(2,5)\cup\mathrm{L}(5,2)$. In the figure below lattice points of $(A-B)\cap R$ are marked by $\times$, those of $A\cap B\cap R$ by $\bullet$, those of $(C \cap R)-(A \cup B)$ by $\circ$, and those of $R-C$ by $\otimes$. The set $(B-A) \cap R$ is empty.



In this example the fundamental sets $W \subseteq R$ whose maximal points are not in C are just R, $L(2,5)$, and $L(5,2)$, and it is easily calculated that $C(R)/Q_n(R) < [A(W)+B(W)]/Q_2(W)$ for each of these. However, we see that several fundamental sets W in R satisfy the condition $C(R)/Q_2(R) \geqslant [A(W)+B(W)]/Q_2(W)$. The smallest of these is the set $\{(1,0),(2,0),(0,1),(1,1),(0,2)\}$, whose maximal points are all in $R-(A \cup B)$.

If we delete the condition in (I) that the maximal elements of W are not in C, or if we replace it by the condition that the maximal elements of W are not in $A \cup B$, we still obtain statements which are in general false, as will be shown in § 6. Both statements thus obtained are, however, true for important special cases.

§ 2. Statements of theorems. In this section Q denotes a fixed Q_n. Let A and B be subsets of Q, let $C = A+B$, and let R be a fundamental set such that $C(R) < Q(R)$. Let S be a fundamental set such that (i) $S \subseteq \bigcup_{g \in R-C} \mathrm{L}g$, (ii) $C(S) < Q(S)$, (iii) $C(S) \geqslant A(S)+B(S)$, and (iv) if S' is any fundamental set satisfying (i), (ii), (iii), then $Q(S') \leqslant Q(S)$. (The existence of such an S is implied by the Remark following Lemma 3 in § 3.)

Let $Q(R-C) = k$ and $Q(S-C) = s$, $1 \leqslant s \leqslant k$. If $s < k$, hence $S-C \neq R-C$, let $T = R-S$ and let $T-C = \{g_1, \ldots, g_{k-s}\}$. Let $X_i = \mathrm{L}g_i - S$, let $X_i' = X_i - \{g_i\}$, and let $Y_i = \{g_i - x\colon x \in X_i'\}$ for all $i = 1, \ldots, k-s$.

THEOREM 1. *There exists a fundamental set $W \subseteq R$ such that the maximal elements of W are not in $A \cup B$ and*

$$C(R)/Q(R) \geqslant [A(W)+B(W)]/Q(W)$$

if $Q(R)/k \geqslant Q(S)/s$, or if $s < k$ and there exists a nonempty subset $\{g_1, \ldots, g_t\}$ of $T-C$ such that g_i is a minimal element of $T-C$ for all $i = 1, \ldots, t$,

$$Q(R)/k \geqslant Q\Big(\bigcup_{i=1}^{t} X_i\Big)/t,$$

and, if $t > 1$, the set $Y_i' = Y_i - \bigcup_{j=1}^{i-1} Y_j$ is not empty for each $i = 2, \ldots, t$.

We note that when $s = k$ the condition $Q(R)/k \geqslant Q(S)/s$ is satisfied.

THEOREM 2. *There exists a fundamental set $W \subseteq R$ such that the maximal elements of W are not in $A \cup B$ and*

$$C(R)/Q(R) \geqslant [A(W)+B(W)]/Q(W) \quad \text{if} \quad s < k$$

and there exists a linearly ordered subset $\{g_1, \ldots, g_t\}$ of $T-C$ such that

$$g_j \notin \bigcup_{i=1}^{t} X_i, \quad t < j \leqslant k-s,$$

and

$$Q(R)/k \geqslant Q\Big(\bigcup_{i=1}^{t} X_i\Big)/t.$$

§ 3. Preliminary lemmas. The following Lemma 3 is stated and proved in [4] for Q_1. The proof is unchanged for $Q = Q_n$.

LEMMA 1. *Let R be a finite nonempty subset of Q, and let k, s, t be positive integers such that $s+t = k$. Further suppose that R is partitioned by two nonempty sets S and T. Then*

$$[Q(R)-k]/Q(R) \geqslant [Q(T)-t]/Q(T) \Leftrightarrow Q(S)/s \geqslant Q(T)/t \Leftrightarrow Q(R)/k \geqslant Q(T)/t.$$

Proof. $[Q(R)-k]/Q(R) = [Q(S)+Q(T)-s-t]/[Q(S)+Q(T)]$, etc.

LEMMA 2. *Let R, A, B, C be the sets introduced in § 2, and let W be any fundamental set such that $W \subseteq R$, $Q(W) = A(W)+B(W)+w$, $w > 0$. Then*

$$C(R)/Q(R) \geqslant [A(W)+B(W)]/Q(W) \Leftrightarrow Q(R)/k \geqslant Q(W)/w.$$

Proof. Use Lemma 1 and $C(R) = Q(R)-k$.

LEMMA 3. *Let A and B be any subsets of Q, let g be any element of $Q-(A+B)$, let X be any subset of $\mathrm{L}g-\{g\}$, and let $Y = \{g-x\colon x \in X\}$. If $Q(X) = A(X)+B(X)-u$ then $Q(Y) \geqslant A(Y)+B(Y)+u$.*

Remark. If X, Y, g are defined as in Lemma 3, and if $X = Y$, then it is clear that $u \leqslant 0$. This is the case when $X = \mathrm{L}g-\{g\}$, for example. We know $g \notin A \cup B$, so $Q(\mathrm{L}g) \geqslant A(\mathrm{L}g)+B(\mathrm{L}g)+1$ for any $g \in Q-(A+B)$. If g is a minimal point of $Q-(A+B)$ then $C(\mathrm{L}g) = Q(\mathrm{L}g)-1 \geqslant A(\mathrm{L}g)+B(\mathrm{L}g)$.

LEMMA 4. *The sets introduced in § 2 satisfy the following conditions:*

(1) $C(S) = Q(S)-s = A(S)+B(S)$.

(2) *If x is a minimal element of T then $x \in A \cap B$.*

(3) *If $\emptyset \neq V \subseteq T$, if $S \cup V$ is a fundamental set, and if $Q(V) = A(V)+B(V)+v$, then $Q(V-C) \geqslant v+1$.*

(4) *If g_i is a minimal element of $T-C$ then $Q(X_i) \leqslant A(X_i)+B(X_i)$, and $Q(X_i') \leqslant A(X_i')+B(X_i')-1$.*

(5) *If g_i is a minimal element of $T-C$ then* $Q(Y_i) \geqslant A(Y_i)+B(Y_i)+1$.

(6) *For each* $i = 1, \ldots, k-s$, Y_i *is a fundamental set.*

Proof. Statements (1), (2), (3) follow directly from the definition of S. From (2) we have $X_i' \neq \emptyset$. Statement (4) then follows from the definition of S since $S \cup X_i$ and $S \cup X_i'$ are fundamental sets properly containing S. Statement (5) follows from (4) and Lemma 3.

To prove (6) we note that $X_i' \neq \emptyset$ implies $Y_i \neq \emptyset$. For $y_0 \in Y_i$ and $y \in \mathrm{L}y_0$ we have $g_i - y_0 \leqslant g_i - y$. Thus $g_i - y_0 \notin S$ and S is a fundamental set imply $g_i - y \notin S$. But $g_i - y < g_i$, hence $g_i - y \in \mathrm{L}g_i - \{g_i\}$. Therefore $g_i - y \in X_i'$ and $y \in Y_i$.

§ 4. Proof of Theorem 1. Assume $Q(R)/k \geqslant Q(S)/s$. Since $C(S) < Q(S)$ there exists $g \in S-C$, hence there exists $g \in S-(A \cup B)$. Let $W = \bigcup \mathrm{L}g$, taken over all $g \in S-(A \cup B)$. Then W is a fundamental set such that $W \subseteq S \subseteq R$ and the maximal elements of W are not in $A \cup B$. Also $Q(S)-s = C(S) = A(S)+B(S)$ implies

$$Q(W)-s = C(W) \geqslant A(W)+B(W).$$

Finally, $Q(R)/k \geqslant Q(S)/s \geqslant Q(W)/s$ and Lemma 2 imply

$$C(R)/Q(R) \geqslant [A(W)+B(W)]/Q(W).$$

Assuming the second set of hypotheses, let $W = \bigcup_{i=1}^{t} Y_i$. Then the maximal elements of W are not in $A \cup B$ from Lemma 4 (2), W is a fundamental set from Lemma 4 (6), and $Q(Y_1) \geqslant A(Y_1)+B(Y_1)+1$ from Lemma 4 (5).

We note that $Q(W) = Q(Y_1)$ if $t = 1$ and that $Q(W) = Q(Y_1)+Q(Y_2')+ +\ldots+Q(Y_t')$ if $t > 1$. Assuming the latter case, let $Z_i = \{g_i - y\colon\ y \in Y_i'\}$, $i = 2, \ldots, t$. Then $\emptyset \neq Z_i \subseteq X_i'$, and $Q(Z_i - C) = 0$ since g_i is minimal in $T-C$.

Suppose $z \in Z_i$, $x \in \mathrm{L}z \cap T$. Then $x \in X_i'$, hence $g_i - x \in Y_i$. Also, $g_i - x \geqslant g_i - z$, and $g_i - z \notin \bigcup_{j=1}^{i-1} Y_j$. But $\bigcup_{j=1}^{i-1} Y_j$ is a fundamental set, hence $g_i - x \in Y_i - \bigcup_{j=1}^{i-1} Y_j = Y_i'$, hence $x \in Z_i$. Therefore $S \cup Z_i$ is a fundamental set and, from Lemma 4 (3), $Q(Z_i) \leqslant A(Z_i)+B(Z_i)-1$. This and Lemma 3 imply $Q(Y_i') \geqslant A(Y_i')+B(Y_i')+1$ for all $i = 2, \ldots, t$. Thus

$$\begin{aligned} Q(W) &= Q(Y_1)+\sum_{i=2}^{t} Q(Y_i') \\ &\geqslant A(Y_1)+B(Y_1)+1+\sum_{i=2}^{t} A(Y_i')+B(Y_i')+1 \\ &= A(W)+B(W)+t. \end{aligned}$$

We have $Q(\bigcup_{i=1}^{t} X_i) > Q(W)$. (See [6], Lemma 1, and note that the set S' there remains unchanged if the δ_j's are chosen to be maximal, instead of minimal, in S.) Therefore $Q(R)/k \geqslant Q(\bigcup_{i=1}^{t} X_i)/t > Q(W)/t$, and, from Lemma 2, $C(R)/Q(R) \geqslant [A(W)+B(W)]/Q(W)$.

§ 5. Proof of Theorem 2. Since $\{g_1, \ldots, g_t\}$ is linearly ordered, we may assume $g_1 < g_2 < \ldots < g_t$. Then $\bigcup_{i=1}^{t} X_i = X_t$. Except for notation the proof of this theorem is the same as that of the Theorem in [4], beginning with Case 1.2. The set T_1 there would now be replaced by X_t, t_1 by t, P by Q_n, g_{s_1+1} by g_1, (x_1, g_{s_1+1}) by X_1, X_1 by X_1', Y_2 by Y_1, R_2 by Y_t, $(g_{s_1+i} - x_1 + 1, g_{s_1+i+1} - x_1)$ by $Y_{i+1} - Y_i$, etc.

It will be noted that when Case 1.2 holds in the proof of the Theorem in [4] then the desired fundamental set W is obtained because Case 2.1 must ultimately hold, possibly after many repetitions of the type described in the later cases. The set S_2 of Case 2.1 contains an integer $g_{s_1+s_2} - x_1$ which is not in $A \cup B$ (since $x_1 \in A \cap B$ and $g_{s_1+s_2} \notin C$), hence there exists a largest integer $h \in S_2-(A \cup B)$. If $W = \mathrm{L}h$ then the largest element of W is not in $A \cup B$ and $C(R)/P(R) \geqslant [A(W)+B(W)]/P(W)$. In the n-dimensional case the set corresponding to S_2 will contain one of the sets Y_u, whose maximal elements have the form $g_u - x$ where $1 \leqslant u \leqslant t$ and x is a minimal element of X_u, therefore a minimal element of T. Thus there exists $h \in S_2-(A \cup B)$ in the n-dimensional case (Lemma 4 (2)), and the desired fundamental set W may be defined to be $W = \bigcup \mathrm{L}h$, taken over all $h \in S_2-(A \cup B)$.

§ 6. Examples in Q_2. In this section we give two examples of a fundamental set R and sets $A, B, C = A+B$ in $Q = Q_2$ such that $C(R) < Q(R)$ and there does not exist a fundamental set $W \subseteq R$ for which $C(R)/Q(R) \geqslant [A(W)+B(W)]/Q(W)$.

EXAMPLE 1. Let $R = \bigcup_{i=1}^{7} \mathrm{L}g_i$, where $g_1 = (24, 54)$, $g_2 = (25, 53)$, $g_3 = (26, 52)$, $g_4 = (51, 27)$, $g_5 = (52, 26)$, $g_6 = (53, 25)$, $g_7 = (54, 24)$. Note that the points g_i are all on the line $x+y = 78$, and no other points of R are on or above this line. Let $R \cap A$ be the set of all lattice points of R except those on the lines $x+y = 24$, $x+y = 55$, and $x+y = 78$. Let $R \cap B$ consist of just those lattice points of R on the lines $x+y = 23$ and $x+y = 54$. Then $R-C = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$. All lattice points with nonnegative coordinates of the lines $x+y = 23$, $x+y = 24$, and $x+y = 54$ are in R. All lattice points with positive coordinates on the line $x+y = 55$, except $(27, 28)$, are in R. Thus we have $Q(R) = 2259$, $C(R) = 2252$, $A(R) = 2174$, $B(R) = 79$.

Lemma 2 and $C(R)/Q(R) < 1$ imply that if W is a fundamental set in R then $C(R)/Q(R) \geqslant [A(W)+B(W)]/Q(W)$ if and only if $Q(W) = A(W)+B(W)+w$, $w > 0$, and $Q(R)/7 \geqslant Q(W)/w$. Thus, for each fixed positive integer w we need to consider only a smallest fundamental set W in R, if one exists, for which $Q(W) = A(W)+B(W)+w$. The following table exhibits these minimal W's for this example, and it is clear that $Q(W)/w > Q(R)/7$ in each case.

w	W	$Q(W)$
1	$\{(x, y) \in Q\colon\ x+y \leqslant 24\}$	324
2	$Lg_1 \cup Lg_2, Lg_6 \cup Lg_7$	1428
3	$Lg_1 \cup Lg_2 \cup Lg_3, Lg_5 \cup Lg_6 \cup Lg_7$	1481
4	$Lg_4 \cup Lg_5 \cup Lg_6 \cup Lg_7$	1533
5	$Lg_2 \cup Lg_5 \cup Lg_6 \cup Lg_i \cup Lg_j \cup Lg_k$, i, j, k in the set $\{1, 3, 4, 7\}$	2234
6	R	2259
$\geqslant 7$	No W exists	—

In Example 1 the lattice points $(1, 0)$ and $(0, 1)$ are not in B, hence the density of B is 0. The sets A and B have positive density in Example 2; otherwise Example 2 is similar to Example 1.

Example 2. Let $R = \bigcup_{i=1}^{7} Lg_i$, where $g_1 = (85, 186)$, $g_2 = (88, 183)$, $g_3 = (91, 180)$, $g_4 = (177, 94)$, $g_5 = (180, 91)$, $g_6 = (183, 88)$, $g_7 = (186, 85)$. Let A consist of all elements of $Q-R$ and all elements of R except $(2, 0)$, $(0, 2)$, and those on the lines $x+y = 85$, $x+y = 187$, $x+y = 270$, $x+y = 271$. Let B consist of all elements of $Q-R$, all elements of R which are on the lines $x+y = 1$, $x+y = 84$, $x+y = 186$, and all $g_i-(2, 0)$, $g_i-(0, 2)$, $i = 1, \ldots, 7$. Then $R-C = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $Q(R) = 26,147$, $C(R) = 26,140$, $A(R) = 25,853$, $B(R) = 288$.

References

[1] L. P. Cheo, *On the density of sets of Gaussian integers*, Amer. Math. Monthly 58 (1951), pp. 618–620.

[2] A. R. Freedman, *An inequality for the sum of sets of vectors in n-dimensional space*, Pacific Journ. Math. 19 (1966), pp. 265–267.

[3] — *A density inequality for the sum of two sets of lattice points*, J. Reine Angew. Math. 223 (1966), pp. 138–141.

[4] B. K. Garrison, *A nontransformation proof of Mann's density theorem*, J. reine angew. Math. (to appear).

[5] B. Kvarda, *On densities of sets of lattice points*, Pacific Journ. Math. 13 (1963), pp. 611–615.

[6] — *An inequality for the number of elements in a sum of two sets of lattice points*, Pacific Journ. Math. 15 (1965), pp. 545–550.

[7] H. B. Mann, *A proof of the fundamental theorem on the density of sums of sets of positive integers*, Ann. of Math. 43 (1942), pp. 523–527.

[8] B. Müller, *Einige Abschätzungen der Dichte von Summenmengen in lokalkompakten abelschen Gruppen*, J. reine angew. Math. 206 (1961), pp. 39–44.

[9] — *Halbgruppen und Dichteabschätzungen in lokalkompakten abelschen Gruppen*, J. reine angew. Math. 210 (1962), pp. 89–104.

[10] L. Schnirelmann, *Über additive Eigenschaften von Zahlen*, Math. Ann. 107 (1933), pp. 649–690.

Received on 30. 8. 1969