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On the sum of the number of divisors in a short segment

by

YOICHI MOTOHASHI (Tokyo)

Let $\tau(n)$ be the number of the divisors of n. Erdös [1] has proved the following interesting result:

Let $h(x)$ be an arbitrary increasing function which tends to infinity with x, and let

$$f(x) > (\log x)^{2\log 2-1}\exp\big(h(x)(\log\log x)^{1/2}\big).$$

Then it holds that for "almost all x"

$$\sum_{j\leqslant f(x)} \tau(x+j) = \big(1+o(1)\big)f(x)\log x.$$

In Erdös' proof a result of Ingham in the theory of the additive divisor problem, namely the asymptotic formula for the sum

$$\sum_{n\leqslant x} \tau(n)\tau(n+k) \qquad (k\neq 0) \qquad (\text{as } x\to\infty)$$

plays an important role.

The purpose of this note is to strengthen the above result for slightly longer intervals. Our result is as follows:

THEOREM. *Let $g(x)$ be an arbitrary increasing function such that*

$$g(x)\leqslant (\log x)^2$$

and $g(x)$ tends to infinity with x, and let

$$t = g(x)(\log x)^3.$$

Then for almost all $n\leqslant x$ the asymptotical equality

$$\sum_{j\leqslant t} \tau(n+j) = t(\log n+2\gamma)+o(t)$$

holds, where γ is the Euler constant.

For the proof of the theorem let us consider the dispersion

$$I(x) = \sum_{n\leqslant x}\Big\{\sum_{j\leqslant t}\tau(n+j)-t(\log n+2\gamma)\Big\}^2.$$

This double sum is decomposed into four parts:

$$\Sigma_1 = \sum_{n\leqslant x}\sum_{j\leqslant t}\tau^2(n+j),$$

$$\Sigma_2 = 2\sum_{n\leqslant x}\sum_{j_1<j_2\leqslant t}\tau(n+j_1)\tau(n+j_2),$$

$$\Sigma_3 = -2t\sum_{n\leqslant x}(\log n+2\gamma)\sum_{j\leqslant t}\tau(n+j),$$

$$\Sigma_4 = t^2\sum_{n\leqslant x}(\log n+2\gamma)^2.$$

Now we will calculate these sums. It is easy to see that

(1) $$\Sigma_1 = t\sum_{n\leqslant x}\tau^2(n)+O\Big\{\sum_{n\leqslant t}\tau^2(n)+\sum_{x<n\leqslant x+t}\tau^2(n)\Big\}$$
$$= \big(1+o(1)\big)\frac{t}{\pi^2}x(\log x)^3.$$

And also we have

$$\frac{1}{t^2}\Sigma_4 = 4\gamma^2x+4\gamma\sum_{n\leqslant x}\log n+\sum_{n\leqslant x}\log^2 n+O(1)$$
$$= 4\gamma^2x+4\gamma(\log x-1)x+(\log^2x-2\log x+2)x+O(x^{3/4}).$$

Hence we get

(2) $$\Sigma_4 = t^2x\{(\log x+2\gamma-1)^2+1\}+O(x^{3/4}).$$

The sum $-\frac{1}{2t}\Sigma_3$ is equal to

$$\sum_{j\leqslant t}\sum_{x^{1/2}<n\leqslant x}\big(\log(n+j)+2\gamma\big)\tau(n+j)+\sum_{n\leqslant x^{1/2}}(\log n+2\gamma)\sum_{j\leqslant t}\tau(n+j)-$$
$$-\sum_{j\leqslant t}\sum_{x^{1/2}<n\leqslant x}\log\Big(1+\frac{j}{n}\Big)\tau(n+j)$$
$$= t\sum_{n\leqslant x}(\log n+2\gamma)\tau(n)+O\big(tx^{1/2}(\log x)^2\big)+O\Big(t^2\sum_{n\leqslant x}\frac{\tau(n)}{n}\Big)$$
$$= t\sum_{n\leqslant x}\tau(n)\log n+2\gamma tx(\log x+2\gamma-1)+O\big(tx^{1/2}(\log x)^2\big).$$

Here the last sum can be represented by the integral

$$-\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\big(\zeta^2(s)\big)'\frac{x^s}{s}ds,$$

and hence by the routine method we get

$$\sum_{n\leqslant x}\tau(n)\log n = x(\log x)^2+2(\gamma-1)x\log x-2(\gamma-1)x+O(x^{3/4}).$$

Therefore we find

(3) $$\Sigma_3 = -2t^2x\{(\log x+2\gamma-1)^2+1\}+O(tx^{3/4}).$$

Now let turn to the most difficult sum Σ_2. Obviously this sum is equal to

(4) $$2\sum_{j_1<j_2\leqslant t}\sum_{n\leqslant x}\tau(n)\tau(n+j_2-j_1)+O(t^2x^{3/4}).$$

To the inner sum we will apply the deep result of Estermann [2] instead of Ingham's. Estermann's assertion is as follows:

It holds that for $k>0$

$$\sum_{n\leqslant x}\tau(n)\tau(n+k) = \frac{6}{\pi^2}\sigma_{-1}(k)x\log^2x+a_1(k)x\log x+a_2(k)x+O\big(x^{11/12}(\log x)^3\big),$$

where

$$a_1(k) = \Big\{\frac{12}{\pi^2}(2\gamma-1)-\frac{144}{\pi^4}\zeta'(2)\Big\}\sigma_{-1}(k)-\frac{24}{\pi^2}\sigma'_{-1}(k),$$

$$a_2(k) = \Big\{\frac{6}{\pi^2}(2\gamma-1)^2+\frac{6}{\pi^2}-\frac{144}{\pi^4}(2\gamma-1)\zeta'(2)-\frac{144}{\pi^4}\zeta''(2)+$$
$$+\frac{1728}{\pi^6}\big(\zeta'(2)\big)^2\Big\}\sigma_{-1}(k)-\Big\{\frac{24}{\pi^2}(2\gamma-1)-\frac{288}{\pi^4}\zeta'(2)\Big\}\sigma'_{-1}(k)+\frac{24}{\pi^2}\sigma''_{-1}(k),$$

and

$$\sigma_{-1}(k) = \sum_{l|k}l^{-1},\quad \sigma'_{-1}(k) = \sum_{l|k}l^{-1}\log l,\quad \sigma''_{-1}(k) = \sum_{l|k}l^{-1}\log^2 l.$$

In order to apply this asymptotic formula to the sum (4), we must verify the uniformity of the error term $O\big(x^{11/12}(\log x)^3\big)$ for $k\leqslant t\ (\leqslant(\log x)^5)$, but this can be easily seen from the proof of Estermann with slightly careful re-reading.

Hence we get

$$\Sigma_2 = \frac{12}{\pi^2} x(\log x)^2 \sum_{j_1<j_2\leqslant t} \sigma_{-1}(j_2-j_1)+2x\log x \sum_{j_1<j_2\leqslant t} a_1(j_2-j_1)+$$
$$+2x \sum_{j_1<j_2\leqslant t} a_2(j_2-j_1)+O\big(t^2x^{11/12}(\log x)^3\big).$$

Also we have

$$\sum_{j_1<j_2\leqslant t} \sigma_{-1}(j_2-j_1) = \sum_{l\leqslant t} l^{-1} \sum_{\substack{j_1\equiv j_2 (\mathrm{mod}\, l)\\ j_1<j_2\leqslant t}} 1$$
$$= \frac{t^2}{2}\sum_{l\leqslant t} l^{-2}+O(t\log t) = \frac{\pi^2}{12}t^2+O(t\log t),$$

and similarly

$$\sum_{j_1<j_2\leqslant t} \sigma'_{-1}(j_2-j_1) = \frac{t^2}{2}\sum_{l=1}^{\infty} l^{-2}\log l+O(t\log^2 t) = -\frac{t^2}{2}\zeta'(2)+O(t\log^2 t),$$
$$\sum_{j_1<j_2\leqslant t} \sigma''_{-1}(j_2-j_1) = \frac{t^2}{2}\sum_{l=1}^{\infty} l^{-2}\log^2 l+O(t\log^3 t) = \frac{t^2}{2}\zeta''(2)+O(t\log^3 t).$$

From these relations we find, after some elementary computations, that

$$\sum_{j_1<j_2\leqslant t} a_1(j_2-j_1) = (2\gamma-1)t^2+O(t\log^2 t)$$

and

$$\sum_{j_1<j_2\leqslant t} a_2(j_2-j_1) = \frac{t^2}{2}\{(2\gamma-1)^2+1\}+O(t\log^3 t).$$

Therefore we obtain

$$(5)\qquad \Sigma_2 = xt^2(\log x)^2+2(2\gamma-1)t^2x\log x+$$
$$+\{(2\gamma-1)^2+1\}t^2x+O\big(tx(\log x)^2\log t\big)$$
$$= t^2x\{\big((2\gamma-1)+\log x\big)^2+1\}+O\big(tx(\log x)^2\log t\big).$$

Finally, collecting the results (1), (2), (3) and (5), we get at once the asymptotic formula

$$I(x) = \big(1+o(1)\big)\frac{1}{\pi^2}xt\log^3 x,$$

By the familiar assertion of Čebyšev's inequality we now complete the proof of the theorem.

Remark. Let $D(x) = x(\log x+2\gamma-1)$, then it is well known that

$$\sum_{n\leqslant x}\tau(n) = D(x)+O(x^{1/3}).$$

On the other hand

$$D(x+t)-D(x) = t(\log x+2\gamma)+o(t).$$

Hence the theorem tells that for almost all n

$$\sum_{j\leqslant t}\tau(n+j) \sim D(n+t)-D(n).$$

References

[1] P. Erdös, *Asymptotische Untersuchungen über die Anzahl der Teiler von n*, Math. Ann. 169 (1967), pp. 230–238.

[2] T. Estermann, *Über die Darstellung einer Zahl als Differenz von zwei Produkten*, J. Reine Angew. Math. 164 (1931), pp. 173–182.

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