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A metric inequality associated with valuated fields

by

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1. Introduction. Suppose that $\boldsymbol{F}$ is a field with a valuation $\|\ \|$, mapping $\boldsymbol{F}$ into $\boldsymbol{R}$, the real numbers. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two points in the cartesian product $\boldsymbol{F}^n = \boldsymbol{F}\times\ldots\times\boldsymbol{F}$, with coordinates $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ respectively. We can define a function from $\boldsymbol{F}^n\times\boldsymbol{F}^n$ to $\boldsymbol{R}$ as follows:

$$d(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \min_{\sigma\in S_n} \max_{1\leqslant j\leqslant n} \|\alpha_j - \beta_{\sigma(j)}\|,$$

where S_n is the symmetric group on n objects. It is clear that if we write $\sigma\boldsymbol{\alpha} = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})$, then for any σ, τ in S_n we have $d(\sigma\boldsymbol{\alpha}, \tau\boldsymbol{\beta}) = d(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

It follows that $d(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the triangle inequality since we may suppose, by taking a suitable permutation of the coordinates if necessary, that

$$d(\boldsymbol{\alpha}, \boldsymbol{\gamma}) = \max_{1\leqslant j\leqslant n} \|\alpha_j - \gamma_j\|,$$

and

$$d(\boldsymbol{\gamma}, \boldsymbol{\beta}) = \max_{1\leqslant j\leqslant n} \|\gamma_j - \beta_j\|.$$

Hence

$$d(\boldsymbol{\alpha}, \boldsymbol{\beta}) \leqslant \max_{1\leqslant j\leqslant n} \|\alpha_j - \beta_j\| \leqslant \max_{1\leqslant j\leqslant n} \|\alpha_j - \gamma_j\| + \max_{1\leqslant j\leqslant n} \|\gamma_j - \beta_j\| = d(\boldsymbol{\alpha}, \boldsymbol{\gamma}) + d(\boldsymbol{\gamma}, \boldsymbol{\beta}).$$

Thus $d(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is a pseudo-metric on $\boldsymbol{F}^n$.

We define the real quantities

$$\mathfrak{M}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathfrak{M} = \max_{1\leqslant j\leqslant n} \max\{\|\alpha_j\|, \|\beta_j\|\},$$

$$\mathfrak{N}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathfrak{N} = \prod_{j,k} \|\alpha_j - \beta_k\|.$$

In this paper we seek lower bounds on $d(\boldsymbol{\alpha}, \boldsymbol{\beta})$ in terms of $\mathfrak{M}$ and $\mathfrak{N}$. If $\|\ \|$ is a non-archimedean valuation, then it readily follows that if $\mathfrak{M} > 0$, then

$$d(\boldsymbol{\alpha}, \boldsymbol{\beta}) \geqslant \frac{\mathfrak{N}^{1/n}}{\mathfrak{M}^{n-1}}.$$

We will be concerned with the corresponding inequality for the archimedean case.

It is well known that a field $\boldsymbol{F}$ with an archimedean valuation can be imbedded in the complex numbers $\boldsymbol{C}$ by a map $f\colon \boldsymbol{F}\to\boldsymbol{C}$, which has the property that there exists a fixed real number ϱ $(0<\varrho\leqslant 1)$ such that

$$\|a\| = |f(a)|^{\varrho}$$

for all a in $\boldsymbol{F}$, and where $|\ |$ is the ordinary absolute value (see O'Meara [3], § 12).

We have the following result.

THEOREM. *If $\|\ \|$ is an archimedean valuation on the field $\boldsymbol{F}$, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are not both $\boldsymbol{0}$, then for some real ϱ $(0<\varrho\leqslant 1)$ depending upon the field $\boldsymbol{F}$, we have*

$$d(\boldsymbol{\alpha},\boldsymbol{\beta}) \geqslant \frac{\min\{\mathfrak{R}^{1/n},\mathfrak{M}^{n}\}}{(1.04n)^{\varrho}\mathfrak{M}^{n-1}}. \tag{1}$$

If $\boldsymbol{F}$ is contained in the field of complex numbers and $\|\ \|$ is the ordinary absolute value, then we have $\varrho = 1$ in (1).

COROLLARY. *If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are non-conjugate algebraic integers of degree n, with conjugates $\alpha_1,\ldots,\alpha_n$ and $\beta_1,\ldots,\beta_n$ respectively, then (with the obvious notation) $d(\boldsymbol{\alpha},\boldsymbol{\beta})$ is a discrete metric for any set of non-conjugate algebraic integers of degree n, and*

$$d(\boldsymbol{\alpha},\boldsymbol{\beta}) > \frac{1}{1.04n\mathfrak{M}^{n-1}}.$$

The corollary follows directly from the theorem, for we note that when $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are non-conjugate we have $\mathfrak{R}\geqslant 1$ by a well known result on symmetric polynomials of conjugates of algebraic integers, and $\mathfrak{M}\geqslant 1$ by a theorem of Kronecker [2]. There is no significance in the value 1.04, and it could probably be replaced by some function $g(n)\sim 1$ as $n\to\infty$. That no better result than this can be found is seen by considering the roots of $x^n = m$ and $x^n = m+1$, for which we have

$$\lim_{m\to\infty}\frac{d(\boldsymbol{\alpha},\boldsymbol{\beta})\,n\mathfrak{M}^{n-1}}{\min\{\mathfrak{R}^{1/n},\mathfrak{M}^{n}\}} = 1.$$

We remark at this stage that in the case when $\max\{\|\alpha_j\|,\|\beta_j\|\} = \mathfrak{M}$, and $\min\{\|\alpha_j\|,\|\beta_j\|\} = \mathfrak{m}$ for all $j = 1,\ldots,n$, the proof that follows will yield the theorem with 1.04 in (1) replaced by 1. It may also be worthwhile to note that we actually prove the following result. If, in the above notation, $d(\boldsymbol{\alpha},\boldsymbol{\beta}) = \max\limits_{1\leqslant j\leqslant n}\|\alpha_j-\beta_j\|$, then

$$\prod_{1\leqslant j\leqslant n}\|\alpha_j-\beta_j\|^{1/n} \geqslant \frac{\min\{\mathfrak{R}^{1/n},\mathfrak{M}^{n}\}}{(1.04n)^{\varrho}\mathfrak{M}^{n-1}}.$$

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2. We need only prove the theorem for the case when $\boldsymbol{F}$ consists of complex numbers, and $\|\ \|$ is the ordinary absolute value. The more general result follows from the result in valuation theory already mentioned in the previous section.

DEFINITION. We shall say that the real valued function $c(n)>0$ *has the property* P if, for all sets of complex numbers $z_1,\ldots,z_n,w_1,\ldots,w_n$, with

$$|z_j|\leqslant 1,\qquad |w_j|\leqslant 1,\qquad |z_j-w_j|\leqslant\frac{1}{c(n)}$$

$(1\leqslant j\leqslant n)$, we have the inequality

$$\prod_{j\neq k}|z_j-w_k|\leqslant\{c(n)\}^n.$$

We have the lemma:

LEMMA 1. *If $c(n)$ has the property* P, *then*

$$d(\boldsymbol{\alpha},\boldsymbol{\beta})\geqslant\frac{\min\{\mathfrak{R}^{1/n},\mathfrak{M}^{n}\}}{c(n)\mathfrak{M}^{n-1}}.$$

Proof. By a remark made in the previous section we may suppose that

$$|\alpha_j-\beta_j|\leqslant d(\boldsymbol{\alpha},\boldsymbol{\beta})\qquad(1\leqslant j\leqslant n).$$

Now

$$\mathfrak{R}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \prod_{j,k}|\alpha_j-\beta_k|\leqslant\{d(\boldsymbol{\alpha},\boldsymbol{\beta})\}^n\mathfrak{M}^{n(n-1)}\prod_{j\neq k}|z_j-w_k|,$$

where $z_j = \alpha_j/\mathfrak{M}$, $w_j = \beta_j/\mathfrak{M}$.

Suppose $\mathfrak{R}>0$; then if

$$d(\boldsymbol{\alpha},\boldsymbol{\beta})<\frac{\min\{\mathfrak{R}^{1/n},\mathfrak{M}^{n}\}}{c(n)\mathfrak{M}^{n-1}},$$

we have

$$|z_j-w_j|\leqslant\frac{\min\{\mathfrak{R}^{1/n},\mathfrak{M}^{n}\}}{c(n)\mathfrak{M}^{n}}\leqslant\frac{1}{c(n)}.$$

Now since $c(n)$ has the property P,

$$\mathfrak{R}<\frac{\mathfrak{R}}{\{c(n)\}^n\mathfrak{M}^{n(n-1)}}\mathfrak{M}^{n(n-1)}\{c(n)\}^n = \mathfrak{R},$$

a contradiction.

3. In this section we shall prove two lemmas which will later enable us to show that $c(n) = 1.04n$ has the property P. The lemmas are modifications of Lemmas 1 and 2 in Cassels' paper [1]. First let us remark that when $n=2$ we have $2^{n(n-1)}=n^n=4$, and $c(n)$ clearly has the required property P. We shall henceforth assume that $n>2$.

LEMMA 2. A. *Let a and β be real numbers, $\lambda=\frac{1}{2}(a+\beta)$ and $1\leqslant r\leqslant n/(n-1)$. If $a\geqslant 0$, $\beta\geqslant 0$, and $2/n<\lambda<3\pi/2$, then the inequality*

$$|re^{ia}-1|^{k-1}|re^{i\beta}-1|<|re^{i\lambda}-1|^k \tag{2}$$

holds for $k=2$, $a\neq\beta$. Equality holds in (2) if $a=\beta=\lambda$.

Suppose that $\lambda=\big((k-1)a-\beta\big)/k$, $\beta=1/n$ and $1\leqslant r\leqslant n/(n-1)$. Then we have

B. *if $8/n\leqslant a\leqslant 2\pi-5/n$, $n\geqslant 3$, then (2) holds with $k=2$;*

C. *if $5/n\leqslant a<8/n$, $n\geqslant 9$, then (2) holds with $k=8$;*

D. *if $41/7n<a<8/n$, $5\leqslant n\leqslant 8$, then (2) holds with $k=5$;*

E. *if $7/n<a<8/n$, $n=4$, then (2) holds with $k=4$.*

Proof. The derivative with respect to r of

$$\left|\frac{re^{ia}-1}{re^{i\lambda}-1}\right|^{2(k-1)}\left|\frac{re^{i\beta}-1}{re^{i\lambda}-1}\right|^2$$

has the same sign as

$$(\cos\beta-\cos\lambda)\left(\frac{r^2+1-2r\cos a}{r^2+1-2r\cos\beta}-\frac{(k-1)(\cos\lambda-\cos a)}{\cos\beta-\cos\lambda}\right)$$

for $1\leqslant r\leqslant n/(n-1)$. By examining the sign of this expression in each of the cases A–E of the enunciation of the lemma, we find that we need only prove the results when $r=n/(n-1)$.

A. This follows directly from Lemma 1 of Cassels [1], since $\lambda>2/n$ and $1\leqslant r\leqslant n/(n-1)$ imply $\cos\lambda<r/(r^2+1-r)$.

B. Put $a=\lambda+\mu$, $\beta=1/n=\mu-\lambda$. Thus, as in Cassels [1], the required inequality is equivalent to

$$1+\cos\mu-\frac{r^2+1}{r}\cos\lambda>0,$$

and this clearly holds if $\pi/2\leqslant\lambda<\pi-2.5/n$.

Now for $a\geqslant 8/n$, we have $3.5/n\leqslant\lambda<\pi/2$, and

$$\begin{aligned}
&1+\cos\mu-\left(2+\frac{1}{n(n-1)}\right)\cos\lambda\\
&=1+\cos\lambda\cos\frac{1}{n}-\sin\lambda\sin\frac{1}{n}-\left(2+\frac{1}{n(n-1)}\right)\cos\lambda\\
&>1-\left(2-\cos\frac{1}{n}+\frac{1}{n(n-1)}\right)\cos\lambda-\frac{\sin\lambda}{n}\\
&\geqslant 1-\left(1+\frac{1}{2n^2}+\frac{1}{n(n-1)}\right)\left(1-\frac{1}{2}\left(\frac{3.5}{n}\right)^2+\frac{1}{24}\left(\frac{3.5}{n}\right)^4\right)-\frac{3.5}{n^2}>0.
\end{aligned}$$

C. We shall first exhibit two inequalities which will be useful in the subsequent parts of the lemma. We put $a=a/n$ and $\lambda=b/n$, whence $b=\big(a(k-1)-1\big)/k$. Now

$$1-\cos\frac{1}{n}<\frac{1}{2n^2}\quad\text{and}\quad 1-\cos\frac{a}{n}>\frac{d}{2n^2},$$

where $d=a^2-a^4/12n^2$.

Hence, with $r=n/(n-1)$,

$$\begin{aligned}
\left|\frac{re^{i\beta}-1}{re^{ia}-1}\right|^2&=\frac{r^2+1-2r\cos(1/n)}{r^2+1-2r\cos(a/n)}\\
&=\frac{(r-1)^2+2r\big(1-\cos(1/n)\big)}{(r-1)^2+2r\big(1-\cos(a/n)\big)}<\frac{2n-1}{n(d+1)-d}.
\end{aligned}$$

Consequently

$$\left|\frac{re^{i/n}-1}{re^{ia/n}-1}\right|<\left(\frac{2n-1}{n(d+1)-d}\right)^{1/2}, \tag{3}$$

where $d=a^2-a^4/12n^2$.

Similarly we have, since $a>\lambda$,

$$\begin{aligned}
\left|\frac{re^{ia}-1}{re^{i\lambda}-1}\right|^2&=\frac{r^2+1-2r\cos a}{r^2+1-2r\cos\lambda}=\left(\frac{a}{b}\right)^2\frac{(r-1)^2/ra^2+(1-\lambda^2/12+\ldots)}{(r-1)^2/r\lambda^2+(1-\lambda^2/12+\ldots)}\\
&<\frac{(n-1)a^2+n}{(n-1)b^2+n}.
\end{aligned}$$

Thus

$$\left|\frac{re^{ia/n}-1}{re^{ib/n}-1}\right|<\left(\frac{(n-1)a^2+n}{(n-1)b^2+n}\right)^{1/2}<\left(\frac{a^2+1}{b^2+1}\right)^{1/2}. \tag{4}$$

Suppose now that $n\geqslant 9$ and $k=8$. Then (3) and (4) imply that

$$\left|\frac{re^{ia/n}-1}{re^{ib/n}-1}\right|^8\left|\frac{re^{i/n}-1}{re^{ia/n}-1}\right|<\left(\frac{n(a^2+1)-a^2}{n(b^2+1)-b^2}\right)^4\left(\frac{2n-1}{n(d+1)-d}\right)^{1/2}.$$

This last expression is decreasing in both a and n (considering b and d as functions of a). Since we have $n\geqslant 9$, $a\geqslant 5$, $b\geqslant 4.25$ and $d>24.3$, it is majorized by

$$\left(\frac{9(26)-25}{9(19.06)-18.06}\right)^4\left(\frac{17}{9(25.3)-24.3}\right)^{1/2}<1.$$

D. Similarly, in this case we have $a>5.85$, $b>4.48$, $n\geqslant 5$ and $d>30$, which imply that

$$\frac{|re^{ia}-1|^4|re^{i\beta}-1|}{|re^{i\lambda}-1|^5}<\left(\frac{a^2+1}{b^2+1}\right)^{5/2}\left(\frac{2n-1}{31n-30}\right)^{1/2}<\left(\frac{35.5}{21}\right)^{5/2}\left(\frac{9}{125}\right)^{1/2}<1.$$

E. In this case $a>7$, $n=4$ and $d>36$; thus we have

$$\frac{|re^{i\alpha}-1|^3\,|re^{i\beta}-1|}{|re^{i\lambda}-1|^4}<\left(\frac{49}{25}\right)^2\left(\frac{7}{112}\right)^{1/2}<1.$$

This completes all cases of the lemma.

LEMMA 3. *Let $n\geqslant 3$ be an integer and $\theta_1,\ldots,\theta_n$ be real numbers such that*

$$-\frac{1}{n}\leqslant\theta_j\leqslant 2\pi+\frac{1}{n}\qquad(1\leqslant j\leqslant n).$$

Define ψ by

(5) $$\psi=\frac{1}{2n}\sum_{j=1}^{n}\theta_j.$$

Now if

(6) $$\frac{2.5}{n}<\psi<\pi-\frac{2.5}{n},$$

and

$$1\leqslant r\leqslant n/(n-1),$$

then

(7) $$\prod_{1\leqslant j\leqslant n}|re^{i\theta_j}-1|\leqslant|re^{2i\psi}-1|^n,$$

with equality only when $\theta_1=\theta_2=\ldots=\theta_n=2\psi$.

Proof. We shall sketch the proof which is modelled on that of Lemma 2 in [1]. For a given ψ, the left hand side of (7) attains its upper bound for θ_j in the range $[-1/n, 2\pi+1/n]$. So we shall suppose that the θ_j give this upper bound. If all the θ_j are equal in value, then the result holds, and so we may confine our attention to the case when the upper bound is attained at unequal values of the θ_j. The condition (6) implies that $|\cos\psi|<r/(r^2-r+1)$, and so if $0\leqslant\theta_j\leqslant 2\pi$ for all j, then the result becomes Lemma 2 of [1]. Also, if for each j either one of the inequalities $|\theta_j|\leqslant 5/n$ or $|2\pi-\theta_j|\leqslant 5/n$ holds, then clearly (7) follows from (6).

There remain the following cases.

(a) Suppose that there are θ_j in both of the intervals $[-1/n, 0)$ and $(2\pi, 2\pi+1/n]$. If $\alpha=\max_{1\leqslant j\leqslant n}\theta_j$, $\beta=\min_{1\leqslant j\leqslant n}\theta_j$, and $\lambda=\frac{1}{2}(\alpha+\beta)$, then we have

$$|re^{i\alpha}-1|\,|re^{i\beta}-1|=|re^{i(\alpha-2\pi)}-1|\,|re^{i(\beta+2\pi)}-1|<|re^{i\lambda}-1|^2,$$

by Lemma 2A. Consequently we may increase the left hand side of (7) by replacing each of α and β by λ, without changing the value of ψ, and so contradict the assumption that an upper bound was already attained for this given ψ. Thus, by replacing θ_j by $2\pi-\theta_j$ and ψ by $\pi-\psi$, if necessary, we may suppose that $-1/n\leqslant\theta_j\leqslant 2\pi$ for all j, and $\theta_j<0$ for at least one j.

(b) By an earlier remark we can suppose that there is some θ_j, say α, for which $5/n<\alpha<2\pi-5/n$. If there is some β among the θ_j in the range $[0, 2\pi]$ $(\alpha\neq\beta)$, then we may again apply Lemma 2A and obtain a contradiction. If for any β in the interval $[-1/n, 0)$ we have $\lambda=\frac{1}{2}(\alpha+\beta)\geqslant\pi/2-1/2n$, then we have as in the proof of Lemma 2B,

$$1+\cos\mu-\left(2+\frac{1}{n(n-1)}\right)\cos\lambda>0.$$

Thus we may suppose for any such β, that $\lambda<\pi/2-1/2n$, and it readily follows that (7) holds in general provided we can prove the result when all the negative θ_j take the value $-1/n$. For the remainder of the proof we shall suppose that we have m of the θ_j equal to α for $5/n<\alpha<2\pi-5/n$, and $n-m$ of the θ_j equal to $-1/n$. We take the following subcases.

(i) If $8/n\leqslant\alpha<2\pi-5/n$, then for $\lambda=\frac{1}{2}(\alpha-1/n)$ we have by Lemma 2B,

$$|re^{i\alpha}-1|\,|re^{i/n}-1|<|re^{i\lambda}-1|^2,$$

and so by replacing both an α and $-1/n$ by λ, we again obtain a contradiction.

(ii) If $5/n\leqslant\alpha<8/n$, we note that the inequality (6) implies $5<2n\psi=m\alpha-(n-m)/n$, and so $m/n>6/(a+1)$, where $\alpha=a/n$. Thus if $n\geqslant 9$ we must have $m\geqslant 7$, since m is an integer. Lemma 2C then implies

$$|re^{i\alpha}-1|^7\,|re^{i/n}-1|<|re^{i\lambda}-1|^8,$$

with $\lambda=(7\alpha-1/n)/8$. We again obtain our contradiction by replacing seven α and one $-1/n$ each by λ.

(iii) If $a\leqslant 41/7n$, then $m/n>7/8$, contradicting $m/n\leqslant(n-1)/n\leqslant 7/8$ when $n\leqslant 8$. If $\alpha\leqslant 7/n$, $n\leqslant 4$, or $\alpha<8/n$, $n=3$, we obtain similar contradictions. We deal with the two remaining cases $41/7<a<8$, $5\leqslant n\leqslant 8$, and $7<a<8$, $n=4$ as in (i) and (ii) by using Lemmas 2D and 2E respectively.

4. We are now in a position to prove the final lemma, which will imply the theorem as a corollary.

LEMMA 4. *The function $c(n)=1.04n$ has the property* P.

Proof. Suppose that $|z_j|\leqslant 1$, $|w_j|\leqslant 1$ and $|z_j-w_j|\leqslant 1/c(n)$ for all j. By the maximum modulus principle for subharmonic functions, we may suppose that one of z_j or w_j (for each j) lies on the unit circle, while the other lies on the boundary of the region formed by taking the union of the unit disc with the disc of radius $1/c(n)$ about the former point. Thus we may suppose that all the z_j and w_j lie within the annulus $1-1/c(n) \leqslant |z| \leqslant 1$.

Following [1], we define for all $1 \leqslant t \leqslant n-1$

$$P_t = \prod_{j-k\equiv t (\mathrm{mod}\, n)} |z_j - w_k|,$$

so that

$$\prod_{1\leqslant t\leqslant n-1} P_t = \prod_{j\neq k} |z_j - w_k|.$$

By rotating all the points z_j and w_j about the origin, if necessary, we may suppose that for all j,

$$\arg z_j = \varphi_j + \lambda_j, \qquad \arg w_j = \varphi_j,$$

where $0 \leqslant \varphi_1 \leqslant \varphi_2 \leqslant \ldots \leqslant \varphi_n < 2\pi$. Since $n \geqslant 3$ it follows that $|\lambda_j| < 1/n$.

As in [1], for each fixed t and $j-k \equiv t (\mathrm{mod}\, n)$ put

$$\theta_j = \begin{cases} \varphi_j - \varphi_k & \text{if} \quad j > k, \\ \varphi_j - \varphi_k + 2\pi & \text{if} \quad j < k. \end{cases}$$

Now for any real $r > 0$ and $s > 0$, and real θ and φ, we have

$$|re^{i\theta} - se^{i\varphi}| = |re^{i\varphi} - se^{i\theta}|.$$

Thus we may suppose for this particular t and k that $|z_j| \leqslant |w_k|$. Hence for some r_j satisfying

$$1 - \frac{1}{c(n)} \leqslant r_j \leqslant 1,$$

we have

$$P_t \leqslant \prod_{1\leqslant j\leqslant n} |r_j e^{i(\theta_j+\lambda_j)} - 1|.$$

Put $s = 1 - 1/c(n)$; then it follows for such r_j and any θ that

$$\left| \frac{r_j e^{i\theta} - 1}{se^{i\theta} - 1} \right| \leqslant \left(1 - \frac{1}{2c(n)}\right)^{-1}.$$

Thus

$$P_t \leqslant \left(1 - \frac{1}{2c(n)}\right)^{-n} \prod_{1\leqslant j\leqslant n} |se^{i(\theta_j+\lambda_j)} - 1|.$$

Since

$$\prod_{1\leqslant j\leqslant n} |se^{i(\theta_j+\lambda_j)} - 1| = s^n \prod_{1\leqslant j\leqslant n} |s^{-1} e^{i(\theta_j+\lambda_j)} - 1|,$$

we are now in a position to apply Lemma 3, with θ_j replaced by $\theta_j + \lambda_j$, for $-1/n \leqslant \theta_j + \lambda_j \leqslant 2\pi + 1/n$, and

$$2n\psi = \sum_{j=1}^{n} \theta_j + \sum_{j=1}^{n} \lambda_j = 2\pi t + \Lambda,$$

say, where $|\Lambda| < 1$. Thus

$$\frac{2.5}{n} < \psi < \pi - \frac{2.5}{n},$$

and for $c(n) = 1.04n$ we have $s^{-1} = \dfrac{c(n)}{c(n)-1} < \dfrac{n}{n-1}$. The lemma gives

$$\prod_{1\leqslant j\leqslant n} |se^{i(\theta_j+\lambda_j)} - 1| \leqslant s^n |s^{-1} e^{(i2\pi t+\Lambda)/n} - 1|^n = |se^{i(2\pi t+\Lambda)/n} - 1|^n.$$

Thus

$$\prod_{j\neq k} |z_j - w_k| \leqslant \left(1 - \frac{1}{2c(n)}\right)^{-n(n-1)} \prod_{1\leqslant t\leqslant n-1} |se^{i(2\pi t+\Lambda)/n} - 1|^n$$

$$= \left| \frac{1 - s^n e^{i\Lambda}}{1 - se^{i\Lambda/n}} \right|^n \left(1 - \frac{1}{2c(n)}\right)^{-n(n-1)}$$

$$\leqslant \left\{ \frac{1-s^n}{1-s} \left(1 - \frac{1}{2c(n)}\right)^{-(n-1)} \right\}^n$$

$$= \left\{ c(n) \cdot \frac{1 - \left(1 - \dfrac{1}{c(n)}\right)^n}{\left(1 - \dfrac{1}{2c(n)}\right)^{n-1}} \right\}^n$$

$$< \left\{ 1.04n\, 2\sinh \frac{1}{2.08} \right\}^n < \{1.04n\}^n,$$

as required.

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