The cyclotomic numbers of order twenty

by

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1. Introduction. Let \( p = ef + 1 \) be an odd prime. Let \( g \) be a fixed primitive root of \( p \). The cyclotomic number of order \( e \)

\[ (h, k) = (\tilde{h}, \tilde{k})_e \]

is defined as the number of solutions of the congruence

\[ g^{h+k} + 1 = g^{h+k} (\mod p), \quad 0 \leq h, k \leq e-1, \quad 0 \leq s, t \leq f-1. \]

A central problem in the theory of cyclotomy is to obtain formulas for the cyclotomic numbers. The derivation of these formulas has three phases.

The first step is to express the cyclotomic numbers as linear combinations of coefficients of Jacobi sums of order \( e \) and cyclotomic numbers of orders which are divisors of \( e \). The Jacobi sums \( \psi(e'; e') = \psi_e(e'; e') \) of order \( e \) are defined as

\[
\psi_e(e'; e') = \sum_{j=1}^{e-1} \beta^{\text{ind}_f(j) + \text{ind}_g(1-t)}, \quad \text{where} \quad \beta = \exp\left(2\pi i/e\right).
\]

In the second phase, equations relating different Jacobi sums are determined. This permits the equating of various coefficients of Jacobi sums. At the conclusion of this phase, each cyclotomic number is represented as a linear combination of a minimal set of Jacobi coefficients and differences of Jacobi coefficients of orders \( e \) and divisors of \( e \). The significance of this is that the relatively few numbers of the minimal set of Jacobi coefficients contain essentially enough information to generate all the cyclotomic numbers of order \( e \) for the prime \( p \).

In the third and final step, one seeks to replace the minimal set of Jacobi coefficients by coordinates of quadratic decompositions of \( p \). The resulting representations of the cyclotomic numbers are more useful for certain applications.

* The research of the first author was supported in part under NSF grants GP 2091 and GP 5308. The research of the second author was sponsored in part under NSF grant G 24006.
It does not follow, however, that if one is given merely the appropriate quadratic decompositions of \( p \), he can compute explicitly the cyclotomic numbers of order \( e \). The labeling of the cyclotomic numbers depends upon the choice of the primitive root \( g \). Some of the identities between Jacobi sums depend upon the indices (mod e) of the prime divisors of \( e \). In addition, for several values of \( e \) there are sign ambiguities in the equations relating certain Jacobi sums.

Where formulas for the cyclotomic numbers of order \( e \) in terms of quadratic decompositions of \( p \) have been obtained, we shall say that a complete solution has been found. Complete solutions for \( e = 2, 3, 4, 5, \) and 6 were given by L. E. Dickson [4], for \( e = 8 \) by Emma Lehmer [11], and for \( e = 10 \) and 12 by A. L. Whiteman [15], [16]. In published solutions for \( e = 9 \) and 18 [2], \( e = 14 \) [12], and \( e = 16 \) [14], the third phase was not carried through completely. It follows from Theorem 6 of [7], however, that the solution for \( e = 16 \) is indeed complete.

The principal result of this paper is a complete solution for \( e = 20 \). As an application, we show that if 5 is a biquadratic residue of a prime \( 1 \pmod{20} \), then the set of twentieth power residues, with or without zero, does not form a difference set. Whether such difference sets exist where 5 is not a biquadratic residue remains unsolved.

2. Cyclotomy. In this section are gathered properties of cyclotomic numbers, Jacobi sums, and the Lagrange resolvent to be used in the study of the cyclotomic numbers of order 20.

If \( h = h' \pmod{\text{mod e}} \) and \( k = k' \pmod{\text{mod e}} \), the cyclotomic numbers \( (h, k) \) and \( (h', k') \) are equal, so that there are just \( e^2 \) cyclotomic numbers to be considered. Application of the well-known identities ([1], pp. 202-203)

\[
(h, k) = (h, k) \quad (f \text{ even}),
\]

\[
(h, k) = (h, k) \quad (f \text{ odd}),
\]

reduces the number of distinct cyclotomic numbers to

\[
[(e^2 + 3e + 6)/6].
\]

Proof of (2.3) where \( f \) is even: By (2.1) and (2.2),

\[
(h, k) = (h, k) = (-h, k - h) = (h - k, -h) = (h - h, -h) = (-h, k - h).
\]

Assume first that \( 3 | e \). There are \((e-1)(e-2)/2 \) ordered pairs \( (\pmod{e}) \) for which

\[
(h, k, h - k) \neq 0 \pmod{e}.
\]

But if (2.5) is satisfied, the six ordered pairs in (2.4) are different. This accounts for \((e-1)(e-2)/6 \) distinct cyclotomic numbers. If exactly one of \( h, k \), and \( h - k \) is divisible by \( e \), then there are three different ordered pairs in (2.4). This accounts for \((e-1)/6 \) distinct cyclotomic numbers. Finally, add one for \((0, 0) \). The total is \((e^2 + 3e + 6)/6 \). (Dickson [5], Theorem 5 and Section 12) was familiar with this result for \( e = 2 \) or \( 3 \) a prime \( \geq 5 \).

If \( 3 | e \), one adjustment must be made — if \( h = e/3, k = 2e/3, \) then there are only two different ordered pairs in (2.4). For all other \( h, k \) satisfying (2.5), the six ordered pairs in (2.4) are different. Together these account for \((e^2 - 3e)/6 \) distinct cyclotomic numbers. Thus the total is

\[
1 + \frac{(e^2 - 3e)}{6} + \frac{e - 1}{6} = \frac{(e^2 + 3e + 6)}{6}.
\]

A similar analysis may be performed when \( f \) is odd.

Henceforth wherever \( e \) is even, let \( e = 2E \). Define the differences

\[
s(h, k) = (h, k) - (h, h + E) \quad t(h, k) = (h, k) - (h + E, k).
\]

Then it follows from (2.2) that

\[
t(h, k) = \begin{cases} 
(s(h, k)) & \text{if even}, \\
(s(h, k)) & \text{if odd}.
\end{cases}
\]

Let \( e = yz \). Dickson showed that ([8], [2])

\[
(h, k)_z = \sum_{m=0}^{y-1} \sum_{n=0}^{x-1} (h + nz, k + rz)_z.
\]

By rearranging this identity in the case \( y = 2, z = E \), Whiteman ([14], Lemma 1) showed that

\[
4(h, k)_z = (h, k)_E + s(h, k) + s(h + E, k) + 2t(h, k).
\]

Turning now to Jacobi sums, we first set \( s = 0 \) in (1.1):

\[
\psi(p', 1) = \begin{cases} 
p - 2 & (e | p), \\
-1 & (e \nmid p).
\end{cases}
\]

Let \( r + s + t = 0 \pmod{e} \). It follows easily from (1.1) (compare [5], [9]) that

\[
\psi(p', p') = \psi(p', p') = -(1)^{rd} \psi(p', p') = -(1)^{rd} \psi(p', p') = -(1)^{rd} \psi(p', p').
\]

If \( e \) is divisible by at most two distinct primes, at least two of the six expressions in (2.10) can be written in the form \( \pm \psi(p', p') \). (By contrast, \( \psi_{pE}(p', p') \) cannot be represented in this manner.) Collecting the exponents
of \( \beta \) which lie in the same residue class (mod \( a \)) gives the following expansion of \( \psi(\beta^m, \beta^m) \) in a finite Fourier series (compare [13], Theorem 3):

\[
\psi(\beta^m, \beta^m) = (-1)^{\mu a} \sum_{j=0}^{\mu - 1} b(\beta, j) \beta^{j}.
\]

The Jacobi coefficients \( b(j, \nu) = b_\nu(j, \nu) \) are Dickson-Hurwitz sums ([13], (6.21)) defined by

\[
b(j, \nu) = \sum_{k=0}^{\nu - 1} (k, j - \nu h).
\]

They satisfy

\[
b(j, \nu) = b(j, \nu - 1 - \nu), \quad (\nu \geq 2), \quad (2.7)
\]

\[
b(j, 0) = \begin{cases} f^{-1} (f | j), & \text{if } j \equiv 0 \pmod{E}, \\ f, & \text{if } j \equiv 1 \pmod{E}, \end{cases}
\]

(11), p. 201.

If \( \nu = \nu z \) it follows easily from (2.11), or alternatively from (2.12) and (2.7), that

\[
b_\nu(j, \nu) = \sum_{r=0}^{\nu - 1} b_\nu(j + rz, \nu).
\]

(2.13)

If \( \nu = 2
\)

\[d(j, \nu) = d_\nu(j, \nu) = b(j, \nu) - b(j + E, \nu).
\]

(2.16)

Note that

\[
d(j + E, \nu) = -d(j, \nu).
\]

(2.17)

If \( \nu = 2
\)

\[\psi(\beta^m, \beta^m) = (-1)^{\mu a} \sum_{j=0}^{\mu - 1} d(j, \nu) \beta^{j}.
\]

(2.18)

If, in addition, \( E \) is odd, (2.13) can be written as

\[
\psi(\beta^m, \beta^m) = (-1)^{\mu a} \sum_{j=0}^{\mu - 1} d(j, \nu) \beta^{j}.
\]

(2.19)

In particular, if \( \nu = E \), applying (2.10) and (2.9) to (2.19) yields ([15], (5.10))

\[
\sum_{j=0}^{E - 1} d(2j, \nu) = -1.
\]

(2.20)

We shall have occasion to refer to the following two lemmas:

**Lemma 1.** If \( \nu = 2
\)

\[b(j, E) + b(j + E, E) = \begin{cases} 2f - 1, & \text{if } j \equiv 0 \pmod{E}, \\ 2f, & \text{if } j \equiv 1 \pmod{E}. \end{cases}
\]

\[
(2.30)
\]

**Lemma 2.** If \( \nu \) is relatively prime to \( \nu \), then \( b(j, \nu) = b(\nu j, \nu) \), where \( \nu \) satisfies \( \nu \equiv 1 \pmod{\nu} \) ([15], Lemma 1).

Lemma 2 implies that if \( \nu = 2\nu \) and \( \nu \) is relatively prime to \( \nu \)

\[
d(j, \nu) = d(\nu j, \nu).
\]

(2.21)

There are two especially interesting cases where \( E \) is even:

\[
d(j, E - 1) = d(j, E - 1, E - 1),
\]

(2.22)

\[
d(j, E + 1) = d(j, E + 1, E + 1) = d(j + E, E + 1) = 0 \quad (j \text{ odd}),
\]

by (2.17).

The resolvent of Lagrange ([1], p. 83)

\[
\tau(\beta^n) = \sum_{z=1}^{\nu - 1} \beta^{zn} \exp(2\pi i z/n)
\]

is associated with the Jacobi sums through the relationship ([1], p. 86)

\[
\psi(\beta^n, \beta^n) = \tau(\beta^n) \tau(\beta^n)/\tau(\beta^{n+1}),
\]

(2.24)

provided \( n + r \) is not divisible by \( \nu \). By means of (2.24) we verify that

\[
\psi(\beta^n, \beta^n) \psi(\beta^{n+r}, \beta^n) = \psi(\beta^n, \beta^n) \psi(\beta^{n+r}, \beta^n).
\]

(2.25)

An important property of the resolvent is ([1], p. 87)

\[
\tau(\beta^n) \tau(\beta^{-n}) = (-1)^n \beta^n,
\]

(2.26)

provided \( n \) is not divisible by \( \nu \). Hence if \( \nu \) does not divide \( n, r, \) or \( n + r, \)

(2.27)

\[
\psi(\beta^n, \beta^n) \psi(\beta^{-n}, \beta^n) = \beta^n.
\]

We shall have occasion to refer to the following two lemmas:

**Lemma 1.** If \( \nu = 2\nu \), then (17), Lemma 3)

\[
b(j, E) + b(j + E, E) = \begin{cases} 2f - 1, & \text{if } j \equiv 0 \pmod{E}, \\ 2f, & \text{if } j \equiv 1 \pmod{E}. \end{cases}
\]

(2.30)

\[
\psi(\beta^n, \beta^n) = \beta^{2\nu} \psi(\beta^n, \beta^{n+1}) \quad (t \equiv 1 \pmod{E(\nu E)}),
\]

(2.31)

in view of (2.24),

\[
\psi(\beta^n, \beta^n) = \beta^{2\nu} \psi(\beta^n, \beta^{n+1}) \quad (t \equiv 0 \pmod{E(\nu E)}).
\]

(2.31)
In (2.31) replace \( t \) by \( 2t \) and combine with (2.30):

\[
(2.32) \quad \psi(\beta^{2t}, \beta^t) = \beta^{2t} \psi(\beta^t, \beta^t) \quad (t \neq 0, \frac{1}{2}E(\text{mod}E)).
\]

3. Cyclotomy where \( \varepsilon \) is four times an odd prime. The primary goal of this section is to derive a general expression for \((h, k)_n\) by carrying a bit further results obtained in Sections 3 and 4 of [16].

Let \( \varepsilon \) denote an odd integer, \( E = 2\varepsilon \), and \( \varepsilon = 4\varepsilon \). Let \( \mathscr{B}(j, v) \), \( B(j, v) \), and \( b(j, v) \) denote coefficients of Jacobi sums of orders \( \varepsilon, E \), and \( \varepsilon \), respectively. Let

\[
D(j, \varepsilon) = B(j, \varepsilon) - B(j + \varepsilon, \varepsilon), \quad d(j, \varepsilon) = b(j, \varepsilon) - b(j + E, \varepsilon).
\]

(This notation differs somewhat from that of earlier papers.)

We commence with a reformulation of Theorems 3 and 2 of [16].

**Theorem 1.**

\[
2 \sum_{n=0}^{\varepsilon-1} [(h, k + 4n) - (h + E, k + E + 4n)] = d(-h - (k - h) \varepsilon, \varepsilon) + d(-h - (k - h) \varepsilon, -\varepsilon).
\]

**Proof.** It follows easily from (2.2) and (2.12) that for an odd integer \( \varepsilon \),

\[
(3.1) \quad d(j, \varepsilon) = \sum_{r=0}^{\varepsilon-1} (j - \varepsilon r, r),
\]

where \( r = h + Ef(\text{mod}e) \). Take \( v = \varepsilon \) in (3.1) and write \( r \) as \( 4n + z \):

\[
(3.2) \quad d(j, \varepsilon) = \sum_{n=0}^{\varepsilon-1} \sum_{z=0}^{3} (j - z \varepsilon, 4n + z).
\]

Similarly, for \( \varepsilon = -\varepsilon \),

\[
(3.3) \quad d(j, \varepsilon) = \sum_{n=0}^{\varepsilon-1} \sum_{z=0}^{3} (j + z \varepsilon, 4n + z).
\]

Thus

\[
d(-h - (k - h) \varepsilon, \varepsilon) + d(-h - (k - h) \varepsilon, -\varepsilon)
\]

\[
= \sum_{n=0}^{\varepsilon-1} \sum_{z=0}^{3} \left[ (-h - (k - h - z) \varepsilon, z + 4n) - (-h - (k - h - z) \varepsilon + E, z + 4n) + \right.
\]

\[
+ (-h - (k - h - z) \varepsilon, z + 4n) - (-h - (k - h - z) \varepsilon + E, z + 4n).\]

Whenever \( k - h - z \) is odd, the first and the fourth terms in the brackets cancel, and the second and the third terms also cancel. Thus \( z \) can be assumed to take on just the two values \( k - h \) and \( k - h + E(\text{mod}4) \).

Then the first and the third terms are the same, and the second and the fourth terms are the same. Hence

\[
(3.4) \quad d(-h - (k - h) \varepsilon, \varepsilon) + d(-h - (k - h) \varepsilon, -\varepsilon)
\]

\[
= 2 \sum_{n=0}^{\varepsilon-1} \left[ ((h, k + 4n) - (h + E, k + E + 4n)) + \right.
\]

\[
+ ((-h + E, k + E + 4n) - (h, k + E + 4n))
\]

\[
= 2 \sum_{n=0}^{\varepsilon-1} ((h, k + 4n) - (h + E, k + E + 4n) + (h + E, k + E + 4n) -
\]

\[
- (h, k + E + 4n)),
\]

by (2.1), q. e. d.

Replace \( -h \) by \( h \) and \( k - h \) by \( k \) in (3.2):

\[
d(h + k \varepsilon, \varepsilon) = d(h + k \varepsilon, -\varepsilon)
\]

\[
= 2 \sum_{n=0}^{\varepsilon-1} ((h, k + 4n) - (h + E, k + E + 4n) + (h + E, k + E + 4n) - (h, k + E + 4n)).
\]

Combining this with Theorem 1 yields

**Theorem 2.**

\[
4 \sum_{n=0}^{\varepsilon-1} ((h, k + 4n) - (h + E, k + E + 4n)) = d(h + k \varepsilon, \varepsilon) + d(h + k \varepsilon, -\varepsilon) + d(-h + (k - h) \varepsilon, \varepsilon) +
\]

\[
+ d(-h + (k - h) \varepsilon, -\varepsilon).
\]

Define

\[
g(n, r) = \begin{cases} 1, & r | n, \\ 0, & \text{otherwise}. \end{cases}
\]

We now state the principal result of this section.

**Theorem 3.** Let \( \varepsilon \) be an odd prime, \( E = 2\varepsilon \), \( \varepsilon = 2E \). Then

\[
4\varepsilon(h, k)_n = \sum_{n=0}^{\varepsilon-1} \sum_{r=0}^{r | n} \theta(h + kv, n) - 4f(\varepsilon - 3) + 1 + (-1)^{k} + (-1)^{h} +
\]

\[
+ \sum_{n=0}^{\varepsilon-1} \left[ D(h + 2kv, 2v) + D(h + 2kv, 2v) + D(h - k + 2kv, 2v) +
\]

\[
+ 2 \sum_{n=0}^{\varepsilon-1} \left[ d(h + 2kv, 2v) + (-1)^{f} d(h + 2kv, 2v) +
\]

\[
+ d(h - k + 2kv, 2v) + d(h, k) + d(h, k) - 4g(h, k) - 4g(h, k) +
\]

\[
- 4g(h, k) - 4g(h, k) - 2[1 + (-1)^{k/2}] g(h, E) +
\]

\[
+ (-1)^{k} D(h, \varepsilon) + D(h, \varepsilon) + (-1)^{h} D(h, \varepsilon) +
\]

\[
+ 2d(h + k \varepsilon, \varepsilon) + 2d(h + k \varepsilon, -\varepsilon) +
\]

\[
+ 2d(-h + (k - h) \varepsilon, \varepsilon) + 2d(-h + (k - h) \varepsilon, -\varepsilon) +
\]

\[
+ (-1)^{k} d(-h + (k - h) \varepsilon, \varepsilon) + d(-h + (k - h) \varepsilon, -\varepsilon).
\]
If (2.14) is applied to the summation in (3.6), it becomes

\[
\begin{align*}
(3.7) \quad 2 \sum_{v=0}^{N-1} & [d(h + 2kv, 2v) + (-1)^v d(h + 2kv, 2v) + d(h - k - 2kv, 2v)] -
\quad -2[(1)^{h-k} g(h, E) + (-1)^{h-k} g(h, E) + (-1)^{h-k} g(h - k, E)]
\end{align*}
\]

\(\varepsilon(h, k)\) has been evaluated in Theorem 2 of [12]. (Note that the \(f\) of that paper is taken to be even here.)

\[
\begin{align*}
(3.8) \quad \varepsilon(h, k) &= \frac{\delta - 1}{\delta} \sum_{v=0}^{\delta-1} \frac{D(h + \delta v, v)}{D(h + \delta v, v)} - 4f(\delta - 3) + 1 + (-1)^k + (-1)^h +
\quad + (-1)^{h-k} D(\delta - 3) + (-1)^{k-h} D(\delta - k, \delta) +
\quad + (-1)^{h-k} D(h, \delta) + \sum_{v=0}^{\delta-1} [D(h + 2kv, 2v) +
\quad + D(h + 2kv, 2v) + D(h - k - 2kv, 2v)] -
\quad -2[g(h, E) + g(h, E) + g(h - k, E)]
\end{align*}
\]

Combining the \(g\) terms in (3.7) and (3.8) gives

\[
\begin{align*}
(3.9) \quad -4g(h, k) - 4g(h - k, E) - 2[1 + (-1)^{h-k} g(h, E)]
\end{align*}
\]

\[
\begin{align*}
(3.10) \quad 4[D(h + h, k) + D(h + k, h) + (h + h, k) - 3(1, k) +
\quad = 4[D(h, k) - 4(h, k)]
\end{align*}
\]

Substituting (3.7) and (3.8), as modified by (3.9), and (3.10) into (3.6) gives the theorem, q.e.d.

Formulas for the cyclotomic numbers of order 4 in (3.4) are tabulated in [15, pp. 156–157]. They are expressed in terms of \(a\) and \(b\) in the quadratic decomposition

\[
\begin{align*}
(3.11) \quad p = a^2 + b^2, \quad a \equiv 1 \pmod{4}
\end{align*}
\]

By means of these formulas one verifies easily

**Theorem 4.** Let \(O(h, k) = 4[D(h, k) - 4(h, k)]\). If \(f\) is even,

\[
\begin{align*}
(0, 0) &= 6 + 6z, \\
(0, 1) &= O(1, 0) = C(3, 3) = 2 - 2x - 8y, \\
(0, 2) &= C(2, 0) = C(2, 2) = -2 - 2x, \\
(0, 3) &= C(3, 0) = C(1, 1) = 2 - 2x + 8y, \\
(1, 2) &= O(2, 1) = C(1, 3) = C(3, 1) = C(2, 3) =
\quad = C(3, 1) = -2 - 2x.
\end{align*}
\]

If \(f\) is odd,

\[
\begin{align*}
(0, 0) &= C(0, 0) = C(2, 2) = -2 - 2x, \\
(0, 1) &= O(1, 3) = C(3, 2) = -2 - 2x + 8y, \\
(0, 2) &= C(0, 2) = -6 + 6x, \\
(0, 3) &= C(1, 2) = C(3, 1) = -2 - 2x - 8y, \\
(1, 0) &= O(1, 1) = C(2, 1) = C(2, 3) = C(3, 0) =
\quad = C(3, 3) = -2 - 2x.
\end{align*}
\]
Note that Theorems 4 and 5 of [16] hold for every \( e \) which is a multiple of 1.

In (2.15), set \( x = 4, y = e, y \) odd. If \( \psi = 4t + 3 \),

\[
\sum_{s=0}^{e-1} b_4(i + 4s, 4t + 3) = b_4(i, 3) = b_4(i, 0) = \begin{cases} 1(p-5) & (4\mid i), \\ 1(p-1) & (4\nmid i), \end{cases},
\]

by (2.15), (2.13), and (2.14). Hence

\[
\sum_{s=0}^{e-1} d_4(4s, 4t + 3) = d_4(0, 3) = 1,
\]

(3.12)

\[
\sum_{s=0}^{e-1} d_4(1 + 4s, 4t + 3) = d_4(1, 3) = 0.
\]

If \( \psi = 4t + 1 \),

\[
\sum_{s=0}^{e-1} b_4(i + 4s, 4t + 1) = b_4(i, 1).
\]

According to Dickson ([4], (50)) \( \mathcal{E}_2(1, 1) = (-1)^y \left([d_4(0, 1) - d_4(1, 1)]/\psi^y\right) = -a + b/\psi^y \), where \( a \) and \( b \) satisfy (3.11). Hence

\[
\sum_{s=0}^{e-1} d_4(4s, 4t + 1) = d_4(0, 1) = (-1)^{y-1} a,
\]

(3.13)

\[
\sum_{s=0}^{e-1} d_4(1 + 4s, 4t + 1) = d_4(1, 1) = (-1)^y b.
\]

4. Cyclotomy for \( e = 20 \). Throughout this section \( e = 20, E = 10, \epsilon = 5. \beta \) is a primitive twentieth root of unity.

Theorem 3 expresses \((h, k)\) as a linear combination of \( p, a \) constant, Jacobi coefficients of orders 5, 10, and 10, and cyclotomic numbers of orders 2 and 4. The contribution of these cyclotomic numbers is given in Theorem 4. The Jacobi coefficients of orders 5 and 10 are expressed in [15] in terms of \( x, y, v, w \), where

\[
(4.1) \quad 16p = x^2 + 50w^2 + 50v^2 + 125wv, \quad y = 1(\mod 5), \quad xw = v^2 - v^2 - 4wv.
\]

The relationships are summarized as

Theorem 5.

20\( \mathcal{E}_2(0, 1) = 4p - 8 + 4x \),

20\( \mathcal{E}_2(1, 1) = 4p - 8 - x + 10u + 20v + 25w \),

(15), (4.7)

20\( \mathcal{E}_2(2, 1) = 4p - 8 - x + 10u - 10v - 25w \),

20\( \mathcal{E}_2(3, 1) = 4p - 8 - x - 20u + 10v - 25w \),

20\( \mathcal{E}_2(4, 1) = 4p - 8 - x - 10u - 20v + 25w \),

\( \mathcal{E}(j, 2) = \mathcal{E}(j, 3), \quad \mathcal{E}(j, 1) = \mathcal{E}(j, 1) \) (Lemma 2, (2.13)),

\( D(2j, 4) = \mathcal{E}(j, 1) + 3T, 1 - 4f \) (15), (6.12),

\( D(j, 2) = D(j + 4T, 4) \) (15), (6.12),

\( D(j, 6) = D(j + 4T, 4) \) (15), (6.8),

\( D(j, 8) = D(j + 2T, 4) \) (15), (6.6).

By virtue of (2.13), it suffices to determine \( d(j, r), 1 \leq r \leq 17, r \text{ odd} \). As we determine a minimal set of Jacobi differences for these \( d(j, r) \), we shall express \( d(j, 1), d(j, 3) \) and \( d(j, 7) \) in terms of \( d(j, 9) \); \( d(j, 13) \) and \( d(j, 17) \) in terms of \( d(j, 11) \); and \( d(j, 5) \) in terms of \( d(j, 5) \). We note that either \( a \) or \( b \), defined in (3.11) and (3.13), will appear in the expression of \( d(j, r) \) in terms of \( d(j, s) \) if \( v = \psi (\mod 1) \). Furthermore, \( d(2j, 11) \) can be expressed in terms of a Jacobi coefficient of order 5. We begin by examining \( d(j, 11) \) and \( d(j, 13) \).

Theorem 6.

\( d(2j + 1, 11) = 0 \), \( d(4j, 11) = D(2j + 4T, 1) \).

Proof. The first statement is a special case of (2.23). Now set \( t = 1 \) in (2.32): \( \psi(b^{\beta}, \beta) = \beta^{\psi} \psi(b^{\beta}, \beta) \).

Expand the left side of the equation by (2.18) and the right side by (2.11):

\[
(-1)^y \sum_{j=0}^{y} d(j, 11) \beta^j = \beta^{\psi} \sum_{j=0}^{y} b(j, 1) \beta^j.
\]

Since 2 is a quadratic residue of \( p = 1(\mod 20) \) if and only if \( f \) is even,

(4.3)

\[
(-1)^y = \beta^{4/p}.
\]

Thus

\[
\sum_{j=0}^{y} d(j, 11) \beta^j = \beta^{4/2} \sum_{j=0}^{y} [b(j, 1) + b(j + 10, 1)] \beta^j.
\]

\[
\sum_{j=0}^{y} d(2j, 11) \beta^j = \beta^{4/2} \sum_{j=0}^{y} b(j, 1) \beta^j = \beta^{4/2} \sum_{j=0}^{y} D(j, 1) \beta^j
\]

\[
= \beta^{4/2} \sum_{j=0}^{y} D(j + 4T, 1) \beta^{j + 4}.
\]

by the first statement of the theorem, (2.15) and (2.16). Hence

\[
\sum_{j=0}^{y} d(4j, 11) \beta^j = \sum_{j=0}^{y} D(j + 4T, 1) \beta^j = \sum_{j=0}^{y} D(2j + 4T, 1) \beta^j,
\]

\[
\sum_{j=0}^{y} [d(4j, 11) - d(0, 11)] \beta^j = \sum_{j=0}^{y} [D(2j + 4T, 1) - D(4T, 1)] \beta^j.
\]

These sums both lie in the cyclotomic field of degree four over the rationals formed by adjoining \( \beta \). A basis for this field is \( \beta, \beta^3, \beta^5, \beta^7 \). Thus it is permissible to equate coefficients of like powers of \( \beta \):

\[
d(4j, 11) - d(0, 11) = D(2j + 4T, 1) - D(4T, 1), \quad j = 0, 1, 2, 3, 4.
\]
Sum this equation over $i = 0, 1, 2, 3, 4$:

$$\sum_{i=0}^{4} d(4i, 11) - 5d(0, 11) = \sum_{i=6} d(2i+4T, 1) - 5D(4T, 1).$$

Both sums equal $-1$, by (3.12) and (2.20), respectively. Hence $d(0, 11) = D(4T, 1)$, so that $d(4i, 11) = D(2i+4T, 1)$, $i = 0, 1, 2, 3, 4$.

By combining Theorems 5 and 6, we obtain

$$d(4j, 11) = D(2j + 4T, 1) = D(2j + 4T, 8) = D(2j + 6T, 4) = \mathcal{O}(3j + 2T, 1) - 4\mathcal{O}.$$  

The study of $d(j, 9)$ is a modification of that in [17], which was based on work of Cauchy. By (2.26),

$$\psi(\beta^3) \psi(\beta^3) = (-1)^t p = \psi(\beta^3) \psi(\beta^3).$$

In (2.29) put $\theta = \beta^4$ and $t = 1$, then apply (4.5):

$$\psi(\beta^3) \psi(\beta^3) \psi(\beta^3) \psi(\beta^3) = \beta^{-2\theta} \psi(\beta^3) \psi(\beta^3) \psi(\beta^3) \psi(\beta^3),$$

$$\psi(\beta^3) \psi(\beta^3) = \beta^{-2\theta} \psi(\beta^3) \psi(\beta^3),$$

in view of (2.24).

In (2.18) with $v = 2$, set $u = 1$, then 3:

$$\psi(\beta^3, \beta^3) = (-1)^t \sum_{j=0}^{9} d(j, 9) \beta^j, \quad \psi(\beta^3, \beta^3) = (-1)^t \sum_{j=0}^{9} d(j, 9) \beta^j.$$  

By (2.22),

$$d(1, 9) = d(9, 9), \quad d(3, 9) = d(7, 9),$$

Hence

$$(-1)^t \psi(\beta, \beta^3) = A + BI + (C + DI)(\theta - \theta^2 - \theta^3 + \theta^4),$$

where $\theta = \beta^4$, $I = \beta^4$,

$$A = 2d(0, 9) - d(2, 9) - d(4, 9),$$

$$B = 2d(5, 9) - d(3, 9) - d(1, 9),$$

$$\theta = d(2, 9) + d(4, 9),$$

$$C = d(3, 9) + d(1, 9).$$

Similarly,

$$(-1)^t \psi(\beta^3, \beta^3) = A - BI + (-C + DI)(\theta - \theta^2 + \theta^3),$$

By the law of quadratic reciprocity, $5p(3) = (5p)/3 = 1$. Hence $p$ is even. If $F = 0 (mod.4)$, equations (4.8) and (4.10) by virtue of (4.6) shows that

$$2BI + 2C(\theta - \theta^2 - \theta^3 + \theta^4) = 0.$$  

Since $I$ is purely imaginary and $\theta - \theta^2 - \theta^3 + \theta^4$ is real, $B = C = 0$. By (2.27)

$$p = |(-1)^t \psi(\beta, \beta)|^2 = |A + DI(\theta - \theta^2 - \theta^3 + \theta^4)|^2 = A^2 + 5D^2.$$  

Similarly, if $F = 2 (mod.4)$, $\psi(\beta^3, \beta^3) = -\psi(\beta^3, \beta^3)$, so $A = D = 0$.

$$p = |BI + C(\theta - \theta^2 - \theta^3 + \theta^4)|^2 = B^2 + 5C^2.$$  

Set $t = 2$ in (3.13) and apply (4.7):

$$(-1)^t + a = d(0, 9) + 2d(4, 9) + 2d(8, 9),$$

$$(-1)^t + b = d(5, 9) + 2d(1, 9) + 2d(13, 9).$$

Let the quadratic decomposition of $p$ given by (4.11) and (4.12) be written as

$$p = c^2 + 5d^2.$$  

If $F = 0 (mod.4)$, let $c = A, d = D$. $B = C = 0$. If $F = 2 (mod.4)$, let $c = B, d = C$. $A = D = 0$. Then the six equations in (4.9) and (4.13) can be solved for the six quantities $d(0, 9), d(4, 9), d(8, 9), d(1, 9), d(5, 9)$, and $d(13, 9)$ in terms of $a, b, c$ and $d$, yielding:

**Theorem 7.** If $5$ is a biquadratic residue of $p$,

$$5d(0, 9) = (-1)^{t+1} a + 4c,$$

$$5d(4, 9) = (-1)^{t+1} a - c \quad (5\n+n),$$

$$5d(5, 9) = (-1)^t b,$$

$$5d(1, 9) = 5d(9, 9) = (-1)^t b + 5d,$$

$$5d(13, 9) = 5d(17, 9) = (-1)^t b - 5d.$$  

If $5$ is a biquadratic nonresidue of $p$

$$5d(5, 9) = (-1)^{t+1} a + 4c,$$

$$5d(5+4t, 9) = (-1)^t b - c \quad (5\n+n),$$

$$5d(0, 9) = (-1)^{t+1} a,$$

$$5d(4, 9) = 5d(16, 9) = (-1)^{t+1} a + 5d,$$

$$5d(8, 9) = 5d(12, 9) = (-1)^{t+1} a - 5d.$$
Corollary. If $5$ is a biquadratic residue of $p$, $5|b$ and $d$ is even.
Otherwise $5|a$ and $c$ is even.

Proof. By (2.13) and (2.16), $d(j, 9) = d(j, 10)$. But (2.16) and Lemma 1 imply that $d(j, 10)$ is odd if and only if $10|j$. Thus if $F = 0 (mod 4)$, the formula for $5d(1, 9)$ shows that $5|b$ and, since $b$ is even, $d$ must be even. If $F = 2 (mod 4)$, look at the formulas for $5d(0, 9)$ and $5d(1, 9)$.

Part of this corollary is not new. See, e.g., ([9], p. 69).

Theorem 8.

$d(1, 1) = d(j+12T, 9)$.

Proof. In (2.31) set $t = 1$ and apply (2.10), then (4.3):

\[ \beta^{2T} \psi(\beta, \beta) = \psi(\beta^0, \beta) = (1)^{2T} \psi(\beta^0, \beta), \]
\[ \psi(\beta, \beta) = \beta^{2T} \psi(\beta^0, \beta). \]

(4.15)

Expand (4.15) by means of (2.18):

\[ \sum_{j=0}^{3} d(j, 1) \beta^j = \beta^{2T} \sum_{j=0}^{3} d(j, 9) \beta^j = \beta^{2T} \sum_{j=0}^{3} d(j+12T, 9) \beta^{j+12T}. \]

By (2.17)

\[ \sum_{i=0}^{4} d(4i+1, 9) \beta^{4i} = d(4i+1, 1) \beta^{4i+1} = \sum_{i=0}^{4} d(4i+12T, 9) \beta^{4i} + d(4i+1+12T, 9) \beta^{4i+1}. \]

(4.16)

\[ \sum_{i=0}^{4} [d(4i, 1) - d(8, 1)] \beta^{4i} = [d(4i+1, 1) - d(9, 1)] \beta^{4i+1} = \sum_{i=0}^{4} [d(4i+12T, 9) - d(8+12T, 9)] \beta^{4i} + d(4i+1+12T, 9) - d(9+12T, 9)] \beta^{4i+1}. \]

Since the coefficients of $\beta^0$ and $\beta^1$ in (4.16) are zero, and since the set $1, \beta, \beta^2, \beta^3, \beta^4, \beta^5, \beta^6, \beta^7, \beta^8$ forms a basis for the cyclotomic field formed by adjoining $\beta$ to the rationals, we may equate coefficients of corresponding powers of $\beta$ in (4.16):

(4.17)

\[ d(4i, 1) - d(8, 1) = d(4i+12T, 9) - d(8+12T, 9), \]
\[ d(4i+1, 1) - d(9, 1) = d(4i+1+12T, 9) - d(9+12T, 9). \]

Sum (4.17) over $i = 0, 1, 2, 3, 4$ and apply the first statement of (3.13) to obtain $d(8, 1) = d(8+12T, 1)$, so that $d(4i, 1) = d(4i+12T, 1).$ Similar use of the second statement of (3.13) after summing (4.18) yields $d(4i+1, 1) = d(4i+1+12T, 9)$.

Lemma 3. Let

(4.19)

\[ \mu \psi(\beta, \beta) = \psi(\beta^0, \beta). \]

Then $\mu^2 = \beta^{2T}$.

Proof. Set $t = 2$ in (2.32), apply (2.10), then use (2.24):

\[ \beta^{2T} \psi(\beta, \beta) = \psi(\beta^0, \beta) = \psi(\beta^0, \beta^0), \]
\[ \psi(\beta, \beta) = \beta^{2T} \psi(\beta^0, \beta) = \beta^{2T} \psi(\beta^0, \beta^0), \]
\[ \tau(\beta) = \frac{\tau(\beta^0)}{\tau(\beta^0)} = \frac{\beta^{2T} \tau(\beta^0)}{\psi(\beta^0, \beta^0)} = \psi(\beta^0, \beta^0), \]
\[ \psi(\beta^0, \beta^0) = \beta^{2T} \psi(\beta^0, \beta^0), \]

by (4.19). But

\[ \psi(\beta^0, \beta^0) = \beta^{2T} \psi(\beta^0, \beta^0) = \beta^{2T} \psi(\beta^0, \beta^0) = \beta^{2T} \psi(\beta^0, \beta^0), \]

by (4.15) with $\beta$ replaced by $\beta^0$, (4.6), and (4.15) as it is written. Hence $\mu^2 = \beta^{2T}$.

Since $F$ is even, $\mu$ is a fourth root of unity. In order to facilitate working with equations containing $\mu$, we introduce the following notation:

(4.20)

\[ \mu = \beta^n, \quad M = (-1)^{h(n+1)}, \quad M' = (-1)^{h(n+1)}, \]

One may assume that $m$ takes on the four values 0, 5, 10 and 15.

Theorem 9.

\[ d(4j+3, 3) = d(4j-1, 1) - d(4j-1, 1)/5, \]
\[ d(4j+1, 3) = d(4j+1, 1) - d(4j+1, 1)/5. \]

Proof. We proceed as in the proof of Theorem 8. Expand (4.19) by (2.18) and use (4.20):

\[ \sum_{j=0}^{4} d(j, 3) \beta^j = \beta^n \sum_{j=0}^{4} d(j, 1) \beta^j = \beta^n \sum_{j=0}^{4} d(j, 1) \beta^{j-m} = \sum_{j=0}^{4} d(j, 1) \beta^j. \]

(4.21)

\[ d(4j, 3) = d(8, 3) = d(4j-1, 1) - d(8-1, 1). \]

(4.22)

\[ d(4j+1, 3) = d(9, 3) = d(4j+1, 1) - d(9-1, 1). \]

Sum (4.21) over $i = 0, 1, 2, 3, 4$:

\[ \sum_{i=0}^{4} d(4i, 3) - 5d(8, 3) = \sum_{i=0}^{4} d(4i-1, 1) - 5d(8-1, 1). \]

The two sums can be evaluated by (3.12) and (3.13), respectively. Then substituting the value of $d(8, 3)$ back into (4.21) yields the first statement of the theorem. The second part is obtained by summing (4.22), evaluating the sums, and substituting back into (4.22).
By combining Theorems 8 and 9, we obtain

\begin{equation}
\begin{aligned}
&d(4j - 12T, 3) = d(4j - m, 9) - [1 + d_4(-m, 1)]/5, \\
&d(4j + 1 - 12T, 3) = d(4j + 1 - m, 9) - d_4(1 - m, 1)/5.
\end{aligned}
\end{equation}

We now evaluate (4.23) by means of Theorem 7, and express the results compactly with the aid of the notation defined in (4.20) to obtain

**Theorem 10.**

\begin{equation}
\begin{aligned}
&d(8T, 3) = (4Mc - 1)/5, \\
&d(4j + 8T, 3) = (-Mc - 1)/5 (5 \not| j), \\
&d(4j + 5 + 8T, 3) = \left(\frac{j}{5}\right) M'd.
\end{aligned}
\end{equation}

Notice that \(a\) and \(b\) do not appear in Theorem 10.

From Theorem 9 and (4.24) we deduce that \(1 + d_4(-m, 1), d_4(1 - m, 1)\) and \(Mc + 1\) are divisible by 5. Combining this with the Corollary to Theorem 7, we obtain the following determination of \(\mu\):

**Lemma 8.**

\begin{itemize}
  \begin{itemize}
    \item \(\mu = 1, \quad m = 0, \quad M = 1, \quad M' = 1: \quad a \equiv (-1)^c(\text{mod} 5), \quad c \equiv 0(\text{mod} 10). \)
    \item \(\mu = -1, \quad m = 0, \quad M = 1, \quad M' = 1: \quad a \equiv (-1)^c(\text{mod} 5), \quad c \equiv 0(\text{mod} 10). \)
    \item \(\mu = \beta^5, \quad m = 5, \quad M = 1, \quad M' = 1: \quad b \equiv (-1)^c(\text{mod} 5), \quad c \equiv 0(\text{mod} 10). \)
    \item \(\mu = \beta^{15}, \quad m = 15, \quad M = 1, \quad M' = 1: \quad b \equiv (-1)^c(\text{mod} 5), \quad c \equiv 0(\text{mod} 10). \)
  \end{itemize}
\end{itemize}

This determination is definitive only if \(F \equiv 0(\text{mod} 4)\), for in this case we use the fact that \(a \equiv 1(\text{mod} 4)\) to ascertain whether \(\mu = 1\) or \(-1\). Since the sign of \(b\) depends upon the choice of the primitive root \(g\), if \(F \equiv 2(\text{mod} 4)\) whether \(\mu = \beta^5\) or \(\beta^{15}\) depends upon \(g\).

Although by (2.21), \(d(i, 7) = d(3j, 3)\), we prefer to evaluate \(d(j, 7)\) by means of

\begin{equation}
\psi(\beta^T \psi(\beta^r, \beta), \beta) = \mu \psi(\beta^T, \beta).
\end{equation}

**Proof of (4.25).** By (4.19) with \(\beta^T\) replacing \(\beta\),

\begin{equation}
\begin{aligned}
\psi(\beta^T, \beta) &= \psi(\beta^T, \beta^5) = \mu^5 \psi(\beta^T, \beta^5) \\
&= \mu^5 \beta^{2T} \psi(\beta^T, \beta^5) = \mu^5 \beta^{4T + 5T} \psi(\beta^T, \beta) = \mu \psi(\beta^T, \beta),
\end{aligned}
\end{equation}

by (4.15) with \(\beta^T\) replacing \(\beta\), (4.6), and Lemma 3.

Expand (4.25) by means of (2.18):

\begin{equation}
\begin{aligned}
&\beta^{2T} \sum_{j=0}^{9} d(j, 7) \beta^j = \beta^{2T} \sum_{j=0}^{9} d(j, 9) \beta^j, \\
&\beta^{2T} \sum_{j=0}^{9} d(j + 4T, 7) \beta^{4j - 47} = \beta^{2T} \sum_{j=0}^{9} d(j + 5, 9) \beta^{4j - 9m}.
\end{aligned}
\end{equation}

Proceeding as in the proof of Theorem 9, we establish

\begin{equation}
\begin{aligned}
&d(4j - 4T, 7) = d(4j - m, 9) - [1 + d_4(-m, 1)]/5, \\
&d(4j + 1 - 4T, 7) = d(4j + 1 - m, 9) - d_4(1 - m, 1)/5.
\end{aligned}
\end{equation}

Comparison with (4.23) shows that \(d(4j - 4T, 7) = d(4j - 12T, 3)\). Hence from Theorem 10 we obtain

**Theorem 11.**

\begin{equation}
\begin{aligned}
&d(-4T, 7) = (4Mc - 1)/5, \\
&d(4j - 4T, 7) = (-Mc - 1)/5 \quad (5 \not| j), \\
&d(4j + 5 + 4T, 7) = \left(\frac{j}{5}\right) M'd.
\end{aligned}
\end{equation}

We turn now to the evaluation of \(d(i, 13)\) and \(d(i, 17)\). By (2.21), (4.28)

\begin{equation}
\begin{aligned}
&d(j, 13) = d(17j, 17), \\
&d(j, 17) = d(17j, 17).
\end{aligned}
\end{equation}

**Theorem 12.** If \(F \equiv 0(\text{mod} 4)\),

\begin{equation}
\begin{aligned}
&d(4j, 17) = Md(4j + 12T, 11) + [(-1)^{j+1} a + M]/5, \\
&d(4j + 5, 17) = (-1)^b/5.
\end{aligned}
\end{equation}

If \(F \equiv 2(\text{mod} 4)\),

\begin{equation}
\begin{aligned}
&d(4j, 17) = (-1)^{j+1} a/5, \\
&d(4j + 5, 17) = Md(4j + 12T, 11) + [(-1)^b + M]/5.
\end{aligned}
\end{equation}

**Proof.** In (2.25), take \(n = 1, r = 2, s = 1\), then divide by (4.19):

\begin{equation}
\begin{aligned}
&\mu \psi(\beta^T, \beta^r) = \psi(\beta^T, \beta^2) = \beta^{2T} \psi(\beta^{11}, \beta), \\
&\mu \psi(\beta^{2T}, \beta) = \beta^{11T} \psi(\beta^{11}, \beta).
\end{aligned}
\end{equation}

by (4.2). Apply (2.10) and (4.3):

\begin{equation}
\begin{aligned}
&\mu \psi(\beta^{2T}, \beta) = \beta^{12T} \psi(\beta^{11}, \beta), \\
&\beta^{11T} \psi(\beta^{11}, \beta) = \beta^{12T} \psi(\beta^{11}, \beta), \\
&\beta^{11T} \psi(\beta^{11}, \beta) = \beta^{12T} \psi(\beta^{11}, \beta).
\end{aligned}
\end{equation}

Expand by means of (2.18):

\begin{equation}
\begin{aligned}
&\beta^{12T} \sum_{j=0}^{9} d(j - m, 11) \beta^{11j - m} = \beta^{12T} \sum_{j=0}^{9} d(j + 12T, 11) \beta^{11j + 12T} \\
&= \sum_{j=0}^{9} d(24j + 12T, 11) \beta^{24j}.
\end{aligned}
\end{equation}
by Theorem 6. Hence
\[
d((j - m, 17) - d(8 - m, 17) = d(4j + 12T, 11) - d(8 + 12T, 11),
\]
\[
d(4j + 1 - m, 17) = d(9 - m, 17) = 0.
\]
Summing over \( j = 0, 1, 2, 3, 4 \), by (2.15) and (3.12), and substituting to obtain
\[
d((j - m, 17) = d(4j + 12T, 11) + [d_1(-m, 1)]/5,
\]
\[
d(4j + 1 - m, 17) = d_4(-m, 1)/5.
\]
To complete the proof, set \( m = 0 \) and 10, then 5 and 15, and apply (3.13).

**Theorem 13.** If \( F \equiv 0 \pmod{4} \),
\[
\sum_{j=0}^{9} d(j, 5) = -d(4j + 5, 15) = d(8j + 5, 5) - (-1)^j a / 5.
\]
If \( F \equiv 2 \pmod{4} \),
\[
\sum_{j=0}^{9} d(j, 5) = -d(4j + 5, 15) = d(8j + 5, 5) - (-1)^j a / 5.
\]

**Proof.** In (2.25), set \( \alpha = 1, \, \beta = 3, \, \gamma = 1 \), then divide by (4.19):
\[
\sum_{j=0}^{9} d(j, 5) = -d(4j + 5, 15) = d(8j + 5, 5) - (-1)^j a / 5.
\]
by (2.10). Expand by means of (3.18):
\[
\beta^m \sum_{j=0}^{9} d(j, 5) = d((j - m, 15) \beta^m = \sum_{j=0}^{9} d(j, 5) \beta^j = \sum_{j=0}^{9} d(7j, 5) \beta^j.
\]
Thus if we take a basis \( \beta, \beta^2, \beta^3, \beta^4, \beta^5, \beta^6, \beta^7, \beta^8, \beta^9 \),
\[
d((j - m, 15) - d(-m, 15) = d(8j, 5) - d(0, 5),
\]
\[
d(4j + 5 - m, 15) - d(5 - m, 15) = d(8j + 5, 5) - d(15, 5).
\]
Summing over \( j = 0, 1, 2, 3, 4 \) and apply (2.15) and (3.13):
\[
d(4j + 5 - m, 15) = d(8j + 5) + [d_1(-m, 3) + (-1)^j a] / 5,
\]
\[
d(4j + 5 - m, 15) = d(8j + 5) + [d_4(-m, 3) + (-1)^j a] / 5.
\]
To complete the proof, set \( m = 0 \) and 10, then 5 and 15, and apply (3.12).

Some of the relationships between Jacobi sums developed in this section were previously stated by Dickson [15], Section 15.

**5. Evaluating the cyclotomic numbers of order twenty.** To produce a formula for any cyclotomic number of order 20, substitute into Theorem 3 values of the appropriate cyclotomic numbers of orders two and four and Jacobi sums of orders 5, 10, and 20 which can be obtained from Theorems 4 through 13 and (4.26). It is necessary to specify only two parameters, \( T \) and \( \mu \). These determine the parity of \( f \) and \( F \pmod{4} \) by (4.3) and Lemma 3, respectively.

The formula could be expressed as a linear combination of twenty variables: \( p, \) the constant 1, \( a, b \) (see (3.11)), \( e, d \) (see (4.14)), \( x, u, v, w \) (see (4.1)), and \( d(j, 5), \) \( 0 \leq j \leq 9. \) According to Theorem 6 of [7],
\[
\sum_{j=0}^{9} d(j, 5)^2 = p,
\]
so the solution of the cyclotomic number problem for \( c = 20 \) is complete according to the definition given in the introduction.

Not all of the variables are linearly independent, however. According to (3.13),
\[
d(0, 5) - d(2, 5) + d(4, 5) + d(8, 5) + d(8, 5) = (-1)^j a,
\]
\[
d(1, 5) - d(3, 5) + d(5, 5) - d(7, 5) + d(9, 5) = (-1)^j b.
\]
Two of these variables must be dropped to form a minimal set. In the interests of symmetry, \( a \) and \( b \) were dropped, instead of two of the \( d(j, 5). \)

Producing a list of all the formulas for the cyclotomic numbers, however, is a formidable task. There are ten choices for \( T \pmod{4} \) and four possibilities for \( \mu \), so that there are forty classes to be considered. For each class 77 formulas must be computed, according to (2.3). We turned to the computer for assistance in deriving the 3800 formulas. An assembly language program was written for the IBM7070 to generate the formulas.

Computers were quite useful in earlier phases of the study as well. First the cyclotomic numbers of order twenty were computed for over 600 primes. From these, values of Jacobi coefficients were computed; examination of these values helped formulate some of the theorems proved in Section 4.

When the cyclotomic numbers were computed for the prime \( p, \) \( g \) was chosen to be the smallest positive primitive root. Unless \( 5 | T \) and \( 4 | F \), the choice of \( g \) determined in part which one of the forty classes of formulas would be applicable to \( p. \) Where the cyclotomic numbers had been computed for at least eighteen primes in a class, the formulas for that class were determined empirically. Specifically, we found the coefficient \( x_{jk} \) of the \( j \)th variable in the formula for 1600 (\( h, k \)).

For eighteen primes \( p_i, 1 \leq i \leq 18, \) in a class, the values of the eighteen variables were computed. This gave rise to a set of eighteen linear equations
\[
\sum_{j=0}^{9} a_j x_{jk} = 1600 (h, k),
\]
where \( a_j \) denotes the value of the \( j \)th variable for the \( i \)th prime, and \( (h, k) \) is the value of \( (h, k) \) for the \( i \)th prime. The scaling factor of 1600 was used to insure rational integral values for the \( x_{jk}. \)

The set of equations (5.1) was solved by computer. The computed
values of the coefficients were rounded to the nearest integer. It was then verified that these coefficients formed the exact solution. These coefficients, computed for nine classes, were very helpful in the detection and elimination of errors in the program which generated all the formulas for the cyclotomic numbers.

It is not appropriate to include all 3680 formulas. Table 1 consists of the formulas for the class \( T = 1 \pmod{10} \), \( f = 1 \), \( \mu = 10 \), \( F = 2 \pmod{4} \).

For each of eighteen primes in this class, the values of sixteen of the

| \( (k, k) \) | \( p \) | \( I \) | \( c \) | \( d \) | \( x \) | \( u \) | \( v \) | \( w \) |
|-----|-----|-----|-----|-----|-----|-----|-----|
| (1, 4) | 4 | -20 | -20 | -20 | -20 | -20 | -20 |
| (1, 5) | 5 | -20 | -20 | -20 | -20 | -20 | -20 |
| (1, 6) | 6 | -20 | -20 | -20 | -20 | -20 | -20 |
| (1, 7) | 7 | -20 | -20 | -20 | -20 | -20 | -20 |
| (1, 8) | 8 | -20 | -20 | -20 | -20 | -20 | -20 |
| (1, 9) | 9 | -20 | -20 | -20 | -20 | -20 | -20 |

It was pointed out in [12] that for several values of \( c \), there are fewer distinct cyclotomic number formulas than the upper bound given by (2.3). This phenomenon has been observed for \( c = 6, 8, 10, 12, 14, 16 \).

For \( c = 20 \), the 77 formulas are all different for only some of the classes.

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for the class \( \ind = 2 \pmod{4}, \mu = 5 \).
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### The cyclotomic numbers of order twenty

J. B. Muskat and A. L. Whitman
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### Table 4. Formulas for the cyclotomic numbers \( \left( k, \theta \right) \)

#### \( \text{folded, } \text{ind} 2 = 1 \text{ (mod 10)} \)

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#### \( \text{folded, } \text{ind} 2 = 1 \text{ (mod 10)} \)

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#### in five classes, used in studying difference sets

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### Notes

- Table 4 provides formulas for the cyclotomic numbers \( \left( k, \theta \right) \) for various values of \(p\) and \(c\).
- The table is divided into two sections: one for \(\text{folded, } \text{ind} 2 = 1 \text{ (mod 10)}\) and another for \(\text{folded, } \text{ind} 2 = 1 \text{ (mod 10)}\).
- In the section for five classes, difference sets are used to study cyclotomic numbers.
- The formulas are presented in a tabular format for easy reading and analysis.

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**References**

- The source of the table is not specified in the image. For more information, consult the source document or bibliographic references provided in the original publication.
A copy of all the formulas may be obtained from the first author. A copy has been placed in the Unpublished Mathematical Tables repository maintained by Mathematics of Computation.

The authors wish to express their appreciation to the University of Pittsburgh's Computer Center for granting access to its IBM 7070/1401, IBM 7090/1401, and IBM 360/50 systems, partially supported under NSF grants G-11369 and GP-2310, and NIH grant FR-00250, respectively.

### Table 4 (continued)

#### $\text{ind} \delta = 0 \pmod{4}$, $c = 1 \pmod{10}$

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6. Application to residue difference sets. A difference set of modulus $v$, order $k$, and multiplicity $\lambda$ is a set of $k$ distinct residues $r_1, r_2, \ldots, r_k \pmod{v}$ such that the congruence $r_i - r_j = d \pmod{v}$ has exactly $\lambda$ solutions for each $d \pmod{v}$.

A residue difference set is a difference set consisting of the non-zero $c$th power residues, modulo a prime $p$. A difference set formed by zero and the $c$th power residues is called a modified residue difference set.
Emma Lehmer proved that if \( f \) is odd it is a necessary condition for the existence of a residue difference set or a modified residue difference set. Necessary and sufficient conditions are given, respectively, by

\[
\begin{align*}
(6.1) \; (i, 0) &= (f-1)/e, \quad i = 0, 1, \ldots, E-1 \\
(6.2) \; 1+(0, 0) &= (i, 0) = (f+1)/e, \quad i = 1, 2, \ldots, E-1
\end{align*}
\]

([10], Theorem III).

Theorem 14. If \( 5 \) is a biquadratic residue of a prime \( p = 1 (\mod 20) \), then the twentieth power residues, with or without zero, do not form a difference set.

Proof. The arguments to be presented hold both for residue difference sets and modified residue difference sets, as no reference is made to \((0, 0)\). It suffices to consider \( \text{ind} 2 = 2, 5 (\text{mod} 10) \), for if \( \text{ind} 2 = 3, 7 \) or \( 9 (\text{mod} 10) \), one could choose another primitive root \( g' \) such that \( \text{ind} g' 2 = 1 (\text{mod} 10) \) and the difference set is independent of the choice of \( g \).

The contradictions are obtained from (4.1). The cyclotomic number formula which are needed have been included in Table 4. Note that by (2.1) and (2.2), \( (h, 0) = (-h, -h) = (10 - h, 10 - h) \).

First consider \( \text{ind} 2 = 5 (\text{mod} 10) \). If \( c = 1 (\text{mod} 10) \),

\[
\begin{align*}
4[(1, 0) - (9, 0) + (2, 0) - (8, 0)] &= \psi + \psi = 0, \\
4[(4, 0) - (6, 0) + (3, 0) + (7, 0)] &= \psi - \psi = 0.
\end{align*}
\]

Thus \( \psi = \psi = 0 \). This implies \( \omega \psi = 0 \), so that \( 16p \) is either a perfect square or divisible by 125; either is impossible.

If \( c = 9 (\text{mod} 10) \),

\[
\begin{align*}
4[(1, 0) - (9, 0) + (2, 0) - (8, 0)] &= \psi + \psi = 0, \\
4[(4, 0) - (6, 0) + (3, 0) - (7, 0)] &= \psi - \psi = 0.
\end{align*}
\]

Again \( \psi = \psi = 0 \), a contradiction.

Now let \( \text{ind} 2 = 1 (\text{mod} 10) \). If \( c = 1 (\text{mod} 10) \),

\[
\begin{align*}
16[-(1, 0) - (3, 0) + (7, 0) + (9, 0) - (2, 0) + (4, 0) - (6, 0) + (8, 0)] &= -\psi + \psi = 0, \\
16[-(1, 0) + (3, 0) + (7, 0) - (9, 0) + (2, 0) + (4, 0) - (6, 0) - (8, 0)] &= 3\psi - 3\psi + 6\psi + \psi = 0.
\end{align*}
\]

Thus \( \psi = \psi = 5\psi, \psi = 3\psi + 5\psi + 6\psi = 6\psi + 9\psi \). Then by (4.1)

\[
\begin{align*}
6u + 9\psi \psi &= (u + 5\psi)^2 - u^2 - 4u(u + 5\psi), \\
16u^2 - 16u - 4u^2 &= 0, \quad (4u^2 - 2u)^2 = 2(2u^2).
\end{align*}
\]

This equation has only the solution \( u = v = 0 \). Then \( v = 0 \), which is impossible.

If \( c = 9 (\text{mod} 10) \),

\[
\begin{align*}
16[(1, 0) - (3, 0) + (7, 0) - (9, 0) - (2, 0) - (4, 0) + (6, 0) + (8, 0)] &= -5u + 5v - 9u = 0, \\
16[(1, 0) + (3, 0) - (7, 0) - (9, 0) - 3(2, 0) + 3(4, 0) + (6, 0) - (8, 0)] &= 7u - v - 2w - x = 0.
\end{align*}
\]

Thus \( 9w = 5v - 5u, 9x = 73w - 19c \). By (4.1),

\[
5(u - v)(73w - 19c) = 81(u^2 - u^2 - 4w^2), \quad 176w^2 - 784w^2 + 284u^2 = 0.
\]

Multiply by \( 11/4 \):

\[
(22v - 49u)^2 = 5(18u)^2.
\]

Then \( u = v = 0 \), which is impossible.

If \( 5 \) is a biquadratic nonresidue of \( p \), it suffices to consider just two classes—\( \text{ind} 2 = 1 \) or \( 5 (\text{mod} 10) \), \( e = 6 (\text{mod} 10) \), because replacing \( g \) by \( g' \), where \( t = 11 (\text{mod} 20) \) and \( t \) is relatively prime to \( p - 1 \), would leave \( \text{ind} 2 (\text{mod} 10) \) unchanged but would yield \( e = 4 (\text{mod} 10) \). The formulas for \( (h, 0), 0 \leq h \leq 9 \), for the two classes mentioned above are included in Tables 1 and 4, respectively.

Efforts to prove that there are no residue difference sets or modified residue difference sets for these two classes were unsuccessful.

References

A metric inequality associated with valued fields

by

P. E. BLANKSBY (Cambridge)

1. Introduction. Suppose that $F$ is a field with a valuation $\| \|$ mapping $F$ into $\mathbb{R}$, the real numbers. Let $\alpha$ and $\beta$ be two points in the cartesian product $F^n = F \times \ldots \times F$, with coordinates $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ respectively. We can define a function from $F^n \times F^n$ to $\mathbb{R}$ as follows:

$$d(\alpha, \beta) = \min_{\sigma \in S_n} \max_{1 \leq i \leq n} \| \alpha_i - \beta_{\sigma(i)} \|,$$

where $S_n$ is the symmetric group on $n$ objects. It is clear that if we write $\sigma = (\sigma_0, \ldots, \sigma_{n-1})$, then for any $\sigma, \tau$ in $S_n$ we have $d(\sigma \alpha, \tau \beta) = d(\alpha, \beta)$.

It follows that $d(\alpha, \beta)$ satisfies the triangle inequality since we may suppose, by taking a suitable permutation of the coordinates if necessary, that

$$d(\alpha, \tau) = \max_{1 \leq i \leq n} \| \alpha_i - \gamma_i \|,$$

and

$$d(\tau, \beta) = \max_{1 \leq i \leq n} \| \gamma_i - \beta_i \|.$$

Hence

$$d(\alpha, \beta) \leq \max_{1 \leq i \leq n} \| \alpha_i - \beta_i \| = \max_{1 \leq i \leq n} \| \alpha_i - \gamma_i \| + \max_{1 \leq i \leq n} \| \gamma_i - \beta_i \| = d(\alpha, \tau) + d(\tau, \beta).$$

Thus $d(\alpha, \beta)$ is a pseudo-metric on $F^n$.

We define the real quantities

$$\mathcal{M}(\alpha, \beta) = \mathcal{M} = \max_{1 \leq i \leq n} \{ \| \alpha_i \|, \| \beta_i \| \},$$

$$\mathcal{R}(\alpha, \beta) = \mathcal{R} = \prod_{i=1}^n \| \alpha_i - \beta_i \|.$$.

In this paper we seek lower bounds on $d(\alpha, \beta)$ in terms of $\mathcal{M}$ and $\mathcal{R}$. If $\| \|$ is a non-archimedean valuation, then it readily follows that if $\mathcal{M} > 0$, then

$$d(\alpha, \beta) \geq \frac{\mathcal{M}^{1/n}}{\mathcal{M}^{n-1}}.$$