is a contradiction for $m = |\beta_i|$. Thus we get for some rational q: $\beta_i = q\alpha_i$ $(1 \leqslant i \leqslant r)$. If $\alpha_i = 0$ $(1 \leqslant i \leqslant r)$ then Theorem 2 holds with $k = t_1$. If for some i, $\alpha_i \neq 0$ then (11) with $m = |\alpha_i|$ implies q integer and $t_{|\alpha_i|} \equiv q \bmod 2$. Hence $\eta = \varepsilon^q$ and Theorem 2 holds with $k = q$.

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On the probability that n and $f(n)$ are relatively prime

by

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It is a well-known theorem of Čebyšev that if n and m are randomly chosen positive integers, then $(n, m) = 1$ with probability $6/\pi^2$. One can expect this to remain true if $m = f(n)$ is a function of n, provided that $f(n)$ does not preserve arithmetic properties of n. Erdös and Lorentz [1] proved that this is so, in the case $f(n) = [f_1(n)]$, where $f_1(x)$ is a smooth function satisfying certain (weak) conditions.

The case $f_1(n) = \alpha n$ was considered by G. L. Watson [6]. For all α, the positive integers n for which $(n, f(n)) = 1$ have a density, and in particular, for irrational α this is $6/\pi^2$.

Suppose now that $f(n)$ is a multiplicative function of n. We set

$$T(x) = \sum_{\substack{n \leqslant x \\ (n, f(n)) = 1}} 1.$$

P. Erdös [2] proved that for $f(n) = \varphi(n)$ or $\sigma(n)$, we have

$$T(x) \sim \frac{xe^{-\gamma}}{\log\log\log x}.$$

The case $f = \varphi$ is of particular interest since $(n, \varphi(n)) = 1$ is a necessary and sufficient condition that there is only one group of order n.

In this paper we consider an additive function, namely the sum of the distinct prime factors of n. We denote this by $g(n)$, and the result is as follows.

THEOREM. *Let $T(x)$ denote the number of integers $n \leqslant x$ for which $(n, g(n)) = 1$. Then*

$$T(x) = \frac{6}{\pi^2} x + O\left(\frac{x}{(\log\log\log x)^{1/4} (\log\log\log\log x)^{3/4}}\right).$$

Thus Čebyšev's result holds in this case, as we might expect, for in general additive functions are more evenly distributed over the arithmetic progressions than multiplicative functions; moreover their prime factors, and other arithmetic properties, bear little relation to those of n itself, except when n is prime.

The first half of the proof, that is Lemmas 1–5, is elementary and deals with the large common factors of n and $g(n)$. The small prime factors of n and $g(n)$, which lead to our main term, are treated analytically. Lemma 6 is, of course, quite standard.

I should like to thank Professor Halberstam for his help and encouragement, and particularly Professor Erdös for suggesting this problem to me.

Notation. p and $\tilde{\omega}$ denote primes. $C_1, C_2, C_3, \ldots$, will denote positive constants, absolute unless written in the form $C_i(\varepsilon)$ when they depend on ε. They will be understood to be either large enough, or in some cases small enough, to ensure the validity of formulae in which they occur.

Lemma 1. *For $p \leqslant \sqrt{x}$, and all a,*

$$\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} |\mu(m)| \leqslant C_1 x\left(\frac{1}{p}+\frac{\log p}{\log x}\right).$$

Proof. Let $t = p^2$, and $\nu_t(m)$ denote the number of prime factors of m which exceed t. Set

$$H_\nu(x, a) = \sum_{\substack{m\leqslant x\\ \nu_t(m)=\nu\\ g(m)\equiv a \bmod p}} |\mu(m)|,$$

$$K_\nu(x) = \sum_{\substack{m\leqslant x\\ \nu_t(m)=\nu}} |\mu(m)|,$$

and

$$K_\nu^{(j)}(x) = \sum_{\substack{m\leqslant x\\ j\nmid m,\ \nu_t(m)=\nu}} |\mu(m)|.$$

Then for $\nu \geqslant 1$,

$$\begin{aligned} H_\nu(x, a) &\leqslant \frac{1}{\nu}\sum_{t<\tilde{\omega}\leqslant x} H_{\nu-1}\left(\frac{x}{\tilde{\omega}}, a-\tilde{\omega}\right)\\ &\leqslant \frac{1}{\nu}\sum_{t<m\leqslant x}\sum_{h=1}^{p} H_{\nu-1}\left(\frac{x}{m}, a-h\right)\{\pi(m, p, h)-\pi(m-1, p, h)\}\\ &\leqslant \frac{1}{\nu}\sum_{h=1}^{p}\Bigg[\sum_{t<m\leqslant x-1}\pi(m, p, h)\left\{H_{\nu-1}\left(\frac{x}{m}, a-h\right)-\right.\\ &\qquad \left.-H_{\nu-1}\left(\frac{x}{m+1}, a-h\right)\right\}+\pi(x, p, h)H_{\nu-1}\left(\frac{x}{[x]}, a-h\right)\Bigg]\\ &\leqslant \frac{C_2}{p\nu}\Bigg[\sum_{t<m\leqslant x-1}\frac{m}{\log(m/p)}\left\{K_{\nu-1}\left(\frac{x}{m}\right)-K_{\nu-1}\left(\frac{x}{m+1}\right)\right\}+\\ &\qquad +\frac{x}{\log(x/p)}\cdot K_{\nu-1}\left(\frac{x}{[x]}\right)\Bigg] \end{aligned}$$

by the theorem of Brun–Titchmarsh. Since $m > p^2$, $\log(m/p) > \frac{1}{2}\log m$. (The parameter t was introduced solely for this purpose.) Hence by Čebyšev's theorem,

$$\begin{aligned} H_\nu(x, a) &\leqslant \frac{C_3}{p\nu}\Bigg[\sum_{t<m\leqslant x-1}\pi(m)\left\{K_{\nu-1}\left(\frac{x}{m}\right)-K_{\nu-1}\left(\frac{x}{m+1}\right)\right\}+\pi(x)K_{\nu-1}\left(\frac{x}{[x]}\right)\Bigg]\\ &= \frac{C_3}{p\nu}\Bigg[\sum_{t<\tilde{\omega}\leqslant x}K_{\nu-1}\left(\frac{x}{\tilde{\omega}}\right)+\pi(t)K_{\nu-1}\left(\frac{x}{t+1}\right)\Bigg]. \end{aligned}$$

Now for $\nu \geqslant 1$,

$$K_\nu(x) = \frac{1}{\nu}\sum_{t<\tilde{\omega}\leqslant x}K_{\nu-1}^{(\tilde{\omega})}\left(\frac{x}{\tilde{\omega}}\right) \geqslant \frac{1}{\nu}\sum_{t<\tilde{\omega}\leqslant x}\left(K_{\nu-1}\left(\frac{x}{\tilde{\omega}}\right)-K_{\nu-2}\left(\frac{x}{\tilde{\omega}^2}\right)\right)$$

where $K_{\nu-2}$ is to be interpreted as zero when $\nu = 1$. In this case there is no contribution to $K_0(x/\tilde{\omega})$ by numbers divisible by $\tilde{\omega}$, a prime exceeding t. Hence for $\nu \geqslant 1$,

$$H_\nu(x, a) \leqslant \frac{C_3}{p}\Bigg[K_\nu(x)+\pi(t)K_{\nu-1}\left(\frac{x}{t+1}\right)+\sum_{t<\tilde{\omega}}K_{\nu-2}\left(\frac{x}{\tilde{\omega}^2}\right)\Bigg].$$

Thus

$$\sum_{\nu=1}^{\infty}H_\nu(x, a) \leqslant \frac{C_3}{p}\left[x+\frac{\pi(t)x}{t+1}+\frac{x}{t}\right] \leqslant \frac{C_1 x}{p}.$$

Let $P_t = \prod_{t<\tilde{\omega}\leqslant x}\tilde{\omega}$. Then by Selberg's upper-bound method as developed by Richert and van Lint [5], we have

$$H_0(x, a) \leqslant \sum_{\substack{m\leqslant x\\ (m, P_t)=1}} 1 \leqslant \frac{15x}{2}\prod_{t<\tilde{\omega}\leqslant x}\left(1-\frac{1}{\tilde{\omega}}\right) \leqslant \frac{C_1 x\log p}{\log x}.$$

Finally,

$$\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} |\mu(m)| = \sum_{\nu=0}^{\infty}H_\nu(x, a) \leqslant C_1 x\left(\frac{1}{p}+\frac{\log p}{\log x}\right).$$

We note that the result of Lemma 1 is valueless unless p is quite small. For larger p we find a similar result more simply.

Lemma 2. *We have, for all $p \leqslant x$,*

$$\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} |\mu(m)| \leqslant C_7 x\frac{\log x}{p}.$$

Proof. Taking $t = 1$ in the definition of $H_\nu(x, a)$ in the previous lemma, we have

$$\nu H_\nu(x, a) \leqslant \sum_{\tilde{\omega}\leqslant x}H_{\nu-1}\left(\frac{x}{\tilde{\omega}}, a-\tilde{\omega}\right).$$

Suppose that for $r < \nu$, and all b, we can show

$$H_r(x, b) \leqslant \frac{2x}{pr!}(\log\log x + C_5)^{r-1},$$

then

$$H_\nu(x, a) \leqslant \frac{2x}{p\nu!}\sum_{\bar\omega\leqslant x}\frac{1}{\bar\omega}(\log\log x + C_5)^{\nu-2},$$

and so the result follows by induction for a suitably chosen C_5. Now

$$H_1(x, a) = \sum_{\substack{\bar\omega\leqslant x\\ \bar\omega\equiv a \bmod p}} 1 = \pi(x; p, a) \leqslant \frac{2x}{p} \quad \text{for} \quad p \leqslant x,$$

and this starts the induction. Hence

$$\sum_{\nu=1}^{\infty} H_\nu(x, a) \leqslant \frac{2x}{p}\sum_{\nu=1}^{\infty}\frac{(\log\log x + C_5)^{\nu-1}}{\nu!} \leqslant \frac{C_6 x\log x}{p}.$$

Finally

$$\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} |\mu(m)| \leqslant 1 + \sum_{\nu=1}^{\infty} H_\nu(x, a) \leqslant \frac{C_7 x\log x}{p}.$$

LEMMA 3. *For all x and $p \leqslant x^{3/5}$, and all a,*

$$\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} 1 \leqslant C_{10}x\left(\frac{\log p}{p}\right)^{1/4}.$$

Proof. By the previous lemmas, provided $p \leqslant x$,

$$\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} |\mu(m)| \leqslant C_8 x\left(\frac{\log p}{p}\right)^{1/2}$$

on applying Lemma 1 if $p\log p \leqslant \log^2 x$, and Lemma 2 otherwise. Thus

$$\begin{aligned}\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} |\mu(m)| m^{-1/4} &= \frac{1}{4}\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} |\mu(m)|\int_m^x t^{-5/4}dt + x^{-1/4}\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} |\mu(m)| \\ &\leqslant \frac{1}{4}\int_1^x t^{-5/4}\sum_{\substack{m\leqslant t\\ g(m)\equiv a \bmod p}} |\mu(m)|\,dt + C_8 x^{3/4}\left(\frac{\log p}{p}\right)^{1/2} \\ &\leqslant \frac{1}{4}\int_1^p t^{-1/4}dt + \frac{1}{4}C_8\int_p^x t^{-1/4}\left(\frac{\log p}{p}\right)^{1/2}dt + \\ &\quad + C_8 x^{3/4}\left(\frac{\log p}{p}\right)^{1/2} \\ &\leqslant \frac{1}{3}p^{3/4} + 2C_8 x^{3/4}\left(\frac{\log p}{p}\right)^{1/2} \leqslant C_9 x^{3/4}\left(\frac{\log p}{p}\right)^{1/2}\end{aligned}$$

if $p \leqslant x^{3/5}$. Next

$$\begin{aligned}\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} m^{-9/16} &\leqslant \sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} \frac{|\mu(m)|}{m^{9/16}}\prod_{\bar\omega|m}\left(1-\frac{1}{\bar\omega^{9/16}}\right)^{-1} \\ &\leqslant \zeta\left(\frac{9}{8}\right)\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} \frac{|\mu(m)|\sigma_{9/16}(m)}{m^{9/8}}.\end{aligned}$$

By the Cauchy–Schwarz inequality we deduce

$$\Big(\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} m^{-9/16}\Big)^2 \leqslant \zeta^2\left(\frac{9}{8}\right)\Big(\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} |\mu(m)| m^{-1/4}\Big)\left(\sum_{m\leqslant x}\frac{\sigma_{9/16}^2(m)}{m^2}\right).$$

We have

$$\sum_{m\leqslant x}\frac{\sigma_{9/16}^2(m)}{m^2} \sim C_{11}x^{1/8}$$

either by straightforward calculation, or from an identity of Ramanujan [4],

$$\sum_{m=1}^{\infty}\frac{\sigma_a(m)\sigma_b(m)}{m^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}.$$

Thus provided $p \leqslant x^{3/5}$, we have

$$\Big(\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} m^{-9/16}\Big)^2 \leqslant C_{12}x^{7/8}\left(\frac{\log p}{p}\right)^{1/2},$$

and finally,

$$\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} 1 \leqslant x^{9/16}\sum_{\substack{m\leqslant x\\ g(m)\equiv a \bmod p}} m^{-9/16} \leqslant C_{10}x\left(\frac{\log p}{p}\right)^{1/4}.$$

This completes the proof.

LEMMA 4. *For $x^{3/5} < p \leqslant x$, we have*

$$\sum_{\substack{n\leqslant x\\ p|g(n)}} 1 \leqslant C_{13}\left(\frac{x\log x}{p}\right)^2.$$

Proof. Except in the case $n = 1$, $p\,|\,g(n)$ implies $g(n) \geqslant p$. Since n has at most $2\log x$ prime factors, it must be divisible by a prime $P \geqslant p/2\log x$. It follows from the relation

$$g(n) = P + \sum_{\substack{\bar\omega|n\\ \bar\omega\neq P}}\bar\omega \equiv 0 \bmod p$$

that

$$P \equiv -h \bmod p, \quad 0 \leqslant h \leqslant \frac{2x\log x}{p},$$

since

$$\sum_{\tilde{\omega}} \tilde{\omega} \leqslant \prod_{\tilde{\omega}} \tilde{\omega},$$

each $\tilde{\omega}$ being greater than or equal to 2. Thus n has a prime factor, and hence a divisor, in one of these residue classes $\bmod p$, and

$$\sum_{\substack{p|g(n)\\ n\leqslant x}} 1 \leqslant \sum_{n\leqslant x}\sum_{h}\sum_{\substack{d|n\\ d\equiv -h \bmod p}} 1 \leqslant 2x\sum_{h}\sum_{\substack{d\leqslant x\\ d\equiv -h\bmod p}} \frac{1}{d} \leqslant \frac{C_{13}x^2\log^2 x}{p^2}.$$

This completes the proof.

COROLLARY. *For $p \leqslant x$ we have*

$$\sum_{\substack{n\leqslant x\\ p|g(n)}} 1 \leqslant C_{10}x\left(\frac{\log p}{p}\right)^{1/4}.$$

LEMMA 5. *For all $\xi > 0$,*

$$\sum_{\substack{n\leqslant x\\ \exists p|(n,g(n))\\ p>\xi}} 1 \leqslant \frac{C_{14}x}{\xi^{1/4}(\log\xi)^{3/4}} + \pi(x).$$

Proof. We have

$$\sum_{\substack{n\leqslant x\\ \exists p|(n,g(n))\\ p>\xi}} 1 \leqslant \sum_{\xi<p\leqslant x}\sum_{\substack{m\leqslant x/p\\ p|g(m)}} 1.$$

It is clear that $g(m) \leqslant m$ and so when $p > \sqrt{x}$ the inner sum on the right is at most 1. Thus the sum on the left does not exceed

$$\sum_{\xi<p\leqslant\sqrt{x}}\sum_{\substack{m\leqslant x/p\\ p|g(m)}} 1 + \pi(x) \leqslant \sum_{\xi<p\leqslant\sqrt{x}} C_{10}\frac{x(\log p)^{1/4}}{p^{5/4}} + \pi(x)$$

$$\leqslant \frac{C_{14}x}{\xi^{1/4}(\log\xi)^{3/4}} + \pi(x).$$

This completes the proof.

LEMMA 6. *For all characters χ to modulus q we have*

$$|\operatorname{Log} L(s,\chi)| \leqslant \frac{6C_{16}\log q}{\log(|t|+2)} + C_{18}\log\log\{q(|t|+4)\}$$

in the region

$$\sigma \geqslant 1 - \frac{C_{15}}{2\log\{q(|t|+4)\}},\qquad |t| \geqslant 2.$$

For $\chi \neq \chi_0$ we have

$$|\operatorname{Log} L(s,\chi)| \leqslant C_{20}(\varepsilon)\log q$$

in the region

$$\sigma \geqslant 1 - C_{19}(\varepsilon)q^{-\varepsilon},\qquad |t| < 2.$$

For $\frac{1}{2} < \sigma < 1$, $|t| < 2$, we have

$$|\operatorname{Log} L(s,\chi_0)| \leqslant C_{21}\left(\log q + \log\frac{1}{1-\sigma}\right).$$

Proof. We use the following information about $L(s,\chi)$ which may be found in Prachar [3]. For

$$\sigma \geqslant \max\left(\frac{1}{2},\, 1 - \frac{C_{15}}{\log\{q(|t|+3)\}}\right)$$

we have $L(s,\chi) \neq 0$, except for a possible Siegel zero on the real line, when χ is (at most) one of the real characters. For $|t| \geqslant 2$, we have, in the same region,

$$|L(s,\chi)| = O\left(q^{C_{15}/\log|t|}\log\{q|t|\}\right)$$

while for $\sigma \geqslant \frac{1}{2}$, $|t| \leqslant 11$, $|L(s,\chi)| = O(q^{1/2})$ provided $\chi \neq \chi_0$. Thus if $C_{16} = \max(3C_{15}, \frac{1}{2}\log 5)$,

$$|L(s,\chi)| = O\left(q^{C_{16}/\log(|t|+3)}\log\{q(|t|+2)\}\right)$$

for all t, except in the case $\chi = \chi_0$, $|t| < 2$. We next apply the Borel–Carathéodory theorem, first when $|t| \geqslant 2$.

Let $s_0 = s + r$ have real part $\sigma_0 > 1$, let $R > r$ and let $A(R)$ denote the maximum of $\log|L(s,\chi)|$ on the circle $|z - s_0| = R$. If $R \leqslant 1$, this circle is within the zero-free region of $L(s,\chi)$ if we set

$$R = (\sigma_0 - 1) + \frac{C_{15}}{\log\{q(|t|+4)\}}.$$

Hence, by the theorem,

$$|\operatorname{Log} L(s,\chi)| \leqslant \frac{2r}{R-r}A(R) + C_{17}\frac{R+r}{R-r}\log\frac{1}{\sigma_0-1}.$$

We set

$$\sigma_0 - 1 = \frac{C_{15}}{\log\{q(|t|+4)\}}.$$

Then for

$$\mathbf{R}s = \sigma \geqslant 1 - \frac{C_{15}}{2\log\{q(|t|+4)\}},$$

$$|\operatorname{Log} L(s,\chi)| \leqslant \frac{6C_{16}\log q}{\log(|t|+2)} + 6\log\log\{q(|t|+3)\} + {} + 7C_{17}\log\left(\frac{\log\{q(|t|+4)\}}{C_{15}}\right)$$

$$\leqslant \frac{6C_{16}\log q}{\log(|t|+2)} + C_{18}\log\log\{q(|t|+4)\}.$$

This estimate is equally true for $|t|<2$ for all χ except χ_0 and the real character (if it exists) for which $L(s,\chi)$ has a Siegel zero; denoting this character by χ_1 we find that for $|t|<2,\ \sigma<1$,

$$|\mathrm{Log}\,L(s,\chi_0)| = O(\log q)+O\left(\log\frac{1}{1-\sigma}\right)$$

and

$$|\mathrm{Log}\,L(s,\chi_1)| = O(\log q)+O\left(\log\frac{1}{|\beta-\sigma|}\right),$$

where β denotes the Siegel zero of $L(s,\chi_1)$. By Siegel's theorem, $1-\beta > 2C_{19}(\varepsilon)q^{-\varepsilon}$, and so if $\sigma>1-C_{19}(\varepsilon)q^{-\varepsilon}$,

$$|L(s,\chi_1)| = O(\log q)\quad \text{for}\quad |t|<2.$$

This completes the proof.

LEMMA 7. *For any $q>1$, let $f_q(n) = g(nq)-g(q)$. Then for all a, and all q satisfying*

$$4C_{30}\sqrt{q}\log q\leqslant \log\log x,$$

we have

$$\Bigg|\sum_{\substack{n\leqslant x\\ f_q(n)\equiv a \bmod q}} 1-\frac{x}{q}\Bigg|\leqslant \frac{C_{32}x}{(\log x)^{1/8}}.$$

Proof. For any $q\geqslant 1$, we note that $f(n) = f_q(n) = g(nq)-g(q)$ is additive. Set

$$F_q(s,t) = \sum_{n=1}^{\infty}\frac{e^{2i\pi f(n)t}}{n^s} = \prod_{p\nmid q}\left(1+\frac{e^{2i\pi pt}}{p^s-1}\right)\prod_{p\mid q}\left(1+\frac{1}{p^s-1}\right).$$

Plainly,

$$\frac{1}{q}\sum_{l=1}^{q}e^{-2i\pi al/q}F_q(s,l/q) = \sum_{\substack{n=1\\ f_q(n)\equiv a \bmod q}}^{\infty} n^{-s}$$

and so

$$\sideset{}{^*}\sum_{\substack{n\leqslant x\\ f_q(n)\equiv a \bmod q}} 1 = \frac{1}{2i\pi q}\sum_{l=1}^{q}e^{-2i\pi al/q}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s}F_q(s,l/q)\,ds,$$

where the * denotes that if x is an integer, the last term in the sum on the left is halved. Thus

$$\sideset{}{^*}\sum_{\substack{n\leqslant x\\ f_q(n)\equiv a \bmod q}} 1 = \frac{1}{q}\sideset{}{^*}\sum_{n\leqslant x}1+\frac{1}{2i\pi q}\sum_{l=1}^{q-1}e^{-2i\pi al/q}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s}F_q(s,l/q)\,ds.$$

Hence

$$\Bigg|\sum_{\substack{n\leqslant x\\ f_q(n)\equiv a \bmod q}} 1-x/q\Bigg|\leqslant \max_{1\leqslant l<q}|W(x;l/q)|+1$$

where

$$W(x;l/q) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s}F_q(s,l/q)\,ds.$$

It is plain that $F_q(s,l/q) = F_q(s,l_1/q_1)$ where $(l_1,q_1)=1$ and $l_1/q_1 = l/q$. In fact our estimate of W is weakest when q_1 is large, and so we may assume that $(l,q)=1$, without loss of generality. Now

$$F_q(s,t) = F(s,t)\prod_{p\mid q}\left(\frac{p^s}{p^s-1+e^{2i\pi pt}}\right) = F(s,t)\hat{F}_q(s,t)$$

say, where

$$F(s,t) = \prod_{p}\left(1+\frac{e^{2i\pi pt}}{p^s-1}\right).$$

From this we see that

$$F(s,l/q) = \prod_{p}\left(1+\frac{e^{2i\pi pl/q}}{p^s}\right)\prod_{p}\left(1+\frac{e^{2i\pi pl/q}}{(p^s-1)(p^s+e^{2i\pi pl/q})}\right)$$

where the second factor is regular and bounded independently of q for $\boldsymbol{R}s\geqslant\frac{1}{2}+\varepsilon$. Setting

$$G(s,l/q) = \prod_{\substack{a=1\\ (a,q)=1}}^{q}\ \prod_{p\equiv a \bmod q}\left(1+\frac{e^{2i\pi al/q}}{p^s}\right)$$

we find that

$$\frac{F(s,l/q)}{G(s,l/q)} = \prod_{p\mid q}\left(1+\frac{e^{2i\pi pl/q}}{p^s}\right)\prod_{p}\left(1+\frac{e^{2i\pi pl/q}}{(p^s-1)(p^s+e^{2i\pi pl/q})}\right).$$

Now

$$\begin{aligned}\mathrm{Log}\,G(s,l/q) &= \sum_{\substack{a=1\\ (a,q)=1}}^{q}e^{2i\pi al/q}\sum_{p\equiv a \bmod q}p^{-s}+H_1(s,l/q)\\ &= \frac{1}{\varphi(q)}\sum_{\chi}\sum_{a}\bar{\chi}(a)e^{2i\pi al/q}\sum_{p}\frac{\chi(p)}{p^s}+H_1(s,l/q)\\ &= \frac{1}{\varphi(q)}\sum_{\chi}\chi(e)\tau_\chi\sum_{p}\left\{-\log\left(1-\frac{\chi(p)}{p^s}\right)\right\}+H_2(s,l/q)\end{aligned}$$

where $H_1(s, l/q)$ and $H_2(s, l/q)$ are regular and bounded, independently of q for $\mathbf{R}s \geqslant \frac{1}{2}+\varepsilon$. We write

$$|H_2(s, l/q)| \leqslant H_2, \qquad \mathbf{R}s \geqslant \tfrac{3}{4}.$$

As usual,

$$\tau_\chi = \sum_{b=1}^{q} \bar{\chi}(b)\, e^{2i\pi b/q}$$

so that

$$|\tau_\chi| \leqslant \sqrt{q}.$$

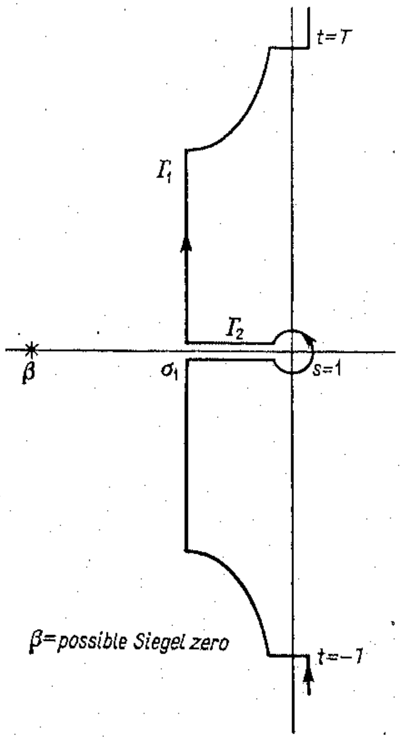
Thus

$$\operatorname{Log} G(s, l/q) = \frac{\mu(q)}{\varphi(q)} \operatorname{Log} L(s, \chi_0) + \frac{1}{\varphi(q)} \sum_{\chi \neq \chi_0} \chi(l)\, \tau_\chi \operatorname{Log} L(s, \chi) + H_2(s, l/q).$$

We conclude that $G(s, l/q)$ and hence $F_q(s, l/q)$ is regular within the zero-free region common to all the L-functions to modulus q, with the exception, when q is squarefree, of a branch point or algebraical singularity at $s = 1$. In fact we shall only be interested in the squarefree case. We set

$$c = 1 + 1/\log x$$

and move the contour of integration to $\Gamma_1 \cup \Gamma_2$, as shown in the figure.



Γ_2 is a lacet around $s = 1$, while on Γ_1 we have $\sigma = c$ for $|t| > T$. For $|t| < T$ set

$$\sigma = 1 - \frac{C_{22}}{M(q, t)}$$

where

$$M(q, t) = \max\{q^{1/4}, \log\{q(|t|+4)\}\}$$

and

$$C_{22} = \min\left(\frac{C_{15}}{2}, C_{19}(\tfrac{1}{4})\right).$$

The contour is completed by horizontal lines at $|t| = T$. Set

$$W(x, l/q) = W_1(x, l/q) + W_2(x, l/q)$$

where

$$W_2(x, l/q) = \frac{1}{2i\pi} \int_{\Gamma_2} \frac{x^s}{s} F_q(s, l/q)\, ds$$

and

$$Z(x, l/q) = \int_0^x W_1(y, l/q) \frac{dy}{y} = \frac{1}{2i\pi} \int_{\Gamma_1} \frac{x^s}{s^2} F_q(s, l/q)\, ds.$$

We estimate the integral over Γ_1 first. Note that for $\varepsilon > 0$,

$$|F_q(s, l/q)/G(s, l/q)| \leqslant C_{23}(\varepsilon) q^\varepsilon$$

uniformly in q, for $\mathbf{R}s \geqslant \frac{3}{4}$. We set

$$Z = Z_1 + Z_2 + Z_3 + Z_4 = \frac{1}{2i\pi}\left(\int_{\substack{|t|<2\\ s \in \Gamma_1}} + \int_{2 \leqslant |t| < T} + \int_{|t|=T} + \int_{|t|>T} \frac{x^s}{s^2} F_q(s, l/q)\, ds\right).$$

For $|t| < 2$, $s \in \Gamma_1$ we have by Lemma 6 that

$$|\operatorname{Log} G(s, l/q)| \leqslant \frac{C_{21}}{\varphi(q)}\left(\log q + \log \frac{q^{1/4}}{C_{22}}\right) + C_{20}\left(\frac{1}{4}\right)\sqrt{q}\log q + H_2$$
$$\leqslant C_{24}\sqrt{q}\log q.$$

Hence

$$Z_1 \leqslant \int_0^2 x^\sigma C_{23}(\varepsilon)\, q^\varepsilon \exp(C_{24}\sqrt{q}\log q)\, dt$$
$$\leqslant x^{1-C_{22}/M(q,2)} \exp(C_{25}\sqrt{q}\log q)$$

on taking $\varepsilon = 1$ and absorbing this factor into the exponential. For $2 \leqslant |t| < T$, we have

$$|\operatorname{Log} G(s, l/q)| \leqslant 6C_{16}\sqrt{q}\log q + C_{18}\sqrt{q}\log\log\{q(T+4)\} + H_2;$$

thus

$$Z_2 \leqslant \int_2^T \frac{x^\sigma}{t^2} C_{23}(1)\, q \exp\{6C_{16}\sqrt{q}\log q + C_{18}\sqrt{q}\log\log\{q(T+4)\} + H_2\}\, dt$$

$$\leqslant x^{1-C_{22}/M(q,T)} \exp\{C_{25}\sqrt{q}\log q + C_{18}\sqrt{q}\log\log\{q(T+4)\}\}.$$

Plainly

$$Z_3 \leqslant \frac{x^c}{T^2} \exp\{C_{25}\sqrt{q}\log q + C_{18}\sqrt{q}\log\log\{q(T+4)\}\},$$

the range of integration not exceeding 1. Finally, by the definition of $F_q(s, l/q)$,

$$Z_4 \leqslant \frac{1}{\pi} \int_T^\infty \frac{x^c}{t^2} \zeta(c)\, dt \leqslant \frac{x\zeta(c)}{T}.$$

We select

$$\log^2 T = \tfrac{1}{2} C_{22} \log x;$$

thus for the range of values of q under consideration, for $x > x_0$, we have

$$M(q, T) = \log\{q(T+4)\} < 2\log T.$$

Hence

$$Z(x, l/q) \leqslant 2x \exp\left\{C_{25}\sqrt{q}\log q + C_{18}\sqrt{q}\log(2\log T) - \frac{C_{22}\log x}{2\log T}\right\} +$$

$$+ x\exp\left\{C_{25}\sqrt{q}\log q - \frac{C_{22}\log x}{q^{1/4}}\right\} + \frac{2x\log x}{T}$$

$$\leqslant x\exp\{-C_{26}\sqrt{\log x}\}.$$

Therefore for all x and for the given q, we have

$$Z(x, l/q) \leqslant x\exp\{-C_{27}\sqrt{\log x}\}.$$

It remains to estimate the integral on Γ_2. Now

$$F_q(s, l/q) = (s-1)^{-\mu(q)/\varphi(q)} F_q^*(s, l/q)$$

where F_q^* is regular in the neighborhood of $s = 1$. If we allow the radius of the loop around $s = 1$ to tend to zero together with the width of the lacet Γ_2 we obtain in the limit,

$$W_2(x, l/q) = \frac{1 - e^{-2i\pi\mu(q)/\varphi(q)}}{2i\pi} \int_1^{\sigma_1} \frac{x^s}{s} (s-1)^{-\mu(q)/\varphi(q)} F_q^*(s, l/q)\, ds$$

where

$$\sigma_1 = 1 - \frac{C_{22}}{q^{1/4}}.$$

In the case $q = 2$, $F_q(s, l/q)$ has a zero at $s = 1$ and the contour of integration is simply Γ_1 continued across the real line. Next,

$$F_q^*(s, l/q) = (s-1)^{\mu(q)/\varphi(q)} \prod_{\chi \neq \chi_0} \{L(s, \chi)\}^{\tau_\chi \chi(l)/\varphi(q)} \{\zeta(s)\}^{\mu(q)/\varphi(q)} \theta(s)$$

where $\theta(s)$ is bounded by

$$C_{29} \prod_{p|q} \left(1 + \frac{1}{p^{\sigma_1}}\right)^3.$$

Since $(s-1)\zeta(s)$ is bounded on Γ_2 we have

$$F_q^*(s, l/q) \leqslant \exp\{C_{30}\sqrt{q}\log q\},$$

on Γ_2. Thus

$$|W_2(x; l/q)| \leqslant x\exp\{C_{30}\sqrt{q}\log q\} \int_0^{1-\sigma_1} u^{-\mu(q)/\varphi(q)} e^{-u\log x}\, du$$

$$\leqslant \frac{x\exp\{C_{30}\sqrt{q}\log q\}}{(\log x)^{1-\mu(q)/\varphi(q)}} \Gamma\left(1 - \frac{\mu(q)}{\varphi(q)}\right)$$

$$\leqslant \frac{C_{31} x}{(\log x)^{1/2}} \exp\{C_{30}\sqrt{q}\log q\}.$$

Thus if $4C_{30}\sqrt{q}\log q \leqslant \log\log x$,

$$|W_2(x; l/q)| \leqslant \frac{C_{31} x}{(\log x)^{1/4}}.$$

It will be seen that the condition on q is much more strict than that involved with W_1, to achieve a weaker result. For this reason no improvement is obtained by taking the contour of integration further to the left, and it is obviously convenient to keep it to the right of the possible Siegel zero, as we have done.

With this restriction on q, we have

$$\int_0^x W(y; l/q) \frac{dy}{y} = Z(x; l/q) + \int_0^x W_2(y; l/q) \frac{dy}{y} \leqslant \frac{2C_{31} x}{(\log x)^{1/4}}.$$

Now

$$W(y; l/q) = \sum_{n \leqslant y}^{*} e^{2i\pi f_q(n) l/q}$$

and so

$$|W(y_1; l/q) - W(y_2; l/q)| \leqslant |y_1 - y_2|.$$

Suppose that

$$\mathbf{R} W(x; l/q) = Mx,$$

where without loss of generality we take $M \geqslant 0$. Then for $y \geqslant x - \frac{1}{2}Mx$,

$$\mathbf{R} W(y, l/q) \geqslant \tfrac{1}{2} Mx.$$

Thus

$$\mathbf{R}\left\{\int_0^x W(y;\, l/q)\frac{dy}{y} - \int_0^{x-\frac{1}{4}Mx} W(y;\, l/q)\frac{dy}{y}\right\} \geqslant \tfrac{1}{4}M^2x.$$

But the left-hand side does not exceed $4C_{31}x(\log x)^{-1/4}$. Hence

$$M \leqslant \frac{4\sqrt{C_{31}}}{(\log x)^{1/8}}.$$

The same argument applies to the imaginary part, and we deduce that

$$|W(x;\, l/q)| \leqslant \frac{C_{32}x}{2(\log x)^{1/8}}.$$

This completes the proof of the lemma.

Proof of the theorem. Let

$$K(n,\omega) = \prod_{\substack{p\leqslant\omega \\ p\mid(n,g(n))}} p.$$

Then

$$T(x) = \sum_{\substack{n\leqslant x \\ K(n,\omega)=1}} 1 + \theta \sum_{\substack{n\leqslant x \\ \exists p>\omega,\, p\mid(n,g(n))}} 1$$

with $|\theta| \leqslant 1$. Thus

$$T(x) - \theta \sum_{\substack{n\leqslant x \\ \exists p>\omega,\, p\mid(n,g(n))}} 1 = \sum_{n\leqslant x}\sum_{q\mid K(n,\omega)} \mu(q) = \sum_{q\mid\Pi}\mu(q) \sum_{\substack{n\leqslant x \\ q\mid n,\, q\mid g(n)}} 1$$

$$= \sum_{q\mid\Pi}\mu(q) \sum_{\substack{m\leqslant x/q \\ g(mq)\equiv 0 \bmod q}} 1 = \sum_{q\mid\Pi}\mu(q) \sum_{\substack{m\leqslant x/q \\ f_q(m)\equiv -g(q) \bmod q}} 1,$$

where $\Pi = \prod_{p\leqslant\omega} p$. Provided $\log\log(x/\Pi) \geqslant 4C_{30}\Pi^{1/2}\log\Pi$, we have, by Lemmas 5 and 7, that

$$T(x) = \sum_{q\mid\Pi}\mu(q)\left\{\frac{x}{q^2} + O\left(\frac{x}{q(\log x)^{1/8}}\right)\right\} + O\left(\frac{x}{\omega^{1/4}(\log\omega)^{3/4}} + \frac{x}{\log x}\right)$$

$$= x\prod_{p\leqslant\omega}\left(1-\frac{1}{p^2}\right) + O\left(\frac{x}{(\log x)^{1/8}}\prod_{p\leqslant\omega}\left(1+\frac{1}{p}\right)\right) + O\left(\frac{x}{\omega^{1/4}(\log\omega)^{3/4}}\right).$$

Now

$$\prod_{p\leqslant\omega} p \leqslant \exp(C_{33}\omega),$$

and so the condition on Π is satisfied if we set

$$\omega = C_{34}\log\log\log x.$$

We substitute this into the expression for $T(x)$ above, and this completes the proof.

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