is a contradiction for $m = |b_k|$. Thus we get for some rational $q$: $\beta_i = qa_i$
$(1 \leq i \leq r)$. If $a_i = 0 (1 \leq i \leq r)$ then Theorem 2 holds with $k = t_k$. If for
some $i$, $a_i \neq 0$ then (11) with $m = |a_i|$ implies $q$ integer and $t_{q|t} = q \mod 2$.
Hence $n = a_i$ and Theorem 2 holds with $k = q$.

References

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On the probability that $n$ and $f(n)$ are relatively prime

by

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It is a well-known theorem of Čebyšev that if $n$ and $m$ are randomly
chosen positive integers, then $(n, m) = 1$ with probability $6/\pi^2$. One
can expect (as have noted) that $m = f(n)$ is a function of $n$, provided
that $f(n)$ does not preserve arithmetic properties of $n$. Erdős and
Lorenzo [1] proved that this is so, in the case $f(n) = \lfloor f(n) \rfloor$, where $f(x)$
is a smooth function satisfying certain (weak) conditions.

The case $f(x) = \lfloor x \rfloor$ was considered by G. L. Watson [6]. For all $a$,
the positive integers $n$ for which $(n, f(n)) = 1$ have a density, and in
particular, for irrational $x$ this is $6/\pi^2$.

Suppose now that $f(n)$ is a multiplicative function of $n$. We set

$$T(n) = \sum_{\substack{n \leq x \atop (n, m) = 1}} 1.$$

P. Erdős [2] proved that for $f(n) = \varphi(n)$ or $\sigma(n)$, we have

$$T(n) \sim \frac{ax}{\log \log \log x}.$$

The case $f = \varphi$ is of particular interest since $(n, \varphi(n)) = 1$ is a necessary
and sufficient condition that there is only one group of order $n$.

In this paper we consider an additive function, namely the sum
of the distinct prime factors of $n$. We denote this by $g(n)$, and the result
is as follows.

Theorem. Let $T(n)$ denote the number of integers $n \leq x$ for which
$(n, g(n)) = 1$. Then

$$T(n) = \frac{6}{\pi^2} x + O\left(\frac{x}{(\log \log \log x)^2 (\log \log \log \log x)^2}\right).$$

Thus Čebyšev's result holds in this case, as we might expect, for in
general additive functions are more evenly distributed over the arithmetic
progressions than multiplicative functions; moreover their prime factors,
and other arithmetic properties, bear little relation to those of $n$ itself,
except when $n$ is prime.

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The first half of the proof, that is, Lemmas 1–5, is elementary and deals with the large common factors of $n$ and $g(n)$. The small prime factors of $n$ and $g(n)$, which lead to our main term, are treated analytically. Lemma 6 is, of course, quite standard.

I should like to thank Professor Halberstam for his help and encouragement, and particularly Professor Erdös for suggesting this problem to me.

Notation. $p$ and $\mathfrak{p}$ denote primes, $C_1, C_2, C_3, \ldots$, will denote positive constants, absolute unless written in the form $C_i(x)$ when they depend on $x$. They will be understood to be either large enough, or in some cases small enough, to ensure the validity of formulae in which they occur.

**Lemma 1.** For $p \leqslant \sqrt{x}$, and all $\alpha$,

$$\sum_{\mu(\mathfrak{m}) \equiv \alpha \pmod{p}} |\mu(\mathfrak{m})| \leqslant C \cdot \frac{1}{p} + \frac{\log p}{\log \alpha}.$$

**Proof.** Let $t = p^2$, and $\gamma_r(m)$ denote the number of prime factors of $m$ which exceed $t$. Set

$$H_r(x, \alpha) = \sum_{\mu(\mathfrak{m}) \equiv \alpha \pmod{p}} |\mu(\mathfrak{m})|,$$

$$K_r(\alpha) = \sum_{\gamma_r(m) \equiv \alpha \pmod{p}} |\mu(\mathfrak{m})|,$$

and

$$K_r^{(0)}(\alpha) = \sum_{\gamma_r(m) \equiv \alpha \pmod{p}} |\mu(\mathfrak{m})|.$$

Then for $\nu \geqslant 1$,

$$H_r(x, \alpha) \leqslant \frac{1}{\nu} \sum_{\nu \geqslant 1} H_{\nu-1}\left(\frac{x}{\mathfrak{p}}, a - \mathfrak{p}\right)$$

$$\leqslant \frac{1}{\nu} \sum_{\nu \geqslant 1} \sum_{m \geqslant 1} \frac{1}{\nu} K_{\nu-1}\left(\frac{x}{m}, a - h\right) \{\pi(m, \nu, p) - \pi(m-1, \nu, p)\}$$

$$= \frac{1}{\nu} \sum_{\nu \geqslant 1} \sum_{m \geqslant 1} \pi(m, \nu, p) H_{\nu-1}\left(\frac{x}{m}, a - h\right) - \pi(x, p) H_{\nu-1}\left(\frac{x}{m}, a - h\right)$$

$$\leqslant C \frac{1}{\nu} \sum_{\nu \geqslant 1} \log(m \nu/p) K_{\nu-1}\left(\frac{x}{m}, a - h\right) + \frac{x}{\log(x/p)} K_{\nu-1}\left(\frac{x}{m}, a - h\right)$$

by the theorem of Brun–Titchmarsh. Since $m > p^2$, $\log(m \nu/p) > \frac{1}{2} \log m$. (The parameter $t$ was introduced solely for this purpose.) Hence by Chebyshev's theorem,

$$H_r(x, \alpha) \leqslant C \frac{1}{p \nu} \sum_{\nu \geqslant 1} \pi(m) K_{\nu-1}\left(\frac{x}{m}, a - h\right) - K_{\nu-1}\left(\frac{x}{m}, a - h\right) \leqslant C \frac{1}{p \nu} \sum_{\nu \geqslant 1} K_{\nu-1}\left(\frac{x}{\alpha^2}\right) + \pi(t) K_{\nu-1}\left(\frac{x}{t+1}\right).$$

Now for $\nu \geqslant 1$,

$$K_r(\alpha) = \frac{1}{\nu} \sum_{\nu \geqslant 1} K_{\nu-1}\left(\frac{x}{\alpha}\right) \geqslant \frac{1}{\nu} \sum_{\nu \geqslant 1} \{K_{\nu-1}\left(\frac{x}{\alpha}\right) - K_{\nu-2}\left(\frac{x}{\alpha^2}\right)\}$$

where $K_{\nu-1}$ is to be interpreted as zero when $\nu = 1$. In this case there is no contribution to $K_r(x/\alpha)$ by numbers divisible by $\alpha$, a prime exceeding $t$. Hence for $\nu \geqslant 1$,

$$H_r(x, \alpha) \leqslant C \frac{1}{p \nu} \left[K_r(x) + \pi(t) K_{\nu-1}\left(\frac{x}{t+1}\right) + \sum_{\nu \geqslant 1} K_{\nu-1}\left(\frac{x}{\alpha}\right)\right].$$

Thus

$$\sum_{\nu \geqslant 1} H_r(x, \alpha) \leqslant C \frac{1}{p \nu} \left[x + \frac{\pi(t) x}{t+1} + \frac{x}{\alpha}\right] \leqslant C \frac{x}{p}.$$

Let $P_t = \prod \mathfrak{p}_t$. Then by Selberg's upper-bound method as developed by Richert and van Lint [5], we have

$$H_0(x, a) \leqslant \sum_{\nu \geqslant 1} \frac{1}{2} \frac{15x}{2} \prod_{\nu \geqslant 1} \left(1 - \frac{1}{\mathfrak{p}}\right) \leqslant C \frac{x \log p}{\log x}.$$

Finally,

$$\sum_{\nu \geqslant 1} |\mu(\mathfrak{m})| = \sum_{\nu \geqslant 1} H_r(x, \alpha) \leqslant C \frac{x}{p} + \frac{\log p}{\log x}.$$

We note that the result of Lemma 1 is valueless unless $p$ is quite small. For larger $p$ we find a similar result more simply.

**Lemma 2.** We have, for all $p \leqslant \sqrt{x}$,

$$\sum_{\nu \geqslant 1} |\mu(\mathfrak{m})| \leqslant C \frac{x}{p} - \frac{\log p}{\log x}.$$

**Proof.** Taking $t = 1$ in the definition of $H_r(x, \alpha)$ in the previous lemma, we have

$$\nu H_r(x, \alpha) \leqslant \sum_{\nu \geqslant 1} H_{\nu-1}\left(\frac{x}{\mathfrak{p}}, a - \mathfrak{p}\right).$$
Suppose that for \( r > r \), and all \( b \), we can show
\[
H_r(x, b) \leq \frac{2x}{pr!} (\log \log x + C_3)^{-1},
\]
then
\[
H_r(x, a) \leq \frac{2x}{pr!} \sum_{a \leq x} \left( \log \log x + C_3 \right)^{-1},
\]
and so the result follows by induction for a suitably chosen \( C_3 \). Now
\[
H_r(x, a) = \sum_{a \leq x} 1 = \pi(x; p, a) \leq \frac{2x}{p} \quad \text{for} \quad p \leq x,
\]
and this starts the induction. Hence
\[
\sum_{i=1}^{\infty} H_r(x, a) = \frac{2x}{p} \sum_{i=1}^{\infty} \left( \log \log x + C_3 \right)^{-1} \leq C_4 \log x.
\]
Finally
\[
\sum_{\frac{m \leq x}{\varphi(m) = \text{max} \{m \}} \left( \log p \right)^{1/4}.
\]
Lemma 3. For all \( x \) and \( p \leq x^{1/6} \), and all \( a \),
\[
\sum_{\frac{m \leq x}{\varphi(m) = \text{max} \{m \}}} 1 \leq C_{23} x \left( \frac{\log p}{p} \right)^{1/4}.
\]
Proof. By the previous lemmas, provided \( p \leq x \),
\[
\sum_{\frac{m \leq x}{\varphi(m) = \text{mod} \{m \}}} \left( \log p \right)^{1/4}.
\]
on applying Lemma 1 if \( p \log p \leq \log^2 x \), and Lemma 2 otherwise. Thus
\[
\sum_{\frac{m \leq x}{\varphi(m) = \text{mod} \{m \}}} |\mu(m)| \leq C_6 x \left( \frac{\log p}{p} \right)^{1/4}.
\]
Next
\[
\sum_{\frac{m \leq x}{\varphi(m) = \text{mod} \{m \}}} |\mu(m)| \leq C_6 x \left( \frac{\log p}{p} \right)^{1/4}.
\]
By the Cauchy–Schwarz inequality we deduce
\[
\left( \sum_{\frac{m \leq x}{\varphi(m) = \text{mod} \{m \}}} m^{-3/16} \right)^{2} \leq C_{12} x \left( \sum_{\frac{m \leq x}{\varphi(m) = \text{mod} \{m \}}} \left( \sum_{\frac{m \leq x}{\varphi(m) = \text{mod} \{m \}}} \right)^{3/16} m^{3/16} \right).
\]
We have
\[
\sum_{\frac{m \leq x}{\varphi(m) = \text{mod} \{m \}}} \eta(\mu(m)) \sim C_{25} x^{1/6},
\]
either by straightforward calculation, or from an identity of Ramanujan \([4]\),
\[
\sum_{m=1}^{x} a(m) b(m) = \frac{\zeta(s) \zeta(s-a) \zeta(s-b) \zeta(s-a-b)}{\zeta(2s-a-b)}.
\]
Thus provided \( p \leq x^{1/6} \), we have
\[
\left( \sum_{\frac{m \leq x}{\varphi(m) = \text{mod} \{m \}}} m^{-3/16} \right)^{2} \leq C_{12} x^{1/2} \left( \frac{\log p}{p} \right)^{1/4},
\]
and finally,
\[
\sum_{\frac{m \leq x}{\varphi(m) = \text{mod} \{m \}}} 1 \leq x^{1/4} \sum_{\frac{m \leq x}{\varphi(m) = \text{mod} \{m \}}} m^{-3/16} \leq C_{25} x \left( \frac{\log p}{p} \right)^{1/4}.
\]
This completes the proof.

Lemma 4. For \( x^{1/6} < p \leq x \), we have
\[
\sum_{\frac{m \leq x}{\varphi(m) = \text{mod} \{m \}}} 1 \leq C_{11} \left( \frac{x \log p}{p} \right)^{2}.
\]
Proof. Except in the case \( n = 1 \), \( p \mid g(n) \) implies \( g(n) \mid p \). Since \( n \) has at most \( 2 \log x \) prime factors, it must be divisible by a prime \( P \mid p/2 \log x \). It follows from the relation
\[
g(n) = F + \sum_{\frac{m \leq x}{\varphi(m) = \text{mod} \{m \}}} 0 \leq k \leq \frac{2 \log x}{p},
\]
that
\[
P = -k \text{mod } p, \quad 0 \leq k \leq \frac{2 \log x}{p}.
\]
since
\[ \sum_{\mathfrak{a}} \leq \prod_{\mathfrak{a}} \mathfrak{a}, \]
each \( \mathfrak{a} \) being greater than or equal to 2. Thus \( n \) has a prime factor, and hence a divisor, in one of these residue classes mod \( p \), and
\[ \sum_{n \leq x) \sum_{\mathfrak{a} \mid m \equiv 0 \pmod{p}} \sum_{d \leq \frac{x}{\mathfrak{a}}} \sum_{d \equiv -\mathfrak{a} \pmod{p}} 1 \leq 2x \sum_{n \leq x} \sum_{d \leq \frac{x}{n}} \frac{1}{d} \leq \frac{C_{18} x \log^2 x}{p^3}. \]
This completes the proof.

**Corollary.** For \( p \leq x \) we have
\[ \sum_{n \leq x, \mathfrak{a} \mid m \equiv 0 \pmod{p}} 1 \leq C_{18} x \left( \frac{\log x}{p} \right)^{1/4}. \]

**Lemma 5.** For all \( \xi > 0 \),
\[ \sum_{n \leq x, \mathfrak{a} \mid m \equiv 0 \pmod{p}} \frac{C_{18} x}{\xi^{1/4} \log \xi^{1/4}} + \pi(x). \]

**Proof.** We have
\[ \sum_{n \leq x} 1 \leq \sum_{n \leq x, \mathfrak{a} \mid m \equiv 0 \pmod{p}} \sum_{d \leq \frac{x}{\mathfrak{a}}} 1. \]
It is clear that \( g(m) \leq m \) and so when \( p > \sqrt{x} \) the inner sum on the right is at most 1. Thus the sum on the left does not exceed
\[ \sum_{n \leq x, \mathfrak{a} \mid m \equiv 0 \pmod{p}} \frac{C_{18} x (\log p)^{1/4}}{p^{1/4}} + \pi(x) \leq \frac{C_{18} x}{\xi^{1/4} \log \xi^{1/4}} + \pi(x). \]
This completes the proof.

**Lemma 6.** For all characters \( \chi \) to modulus \( q \) we have
\[ \left| \log L(s, \chi) \right| \leq \frac{6C_{18} \log q}{\log (|t| + 2)} + \frac{C_{18} \log \log q (|t| + 4)}{\log (|t| + 4)} \]
in the region
\[ \sigma \geq 1 - \frac{C_{18}}{2 \log \log q (|t| + 4)}, \quad |t| \geq 2. \]
For \( \chi \not\equiv 1 \) we have
\[ \left| \log L(s, \chi) \right| \leq C_{18}(s) \log q \]
in the region
\[ \sigma \geq 1 - C_{18}(s) q^{-s}, \quad |t| < 2. \]

For \( 1 < \sigma < 1, |t| < 2, \) we have
\[ \left| \log L(s, \chi) \right| \leq C_{18} \left( \log q + \log \frac{1}{1 - \sigma} \right). \]

**Proof.** We use the following information about \( L(s, \chi) \) which may be found in Prachar [3]. For
\[ \sigma \geq \max \left( \frac{1}{2}, 1 - \frac{C_{18}}{\log (|t| + 3)} \right) \]
we have \( L(s, \chi) \neq 0 \), except for a possible Siegel zero on the real line, when \( \chi \) is (at most) one of the real characters. For \( |t| \geq 2 \), we have, in the same region,
\[ \left| \log L(s, \chi) \right| = O \left( q^{C_{18} \log \log \log q} \right) \]
while for \( \sigma \geq \frac{3}{5}, |t| \leq 11, \) \( \left| \log L(s, \chi) \right| = O(q^{1/4}) \) provided \( \chi \neq \chi_0 \). Thus if
\[ C_{18} = \max (3C_{18}, \frac{1}{4} \log 5), \]
\[ \left| \log L(s, \chi) \right| = O \left( q^{C_{18} \log \log \log q} \right) \]
for all \( t \), except in the case \( \chi = \chi_0, |t| < 2 \). We next apply the Borel–Carathéodory theorem, first when \( |t| \geq 2 \).

Let \( s_t = s + r \) have real part \( c > 1 \), let \( R > r \) and let \( A(R) \) denote the maximum of \( \log |L(s, \chi)| \) on the circle \( |z - s_t| = R \). If \( R < 1 \), this circle is within the zero-free region of \( L(s, \chi) \) if we set
\[ E = (c - 1) + \frac{C_{18}}{\log \log q (|t| + 4)}. \]
Hence, by the theorem,
\[ \left| \log L(s, \chi) \right| \leq \frac{2r}{E - r} \log A(R) + \frac{R + r}{E - r} \log \frac{1}{c - 1}. \]
We set
\[ c_0 = 1 + \frac{C_{18}}{\log \log q (|t| + 4)} \]
Then for
\[ R = c_0 - 1 - \frac{C_{18}}{2 \log \log q (|t| + 4)}, \]
\[ \left| \log L(s, \chi) \right| \leq \frac{6C_{18} \log q}{\log (|t| + 2)} + 6C_{18} \log \log q (|t| + 3) + \frac{C_{18} \log \log q (|t| + 4)}{C_{18}} \]
\[ \leq \frac{6C_{18} \log q}{\log (|t| + 2)} + C_{18} \log \log q (|t| + 4). \]
This estimate is equally true for \(|t| < 2\) for all \(\chi\) except \(\chi_0\) and the real character (if it exists) for which \(L(s, \chi)\) has a Siegel zero; denoting this character by \(\chi_s\), we find that for \(|t| < 2, \sigma < 1,\)

\[
|\log L(s, \chi_0)| = O(\log q) + O\left(\log \frac{1}{1 - \sigma}\right)
\]

and

\[
|\log L(s, \chi_s)| = O(\log q) + O\left(\log \frac{1}{|\beta - \sigma|}\right),
\]

where \(\beta\) denotes the Siegel zero of \(L(s, \chi_s)\). By Siegel's theorem, \(1 - \beta > 2C_{\nu}(\sigma)q^{-\tau}\), and so if \(\sigma > 1 - C_{\nu}(\sigma)q^{-\tau}\),

\[
|L(s, \chi_s)| = O(\log q) \quad \text{for} \quad |t| < 2.
\]

This completes the proof.

**Lemma 7.** For any \(q > 1\), let \(f_q(n) = g(qn) - g(q)\). Then for all \(a, q, l\), and all \(q\) satisfying

\[
4C_{\nu}q \log q \ll \log \log q,
\]

we have

\[
\left| \sum_{\substack{n \leq x \leq \varphi(n) \equiv a \mod q}} \frac{\varphi(n)}{n} \right| \leq \frac{C_{\nu}q}{(\log q)^{1/8}}.
\]

**Proof.** For any \(q \geq 1\), we note that \(f(n) = f_q(n) = g(qn) - g(q)\) is additive. Set

\[
F_q(s, t) = \sum_{n=1}^{\infty} \frac{e^{2\pi i q n t}}{n^s} = \prod_{p \mid q} \left(1 + \frac{e^{2\pi i q p t}}{p^s - 1}\right) \prod_{p \nmid q} \left(1 + \frac{1}{p^s - 1}\right).
\]

Plainly,

\[
\frac{1}{q} \sum_{n=1}^{q} e^{-2\pi i q n t} F_q(s, l/q) = \sum_{\substack{n \equiv a \mod q}} n^{-s}
\]

and so

\[
\sum_{\substack{n \equiv a \mod q}} 1 = \frac{1}{2\pi i q} \sum_{p \mid q} \int_{c-i \infty}^{c+i \infty} y^{s-1} F_q(s, l/q) dy,
\]

where the * denotes that if \(s\) is an integer, the last term in the sum on the left is halved. Thus

\[
\sum_{\substack{n \equiv a \mod q}} 1 = \frac{1}{q} \sum_{n \equiv a \mod q} 1 + \frac{1}{2\pi i q} \sum_{p \mid q} \int_{c-i \infty}^{c+i \infty} y^{s-1} F_q(s, l/q) dy.
\]

Hence

\[
\frac{1}{q} \sum_{n \equiv a \mod q} 1 = \max_{l/q < 1} |W(w; l/q)| + 1
\]

where

\[
W(w; l/q) = \frac{1}{2\pi i q} \int_{c-i \infty}^{c+i \infty} y^{s-1} F_q(s, l/q) dy.
\]

It is plain that \(F_q(s, l/q) = F_q(s, l/q)\) where \((l, q) = 1\) and \(l_q = l/q\). In fact our estimate of \(W\) is weakest when \(q\) is large, and so we may assume that \((l, q) = 1\), without loss of generality. Now

\[
F_q(s, t) = F(s, t) \prod_{p \mid q} \left(1 + \frac{e^{2\pi i q p t}}{p^s - 1}\right) = F(s, t) \tilde{F}_q(s, t)
\]

say, where

\[
F(s, t) = \prod_{p} \left(1 + \frac{e^{2\pi i p t}}{p^s - 1}\right).
\]

From this we see that

\[
F(s, l/q) = \prod_{p \nmid q} \left(1 + \frac{e^{2\pi i p l q t}}{p^s - 1}\right) \prod_{p \mid q} \left(1 + \frac{e^{2\pi i q p l t}}{p^s - 1}\right)
\]

where the second factor is regular and bounded independently of \(q\) for \(R_s \geq \frac{1}{2} + \epsilon\). Setting

\[
G(s, l/q) = \prod_{a=1}^{q} \prod_{p \equiv a \mod q} \left(1 + \frac{e^{2\pi i q p l t}}{p^s}\right)
\]

we find that

\[
F(s, l/q) \quad G(s, l/q) = \prod_{p \nmid q} \left(1 + \frac{e^{2\pi i p l q t}}{p^s}\right) \prod_{p \mid q} \left(1 + \frac{e^{2\pi i q p l t}}{p^s}\right)
\]

Now

\[
\log G(s, l/q) = \sum_{a=1}^{q} \sum_{p \equiv a \mod q} p^{-s} - H_{1}(s, l/q)
\]

\[
= \frac{1}{\varphi(q)} \sum_{a} \sum_{p} \frac{x(a) e^{2\pi i q p l t}}{p^s} + H_{1}(s, l/q)
\]

\[
= \frac{1}{\varphi(q)} \sum_{a} x(a) \sum_{p} \left(1 - \frac{\chi(p)}{p^s}\right) + H_{1}(s, l/q)
\]

\[
= \frac{1}{\varphi(q)} \sum_{a} x(a) \sum_{p} \left(1 - \frac{\chi(p)}{p^s}\right) + H_{2}(s, l/q)
\]
where \( H_1(s, l/q) \) and \( H_2(s, l/q) \) are regular and bounded, independently of \( q \) for \( Rs > \frac{1}{2} + \varepsilon \). We write

\[
|H_1(s, l/q)| \ll H_2, \quad Rs > \frac{1}{2}.
\]

As usual,

\[
\tau_x = \sum_{\nu=1}^{q} \chi(b) e^{\nu b/q}.
\]

so that

\[
|\tau_x| \ll V q.
\]

Thus

\[
\log \Theta(s, l/q) = \frac{\mu(q)}{\varphi(q)} \log \Delta(s, \chi_0) + \frac{1}{\varphi(q)} \sum_{\nu=1}^{q} \chi(l) \tau_x \log L(s, \chi) + H_3(s, l/q).
\]

We conclude that \( \Theta(s, l/q) \) and hence \( F_q(s, l/q) \) is regular within the zero-free region common to all the \( L \)-functions to modulus \( q \), with the exception, when \( q \) is squarefree, of a branch point or algebraical singularity at \( s = 1 \). In fact we shall only be interested in the squarefree case. We set

\[
e \equiv 1 + \frac{1}{\log z}
\]

and move the contour of integration to \( \Gamma_1 \cup \Gamma_2 \), as shown in the figure.

\[
\Gamma_2 \text{ is a loop around } s = 1, \text{ while on } \Gamma_3 \text{ we have } s = e \text{ for } |t| > T. \text{ For } |t| \leq T \text{ set}
\]

\[
\sigma = 1 - \frac{C_\infty}{M(q, t)}
\]

where

\[
M(q, t) = \max \{ q^{1/4}, \log \{ q(|t| + 4) \} \}
\]

and

\[
C_{22} = \min \left( \frac{C_{11}}{2}, C_{10} \left( \frac{1}{2} \right) \right).
\]

The contour is completed by horizontal lines at \( |t| = T \). Set

\[
W(x, l/q) = W_1(x, l/q) + W_2(x, l/q)
\]

where

\[
W_1(x, l/q) = \frac{1}{2 \pi i} \int_{T_2}^{x} \frac{\chi}{s} F_q(s, l/q) \, ds
\]

and

\[
Z(x, l/q) = \int_{0}^{x} W_1(y, l/q) \, dy = \frac{1}{2 \pi i} \int_{T_1}^{x} \frac{\chi}{s} F_q(s, l/q) \, ds.
\]

We estimate the integral over \( \Gamma_1 \) first. Note that for \( \varepsilon > 0 \),

\[
|F_q(s, l/q)| \theta(s, l/q) \ll C_\infty(e) q^\varepsilon
\]

uniformly in \( q \), for \( Rs > \frac{1}{2} \). We set

\[
Z = Z_1 + Z_2 + Z_3 + Z_4 = \frac{1}{2 \pi i} \left( \int_{a_1}^{\beta} + \int_{a_2}^{1} + \int_{1}^{\beta} + \int_{b}^{1} \right) \frac{\chi}{s} F_q(s, l/q) \, ds.
\]

For \( |t| < 2, s \in \Gamma_1 \), we have by Lemma 6 that

\[
|\log \Theta(s, l/q)| \ll \frac{C_{21}}{\varphi(q)} \left( \log q + \log \frac{q^{1/4}}{C_{22}} \right) + C_{20} \left( \frac{1}{4} \right) V q \log q + H_3
\]

\[
\leq C_{24} V q \log q.
\]

Hence

\[
Z_1 \ll \int_{a}^{b} C_{23} \left( \frac{e}{4} \right) q^\varepsilon \exp \left( C_{24} V q \log q \right) \, dt
\]

\[
\ll a^{1-C_{22} \log q} \exp \left( C_{24} V q \log q \right)
\]

on taking \( \varepsilon = 1 \) and absorbing this factor into the exponential. For \( 2 \leq |t| < T \), we have

\[
|\log \Theta(s, l/q)| \ll 6 C_{14} V q \log q + C_{10} V q \log \log \{ q(T + 4) \} + H_3;
\]
On the probability that \( n \) and \( f(n) \) are relatively prime

In the case \( q = 2 \), \( F_n(x, l/q) \) has a zero at \( s = 1 \) and the contour of integration is simply \( I_s \) continued across the real line. Next,

\[
F_n(s, l/q) = (s-1)^{\alpha(s)/\alpha(\sigma)} \prod_{\beta < 0} \left[ L(s, \beta) \right]^{\gamma(s)/\alpha(\beta)} \theta(s)^{\alpha(\gamma)/\alpha(s)}
\]

where \( \theta(s) \) is bounded by

\[
C_{33} \sum_{\beta < 0} \left( 1 + \frac{1}{\beta} \right)^{3/4}.
\]

Since \( (s-1)\zeta(s) \) is bounded on \( I_s \) we have

\[
F_n(s, l/q) \leq \exp\left\{ C_{33} \sqrt{q \log q} \right\},
\]

on \( I_s \). Thus

\[
|W_n(x, l/q)| \leq \exp\left\{ C_{33} \sqrt{q \log q} \right\} \int_0^1 \left[ \frac{1}{\log x} \frac{\mu(x)}{\sigma(x)} \left( 1 - \frac{\mu(x)}{\sigma(x)} \right) \right] du
\]

\[
\leq \exp\left\{ C_{33} \sqrt{q \log q} \right\} \Gamma \left( 1 - \frac{\mu(x)}{\sigma(x)} \right)
\]

\[
\leq \frac{C_{33} x}{\log x} \exp\left\{ C_{33} \sqrt{q \log q} \right\}.
\]

Thus if \( 4C_{33} \sqrt{q \log q} \leq \log \log x \),

\[
|W_n(x, l/q)| \leq \frac{C_{33} x}{(\log x)^{1/4}}.
\]

It will be seen that the condition on \( q \) is much more strict than that involved with \( W_n \), to achieve a weaker result. For this reason no improvement is obtained by taking the contour of integration further to the left, and it is obviously convenient to keep it to the right of the possible Siegel zero, as we have done.

With this restriction on \( q \), we have

\[
\int_0^1 W_n(x, l/q) \frac{dy}{y} = Z(x, l/q) + \int_0^1 W_n(x, l/q) \frac{dy}{y} \leq \frac{2C_{33} x}{(\log x)^{1/4}}.
\]

Now

\[
W_n(x, l/q) = \sum_{\beta < 0} e^{\alpha(s)/\alpha(\beta)} \frac{\mu(x)}{\sigma(x)}
\]

and so

\[
|W_n(x_1, l/q) - W_n(x_2, l/q)| \leq |x_1 - x_2|.
\]

Suppose that

\[
RX_n(x, l/q) = Mx,
\]

where without loss of generality we take \( M > 0 \). Then for \( y \gg x - \frac{1}{2} Mx \),

\[
RX_n(y, l/q) \geq \frac{1}{2} Mx.
\]
Thus
\[ R\left( \int_0^x \frac{W(y; l/q)}{y} \frac{dy}{y} - \int_0^x W(y; l/q) \frac{dy}{y} \right) \geq \frac{1}{2} M^2 \omega. \]

But the left-hand side does not exceed \(4C_{1,2}(\log x)^{1/4}\). Hence
\[ M \leq \frac{4V_{\text{c}}(\log x)^{1/2}}{(\log x)^{1/2}}. \]

The same argument applies to the imaginary part, and we deduce that
\[ |W(x; l/q)| \leq \frac{C_{3,1}x}{2(\log x)^{1/2}}. \]

This completes the proof of the lemma.

Proof of the theorem. Let
\[ K(y, \omega) = \prod_{p \leq y; p \nmid n} p. \]

Then
\[ T(x) = \sum_{n \leq x} 1 + \theta \sum_{n \leq x} \frac{1}{\omega(n, \omega)} \]
with \(|\theta| \leq 1\). Thus
\[ T(x) - \theta \sum_{n \leq x} \frac{1}{\omega(n, \omega)} = \sum_{n \leq x} \mu(n) \sum_{q \leq \sqrt{x}} \mu(q) \sum_{\omega((n, q)) \equiv 1} 1 \]
\[ = \sum_{q \leq \sqrt{x}} \mu(q) \sum_{\omega((q, \omega)) \equiv 1} 1 = \sum_{q \leq \sqrt{x}} \mu(q) \sum_{\omega((q, \omega)) \equiv 1} 1, \]
where \(\Pi = \prod_{p \leq x} p\). Provided \(\log \log(x/\Pi) \geq 4C_{1,1}(\log \Pi)^{1/2}\), we have, by Lemmas 5 and 7, that
\[ T(x) = \sum_{q \leq x} \mu(q) \left( \frac{x}{y} + O\left( \frac{x}{(\log x)^{1/2}} \right) \right) + O\left( \frac{x}{(\log x)^{1/2}} \right). \]
\[ = \frac{x}{\Pi} \left( \Pi - 1 - \frac{1}{P} \right) + O\left( \frac{x}{(\log x)^{1/2}} \right) \prod_{p \leq x} \left( 1 + \frac{1}{p} \right) + O\left( \frac{x}{(\log x)^{1/2}} \right). \]

Now
\[ \Pi \prod_{p \leq x} p \leq \exp(O_{1,2} \omega), \]
and so the condition on \(\Pi\) is satisfied if we set
\[ \omega = C_{3,1} \log \log x. \]

We substitute this into the expression for \(T(x)\) above, and this completes the proof.