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The distribution of power residues and certain related results

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1. Introduction In this paper we prove as Theorem 1 a result concerning the distribution of power residues. We have two applications of this theorem immediately in mind.

Let k be a positive integer greater than 1. For each rational prime p which satisfies the relation $p \equiv 1 \pmod{k}$ we define $n_k(p)$ to be the least positive integer which is not a kth-power residue $\pmod{p}$. Similarly let $r_k(p)$ denote the least prime which is a kth-power residue $\pmod{p}$. For other primes we set $n_k(p) = 0 = r_k(p)$. It was proved by Erdös [9], that

$$\frac{1}{\pi(x)} \sum_{p \leqslant x} n_2(p) \to c_2 \quad (x \to \infty),$$

where c_2 is a positive constant, and $\pi(x)$ denotes the number of rational primes not exceeding the real number x. This result was generalized in the papers [5] and [6]. There it was proved that for each rational integer $k > 1$ there are positive constants c_k so that

$$\frac{1}{\pi(x)} \sum_{p \leqslant x} n_k(p) \to c_k \quad (x \to \infty,\ k = 1, 2, \ldots).$$

Moreover, there are constants d_k, e_k, so that

$$\frac{1}{\pi(x)} \sum_{p \leqslant x} r_k^{e_k}(p) \to d_k \quad (x \to \infty,\ k = 1, 2, \ldots).$$

The constants e_k are positive, but unless $k = 2$ they are less than 1 in value. Similar results can be proved with any real number $\delta \leqslant e_k$ in place of e_k. It is implicit in the proof of Erdös [9] that the estimate

$$\frac{1}{\pi(x)} \sum_{p \leqslant x} n_2(p) = c_2 + O\left(\exp\left(-\frac{A \log\log x}{\log\log\log x}\right)\right)$$

holds for any fixed positive value of A. A similar result can be proved with $r_2(p)$ in place of $n_2(p)$ (see for example [6]). As an example in the application of Theorem 1 we shall prove as Theorem 2 (in § 5) that for odd values of k there is a positive constant a so that

$$\frac{1}{\pi(x)}\sum_{p\leqslant x} n_k(p) = c_k + O\left(\exp\left(-(\log x)^a\right)\right).$$

When k is an even integer we only establish an error term similar to that possible in Erdös' theorem. Thus the results of this type are for odd values of k far superior to those for even values of k.

As a second application we note that Theorem 1 can also be used to establish finite probability spaces on which the distribution of L-series formed with Dirichlet characters can be studied. We shall postpone this application to two further papers. There are also applications of results similar in nature to Theorem 1 which are useful in sieve methods. They are of a slightly different quality however, and we shall not consider them here.

There is an old conjecture of Artin which states that the rational primes for which the integer 2 is a primitive root have a positive limiting frequency amongst the sequence of all rational primes. Recently a proof of this conjecture by Hooley [12], conditional upon a certain hypothesis of Riemann type, was given. More exactly his hypothesis analogous to the classical Riemann hypothesis was concerned with the position in the complex plane of the zeros of the Dedekind zeta functions of certain Kummer fields. By means of a representation of such zeta functions in terms of L-series defined over an appropriate cyclotomic subfield we discuss in § 6, the implications of this hypothesis. We also formulate a conjecture of Large Sieve type which if true would imply the truth of Artin's conjectured result.

Before stating our basic results we need some notation, and certain definitions. It will be convenient to denote algebraic number fields which are extensions of the rational number field Q, by $K, L, \ldots$ and so on, $\overline{K}, \overline{L}, \ldots$ will then denote their corresponding rings of algebraic integers. If a is an element of a ring $[a]$ will denote the principal ideal which it generates. A typical ideal will be denoted by a, whilst $\mathfrak{p}$ will denote a prime ideal, in general lying over a rational prime p. $c_1, c_2, \ldots$ will denote positive constants.

Let k be a positive rational integer, α an algebraic integer, and $\mathfrak{p}$ a prime ideal of the ring $Q(\sqrt[k]{1})$ which satisfies $\mathfrak{p}\nmid[\alpha]$. We recall that the kth-power residue symbol at α is defined by

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_k \equiv \alpha^{\frac{1}{k}(N\mathfrak{p}-1)} \pmod{\mathfrak{p}}, \quad \left(\frac{\alpha}{\mathfrak{p}}\right)_k = 1.$$

2. Statement of the main results.

THEOREM 1. *Let k, r, be positive rational integers, and let $a_1, a_2, \ldots, a_r$, be r further integers. These need not necessarily be distinct. Let $\varepsilon_1, \ldots, \varepsilon_r$, be r k-th roots of unity. We define $N(k, r)$ by*

$$N(k,r) = \sum_{\substack{\nu_1=1 \\ a_1^{\nu_1}\ldots a_r^{\nu_r}=\beta^k}}^{k} \cdots \sum_{\nu_r=1}^{k} (\varepsilon_1^{\nu_1}\ldots\varepsilon_r^{\nu_r})^{-1}$$

where in the condition of summation β is an algebraic integer of the cyclotomic field generated by $\sqrt[k]{1}$.

For each real number x let

$$S(x,k,r) = S(x,k;\ a_1,\ldots,a_r;\ \varepsilon_1,\ldots,\varepsilon_r)$$

be the number of prime ideals $\mathfrak{p}$ of $Q(\sqrt[k]{1})$ which satisfy $N\mathfrak{p}\leqslant x$, for which the relations

$$\left(\frac{a_j}{\mathfrak{p}}\right)_k = \varepsilon_j \quad (j=1,\ldots,r)$$

are satisfied. Then the following estimates hold:

(iii) *If k is odd*

$$S(x,k,r) - k^{-r}N(k,r)\pi(x) = \begin{cases} O\left(x\exp\left(-c_1\sqrt{\log x}\right)\right) & \text{if the least common multiple } M \text{ of the } a_j\ (j=1,\ldots,r) \text{ does not exceed } \exp\left(c_2\sqrt{\log x}\right), \\ O\left(x(\log x)^{-B}\right) & \text{if } M \text{ does not exceed } \exp\left(c_3\dfrac{\log x}{(B+6)\log\log x}\right). \end{cases}$$

(iv) *If k is even*

$$S(x,k,r) - k^{-r}N(k,r)\pi(x) = O\left(x\exp\left(-c_4\sqrt{\log x}\right)\right)$$

if M does not exceed some fixed power of $\log x$.

The first of the estimates in (iii) is that which will be used in the proof of Theorem 2. The second estimate is the more convenient for the study of the distribution of the values of L-series formed with Dirichlet characters mentioned earlier, since in this case it turns out to be more useful to have the weaker restriction upon the size of M. In the inequality of (iv), if M does not exceed $(\log x)^A$ say, then the constant c_4 may well depend upon A. This last estimate is non-effective, although

an effective form could be obtained if the size of A is suitably restricted. The estimates of (iii) are both effective.

THEOREM 2. *In the notation of the introduction*

$$\frac{1}{\pi(x)}\sum_{p\leqslant x} n_k(p) - c_k = \begin{cases} O\left(\exp\left(-c_5\frac{\sqrt{\log x}}{\log\log x}\right)\right) & \text{if } k \text{ is odd,} \\ O\left(\exp\left(-A\frac{\log\log x}{\log\log\log x}\right)\right) & \text{for any fixed value of } A \text{ if } k \text{ is even.} \end{cases}$$

Similar results can be proved concerned the functions $r_k(p)$, and indeed other functions of this type (see for example [8]).

3. Some necessary lemmas.

LEMMA 1 (Borel–Carathéodory). *Let the function $f(s)$ be regular in the circle $|s-s_0|\leqslant r$ and satisfy $f(s_0)=0$. Furthermore let $f(s)$ satisfy the inequality $\mathrm{Re} f(s)\leqslant M$ throughout this circle. Then $f(s)$ is bounded by*

$$|f(s)|\leqslant \frac{4r_1}{r-r_1}\max(M,0)$$

inside the circle $|s-s_0|\leqslant r_1<r$.

Proof. As the proof of this lemma is short we give it here. In the region $|s-s_0|\leqslant r$ $f(s)$ has an expansion

$$f(s)=\sum_{n=1}^{\infty}(u_n+iv_n)(\varrho e^{i\theta})^n, \qquad s-s_0=\varrho e^{i\theta}, \qquad 0\leqslant\varrho\leqslant r,$$

so that

$$\mathrm{Re} f(re^{i\theta})=\sum_{n=1}^{\infty}(u_n\cos n\theta - v_n\sin n\theta)r^n.$$

Then for example

$$r^n u_n \operatorname{sign} u_n = \frac{1}{\pi}\int_0^{2\pi}\mathrm{Re} f(re^{i\theta})[1+\cos n\theta\cdot\operatorname{sign} u_n]\,d\theta$$

$$\leqslant \frac{1}{\pi}\max(M,0)\int_0^{2\pi}[1+\cos n\theta\cdot\operatorname{sign} u_n]\,d\theta$$

and therefore

$$|u_n|r^n\leqslant 2\max(M,0) \qquad (n=1,2,\ldots).$$

In a similar way we prove that

$$|v_n|r^n\leqslant 2\max(M,0) \qquad (n=1,2,\ldots).$$

Finally if r_1 satisfies $0\leqslant r_1<r$, and s satisfies $|s-s_0|\leqslant r_1$, then

$$|f(s)|\leqslant\sum_{n=1}^{\infty}(u_n+v_n)r_1^n\leqslant 4\max(M,0)\frac{r_1}{r}\left(1-\frac{r_1}{r}\right)^{-1},$$

which is the desired inequality.

LEMMA 2. *Let K, $\mathfrak{f}$, $L(s,\chi)$ denote respectively any algebraic number field of degree n, any ideal of K, and any Hecke–Landau L-series formed with character $\chi(\mathrm{mod}\,\mathfrak{f})$. Let*

$$D=\Delta N\mathfrak{f}$$

where Δ denotes the discriminant of the field K, and $N\mathfrak{f}$ denotes the norm of the ideal $\mathfrak{f}$ taken from K down to Q. Then there is a positive constant c, depending only upon n, so that the L-series $L(\sigma+it,\chi)$ has no zero in the region

$$\Gamma:\ \sigma\geqslant 1-\frac{c}{\log D(4+|t|)}\geqslant\frac{3}{4}$$

whenever χ is complex, or χ is real but t is not zero. For a real character χ there may be a simple real zero in this region, but nevertheless there is a further positive constant g, depending at most upon n, so that it lies outside the segment

$$\sigma\geqslant 1-gD^{-2n}, \qquad t=0.$$

If K is considered fixed, so that Δ is also, and only $\mathfrak{f}$ varies, then for any value of $\varepsilon>0$ the constant g can be adjusted so that this last inequality can be improved to

$$\sigma\geqslant 1-gD^{-\varepsilon}.$$

Proof. All of these results save for the last one are proved by Fogels [10]. He points out that the exponent $2n$ of D in the first of the half planes can be slightly improved as to the size of the factor 2. This is not of great interest in most applications however. If we assume that K is fixed then a straightforward modification of the proof of Siegel's theorem for real Dirichlet characters as given by Estermann (see for example Prachar [14]) yields the last of the assertions. All of the inequalities of the present lemma are effective except for the last one.

We shall preserve the notation of Lemma 2 for the duration of the following three lemmas.

LEMMA 3. *There is a positive absolute constant B so that the inequality*

$$|L(s,\chi)|\leqslant c_6(D[4+|t|]^n)^B$$

holds uniformly in the strip $-\frac{1}{2}\leqslant\sigma\leqslant 3$.

Proof. This inequality can be obtained on exactly the same lines as Lemma 4 of Fogels' paper [10].

LEMMA 4. *If we replace the constant c in the statement of Lemma 2 by a slightly smaller one then we can be assured that*

$$\left|\frac{L'}{L}(s,\chi)\right| \leqslant c_7(\log D(4+|t|))^4$$

holds in the region Γ, unless possible χ is a real character. When this is the case we assert that

$$\left|\frac{L'}{L}(s,\chi)\right| \leqslant c_8\max\left(D^{c_9 n},(\log D(4+|t|))^4\right)$$

holds in the region common to Γ and the half-plane $\sigma \geqslant 1-gD^{-2n}$.

If K is considered fixed, then we can set $2n=\varepsilon$, and $c_9 n = 4\varepsilon$. In fact by a suitable adjustment of the value of g in these circumstances, we can replace $c_9 n$ by any fixed positive real number.

Proof. We shall give a detailed proof for the case when χ, the character under consideration, is complex, and confine ourselves to a few remarks concerning the modifications necessary in the argument when χ is real.

Let $s=\sigma+it$ be a point of the region

$$\sigma \geqslant 1-\frac{\frac{1}{8}c}{\log D(4+|t|)}, \qquad |t| \geqslant \tau,$$

where we shall presently choose τ to be (in some sense) a large constant. We shall apply Lemma 1 to the function $f(s)=\log L(s,\chi)L(2+it,\chi)^{-1}$, where the value of the logarithm is chosen so that $f(s)$ vanishes at the point $s_0=2+it$. In the notation of that lemma we then take

$$r = 2+\frac{\frac{1}{2}c}{\log D(4+|t|)}, \qquad r_1 = 2+\frac{\frac{1}{4}c}{\log D(4+|t|)}.$$

Since K is of degree n over Q

$$|L(2+it,\chi)| \geqslant \prod_{\mathfrak{p}}\left(1+\frac{1}{(N\mathfrak{p})^2}\right)^{-1}$$

$$\geqslant \prod_{j|n}\prod_{p}\left(1+\frac{1}{p^{2j}}\right)^{-n} \geqslant \prod_{p}\left(1-\frac{1}{p^2}\right)^{n^2} = \left(\frac{6}{\pi^2}\right)^{n^2}.$$

This is a very wasteful argument, and if for example K is normal over Q, then in terms of the dependence upon n a much better inequality can be obtained. This aspect does not concern us here; however. If we choose τ to have a sufficiently large but otherwise fixed value, the circle

$|s-s_0| \leqslant r$ will be contained in the zero-free region Γ guaranteed by Lemma 2. It follows from Lemma 3 that in this circle

$$\operatorname{Re}\log L(s,\chi)L(s_0,\chi)^{-1} \leqslant c_9\log D(4+|t|).$$

By the Borel–Carathéodory theorem applied to the circle $|s-s_0| \leqslant r_1$,

$$f(s) = \log L(s,\chi)L(s_0,\chi)^{-1} = O\left(\log^2 D(4+|t|)\right).$$

Now let s be inside the slightly smaller region

$$\sigma \geqslant 1-\frac{\frac{1}{8}c}{\log D(4+|t|)}, \qquad |t| \geqslant \tau.$$

We can assume that s lies inside the circle

$$\mu = \left\{z;\ |z-s| \leqslant \frac{\frac{1}{8}c}{\log D(4+|t|)}\right\}$$

which in turn lies inside the circle $\{z;\ |z-s_0| \leqslant r\}$. Hence it follows from Cauchy's theorem that (in an obvious notation)

$$\frac{L'}{L}(s,\chi) = \frac{1}{2\pi i}\int_{\mu}\frac{f(z)}{(z-s)^2}\,dz = O\left(\log^4 D(4+|t|)\right).$$

We have proved this result under the condition $|t| \geqslant \tau$. It can be extended trivially into the region of the zero-free strip, provided that the constant c is suitably adjusted, by means of the maximum modulus principle.

If χ is real, then we choose τ to be $\exp(c_4 D^{\varepsilon})$ where c_4 is a suitably large but fixed positive real number, independent of D ant t. The desired upper bound is now obtained on the same lines as for the case of a complex character. The exponent 4ε in the second inequality of the present lemma can clearly be replaced, at the expense of the constant c_3 only, by any fixed positive real number.

LEMMA 5. *In the notation of Lemma 2 the following inequalities are valid for certain constants c_5, c_6, c_7.*

(i) *If χ is complex:*

$$\sum_{N\mathfrak{p}\leqslant x}\chi(\mathfrak{p}) = \begin{cases} O\left(x\exp(-c_4\sqrt{\log x})\right) & \text{if} \quad \Delta N\mathfrak{f} \leqslant \exp(c_5\sqrt{\log x}), \\ O\left(x(\log x)^{-B}\right) & \text{if} \quad \Delta N\mathfrak{f} \leqslant \exp\left(\dfrac{c_6}{B+6}\cdot\dfrac{\log x}{\log\log x}\right). \end{cases}$$

(ii) *If χ is real:*

$$\sum_{N\mathfrak{p}\leqslant x}\chi(\mathfrak{p}) = O\left(x\exp(-c_7\sqrt{\log x})\right)$$

if $\Delta N\mathfrak{f} \leqslant c_8(\log x)^{1/n}$, and Δ may vary, or if Δ is considered fixed and $\mathfrak{f}$ satisfies $N\mathfrak{f} = O\left((\log x)^A\right)$ for any fixed positive real number A.

These estimates are uniform in all the complex or real characters $\chi \pmod{\mathfrak{f}}$ *as the case may be.*

The first of the two estimates involving complex characters is to be used in the proof of Theorem 2. The second inequality of (i) is, because of its weak restriction on the size of $\Delta N\mathfrak{f}$, the more useful in the study of the L-series formed with Dirichlet characters alluded to in the introduction. The condition $\Delta N\mathfrak{f} \leqslant c_8(\log x)^{1/n}$ demanded in the first of the two situations considered in the situation (ii) can be weakened, but not essentially, however.

Proof. It is convenient to begin by estimating a sum which is slightly different in form from that in the statement of the present lemma. We define

$$\psi(x) = \psi(x, \chi) = \sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a}) \Lambda(\mathfrak{a}) \qquad (x \geqslant 0),$$

where

$$\Lambda(\mathfrak{a}) = \begin{cases} \log N\mathfrak{p} & \text{if} \quad \mathfrak{a} = \mathfrak{p}^m, \ \mathfrak{p} \text{ prime}, \\ 0 & \text{otherwise}. \end{cases}$$

This last function is the analogue for algebraic number-fields of von Mangoldt's function. We then set

$$\psi_1(x) = \int_0^x \psi(y)\, dy \qquad (x \geqslant 0).$$

In the half-plane $\sigma > 1$ we have the representation

$$\psi_1(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{x^{s+1}}{s(s+1)} \frac{L'}{L}(s, \chi)\, ds.$$

This relation can be justified by expressing the function

$$\frac{y^s}{s} \frac{L'}{L}(s, \chi)$$

in terms of the series

$$\sum_{\mathfrak{a}} \chi(\mathfrak{a}) \Lambda(\mathfrak{a}) (N\mathfrak{a})^{-s}$$

which is absolutely convergent in the half plane $\sigma > 1$, and integrating term by term with respect to s. This yields the representation

$$\psi(y) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{y^s}{s} \frac{L'}{L}(s, \chi)\, ds.$$

Integration with respect to y over the range $0 \leqslant y \leqslant x$ now yields the required formula.

Let T be a positive real number, satisfying $T \geqslant 2$. We write

$$\psi_1(x) = \left(\frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} + \frac{1}{2\pi i} \int_{\substack{\sigma + it \\ |t| > T}} \right) \frac{x^{s+1}}{s(s+1)} \frac{L'}{L}(s, \chi)\, ds = I_1 + I_2,$$

say, and estimate each integral separately. The second integral is taken over the pair of line segments $\mathrm{Re}(s) = \sigma > 1$, $|t| > T$. Assuming that x satisfies $x \geqslant 3$ we set $\sigma = 1 + (\log x)^{-1}$. The neatest way to estimate the second of these integrals is perhaps that of Halász [11]:

Let y be a real number, and set

$$G(x) = \sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a}) \Lambda(\mathfrak{a}) (N\mathfrak{a})^{-iy}.$$

Then in the region $\sigma > 1$

$$\frac{1}{s} \frac{L'}{L}(s + iy) = \int_1^\infty G(x) x^{-s-1}\, dx = \int_0^\infty G(e^u) e^{-u\sigma} e^{-uit}\, du$$

so that the pair

$$G(e^u) e^{-u\sigma} \quad \text{and} \quad \frac{1}{s} \frac{L'}{L}(s + iy)$$

are Fourier transforms. An appeal to Parseval's inequality yields the relation

$$\int_{\sigma - i\infty}^{\sigma + i\infty} \left| \frac{1}{s} \frac{L'}{L}(s + iy) \right|^2 |ds| = 2\pi \int_0^\infty |G(e^u)|^2 e^{-2u\sigma}\, du.$$

It follows from a simple well-known inequality of Čebyšev that

$$G(x) = O\Big(\sum_{p \leqslant x} \log p \Big) = O(x) \qquad (x \geqslant 3),$$

so that

$$\int_{\sigma + iy - \frac{1}{2}i}^{\sigma + iy + \frac{1}{2}i} \left| \frac{L'}{L}(s) \right|^2 |ds| \leqslant \sqrt{2} \int_{\sigma - i\infty}^{\sigma + i\infty} \left| \frac{1}{s} \frac{L'}{L}(s + iy) \right|^2 |ds|$$
$$= O\Big(\int_0^\infty e^{2u(\sigma - 1)}\, du \Big) = O\left(\frac{1}{\sigma - 1} \right).$$

We can replace the T in I_2 by an integer w satisfying $\frac{1}{2}T \leqslant w \leqslant T$ since the condition $T \geqslant 2$ is satisfied. Hence

$$\int_{\substack{\sigma + it \\ |t| > T}} \left| \frac{1}{s(s+1)} \right| \left| \frac{L'}{L}(s, \chi) \right|^2 |ds| \leqslant \sum_{\pm m = w}^{\infty} \frac{1}{(m^2 - \frac{1}{4})} \int_{\sigma + i(m - \frac{1}{2})}^{\sigma + i(m + \frac{1}{2})} \left| \frac{L'}{L}(s, \chi) \right|^2 |ds|$$
$$= O\left(\frac{1}{T(\sigma - 1)} \right),$$

and so by the Cauchy–Schwarz inequality

$$4\pi^2|I_2|^2 \leqslant \int\limits_{\substack{\sigma+it\\ |t|>T}} \frac{x^{2(\sigma+1)}}{|s(s+1)|}|ds| \int\limits_{\substack{\sigma+it\\ |t|>T}} \left|\frac{1}{s(s+1)}\right| \left|\frac{L'}{L}(s,\chi)\right|^2 |ds| = O\left(\frac{x^4\log x}{T^2}\right).$$

To estimate the first integral I_1, say for a complex character χ, we deform the contour into the partial rectangle

$$\sigma - iT \to 1-\tfrac{1}{8}c[\log D(4+T)]^{-1} - iT \to 1-\tfrac{1}{8}c[\log D(4+T)]^{-1} + iT \to \sigma + iT.$$

On this new contour we appeal to Lemma 4 and so deduce that

$$I_1 = O\left(x^2\exp\left(-\frac{c\log x}{4\log D(4+T)}\right)\log^4 D(4+T) + x^2T^{-1}\log^4 D(4+T)\right).$$

Altogether therefore we obtain the bound

$$\psi_1(x) = O\left(x^2\log D^4(4+T)\left[\frac{D(\log x)}{D(4+T)} + \exp\left(-\frac{c_7\log x}{\log D(4+T)}\right)\right]\right).$$

If D satisfies $D \leqslant \exp(c_5\sqrt{\log x})$ we define T by

$$(\log D(4+T))^2 = 2c_5\log x$$

and this yields the estimate

$$\psi_1(x) = O\left(x^2\exp\left(-c_8\sqrt{\log x}\right)\right).$$

If on the other hand D is assumed merely to satisfy the weaker restriction $(B+6)\log D \leqslant c_6\log x(\log\log x)^{-1}$, then we set $(B+6)\log(4+T) = c_6\log x(\log\log x)^{-1}$, so that for all absolutely large values of x the inequality $T \geqslant 2$ is satisfied. In this case we obtain for a suitably small real number c_6, independent of B, the estimate

$$\psi_1(x) = O\left(x^2(\log x)^{-2B-1}\right).$$

In order to complete the proof of Lemma 5 we first note that

$$\psi(x) = \frac{1}{h}[\psi_1(x) - \psi_1(x-h)] + O(h\log x) \qquad (h \geqslant 3),$$

and we choose a value of h in either of the cases of (i) under consideration so as to make these terms comparable in size. In fact the error term in this last estimate can be replaced, using a sieve method, by $O(h)$. Partial summation now yields both of the bounds stated in the section (i) of the present lemma. This procedure can be justified by slightly decreasing the values of the constants c_5 and c_6.

In order to consider the case when χ is a real character we need only apply the second or third of the estimates of Lemma 4, rather than the first.

We now specialize our interests to the cases when K is a cyclotomic field $Q(\sqrt[k]{1})$. If m is a rational integer, then the kth-power residue symbol at m induces a character on the ideal class group $(\mathrm{mod}[k^2m])$ in $Q(\sqrt[k]{1})$, as follows:

For any ideal $\mathfrak{a}$ of $Q(\sqrt[k]{1})$ which is prime to $[k^2m]$ set

$$\left(\frac{m}{\mathfrak{a}}\right)_k = \prod_{\mathfrak{p}^w\|\mathfrak{a}} \left(\frac{m}{\mathfrak{p}}\right)_k^w.$$

Then a character can be defined by

$$\chi(\mathfrak{a}) = \begin{cases} \left(\dfrac{m}{\mathfrak{a}}\right)_k & \text{if } \mathfrak{a} \text{ and } [k^2m] \text{ are coprime,} \\ 0 & \text{otherwise.} \end{cases}$$

Our next lemma is concerned with the possibility that this character be real.

Lemma 6. *Let k be a positive rational integer, and let m be a further rational integer which is a k-th-power residue* $(\mathrm{mod}\, p)$ *for all but finitely many rational primes $p \equiv 1(\mathrm{mod}\, k)$. Then m is of the form β^k with β an integer of the field $Q(\sqrt[k]{1})$.*

Proof. This result is due essentially to Trost [16]. See also Ankeny and Rogers [2]. In fact the hypothesis can be considerably weakened. If P denotes the set of primes $p \equiv 1(\mathrm{mod}\, k)$ for which m is a kth-power residue it is sufficient for the validity of the conclusion that

$$\lim_{s\to 1+} \sum_{p\in P} p^{-s}\Big(\sum_p p^{-s}\Big)^{-1} = 1/\varphi(k).$$

Corollary. *Let $\chi(\mathfrak{a})$ be the ideal class character* $(\mathrm{mod}[k^2m])$ *in $Q(\sqrt[k]{1})$ induced by the k-th-power residue symbol taken at the rational integer m. Then $\chi(\mathfrak{a})$ is real if and only if k is odd and $m = \beta_1^k$, or k is even and $m = \beta_2^{k/2}$, with either β_j in $Q(\sqrt[k]{1})$.*

4. Proof of Theorem 1. For each rational integer a_j, and kth root of unity ε_j $(j = 1, \ldots, r)$, and each prime ideal $\mathfrak{p}$ of $Q(\sqrt[k]{1})$, it is clear that

$$k^{-1}\sum_{\nu_j=1}^{k}\left(\left(\frac{a_j}{\mathfrak{p}}\right)_k\varepsilon_j^{-1}\right)^{\nu_j} = \begin{cases} -1 & \text{if } \left(\dfrac{a_j}{\mathfrak{p}}\right)_k = \varepsilon_j, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from Lemma 6 that in the notation of Theorem 1

$$S(x,k,r)-k^{-r}N(k,r)\sum_{N\mathfrak{p}\leqslant x}{}'1=k^{-r}\sum_{\substack{r_1=1\\ a_1^{r_1}\dots a_r^{r_r}\neq\beta^k}}^{k}\dots\sum_{r_r=1}^{k}(e_1^{r_1}\dots e_r^{r_r})^{-1}\sum_{N\mathfrak{p}\leqslant x}\left(\frac{a_1^{r_1}\dots a_r^{r_r}}{\mathfrak{p}}\right)_k$$

where $\sum'$ indicates summation over the prime ideals of the ring $Q(\sqrt[k]{1})$ which do not divide $ka_1\dots a_r$, and where $S(x,k,r)$ has the same definition as that of $S(x,k,r)$ in the statement of Theorem 1, save that only rational primes p not dividing k are counted. In order to estimate the innermost sum over $\mathfrak{p}$ on the right hand side we note that if we replace the kth-power residue symbol by the ideal class character $(\mathrm{mod}[k^2a_1^{r_1}\dots a_r^{r_r}])$ which it induces then the condition $'$ in the summation is redundant. We note that the induced character cannot be principal, otherwise $a_1^{r_1}\dots a_r^{r_r}$ would be in the sense of Lemma 6 everywhere a local kth-power, and so by Lemma 6 a global kth-power, a possibility which has been ruled out. Moreover this character is complex if k is odd, no matter what the values of the r_j.

Thus we can apply the inequalities of Lemma 5 (i), to deduce for example that

$$\sum_{N\mathfrak{p}\leqslant x}{}'\left(\frac{a_1^{r_1}\dots a_r^{r_r}}{\mathfrak{p}}\right)_k=O\big(x\exp(-c_7\sqrt{\log x})\big).$$

In order to complete the proof of the estimations (iii) of Theorem 1 it suffices to prove that

$$\sum_{N\mathfrak{p}\leqslant x}{}'1=\pi(x)+O\big(x\exp(-c_7\sqrt{\log x})\big).$$

This is an easy matter, since

$$\sum_{N\mathfrak{p}\leqslant x}{}'1=\sum_{f\mid\varphi(k)}\frac{\varphi(k)}{f}\sum_{\substack{p^f\leqslant x\\ p^f\equiv1(\mathrm{mod}\,k)}}1+O(1)$$

$$=\varphi(k)\sum_{\substack{p\leqslant x\\ p\equiv1(\mathrm{mod}\,k)}}1+O\Big(\sum_{p^2\leqslant x}1+\sum_{p^3\leqslant x}1+\dots\Big).$$

Finally, an application of Dirichlets theorem (see for example Prachar [14], Chapter IV, Satz 8.3), these last sums are

$$\pi(x)+O\big(x\exp(-c_7\sqrt{\log x})\big)+O(x^{1/2}).$$

Lastly, if k is even, then a similar proof can be given of the estimate (iv) of Theorem 1. Note that $a_1^{k/2}\dots a_r^{k/2}$ satisfies the condition of Lemma 6 which is sufficient to guarantee that the kth-power residue symbol at $a_1^{k/2}\dots a_r^{k/2}$ induces a real ideal class character. We must fall back therefore upon the weaker bounds (ii) of Lemma 5. These lead to the estimate (iv) of Theorem 1, and our proof of Theorem 1 is complete.

We note that by means of some recent results of Sokolovskii [15] the error term in the estimates (iii) of Theorem 1 can in some respects be improved.

5. Proof of Theorem 2. We recall some results from [5]. It was proved in that paper that if $q_1, q_2, \dots,$ are the rational primes in increasing order, then

$$(\alpha)\qquad S(x,k,r)=S(x,k,r,q_1,\dots,q_r;\,1,\dots,1)$$
$$=\big(1+o(1)\big)d_r\pi(x)\qquad(x\to\infty),$$

where the number d_r satisfies an inequality $d_r\leqslant c_9 2^{-r}$. Furthermore it was shown that

$$(\beta)\qquad S(x,k,r)\leqslant c_{10}d_r\pi(x)$$

holds uniformly for all those integers r corresponding to which the primes q_r do not exceed a certain positive constant multiple of $\log x$; say $q_r\leqslant c_{11}\log x$. Finally we shall need the fact, proved in the same paper, that

$$(\gamma)\qquad\frac{1}{\pi(x)}\sum_{\substack{p\leqslant x\\ n_k(p)>c_{11}\log x}}n_k(p)=O\left(x\exp\left(-c_{12}\frac{\log x}{(\log\log x)^2}\right)\right).$$

We shall consider in detail the case when k is odd. Simple modifications will then yield the case of k even.

If c_{13} is chosen to be a suitable positive constant, and q_r does not exceed $c_{13}\sqrt{\log x}$, then the first hypothesis of Theorem 1 (iii) is satisfied, so that $k^{-r}N(k,r)=d_r$, and

$$S(x,k,r)=d_r\pi(x)+O\big(x\exp(-c_1\sqrt{\log x})\big).$$

Hence

$$\frac{1}{\pi(x)}\sum_{\substack{p\leqslant x\\ n_k(p)\leqslant c_{13}\sqrt{\log x}}}n_k(p)=\frac{1}{\pi(x)}\sum_{q_r\leqslant c_{13}\sqrt{\log x}}q_r[S(x,k,r-1)-S(x,r)]$$

$$=\sum_{q_r\leqslant c_{13}\sqrt{\log x}}q_r(d_{r-1}-d_r)+O\big(\log x\cdot\exp(-c_1\sqrt{\log x})\big)$$

$$=c_k+O\left(\exp\left(-\frac{c_5\sqrt{\log x}}{\log\log x}\right)\right),$$

the last step making use of the upper bounds for d_r in (α). Combining this estimate with the inequality (γ), and

$$\frac{1}{\pi(x)} \sum_{\substack{p\leqslant x \\ c_{13}\sqrt{\log x}<n_k(p)\leqslant c_{11}\log x}} n_k(p) = O\Big(\sum_{q_r>c_{13}\sqrt{\log x}} q_r d_{r-1}\Big) = O\left(\exp\left(-\frac{c_5\sqrt{\log x}}{\log\log x}\right)\right)$$

the proof of Theorem 2 is complete.

6. Remarks concerning Artin's conjecture. We confine ourselves for simplicity to the case when primes for which 2 is a primitive root are sought. To estimate the number of those primes not exceeding a given real number x is essentially the same as considering the prime ideal theorem in the Kummer fields $Q(\sqrt[k]{1}, \sqrt[k]{2})$ generated by squarefree integers k. In his paper [12] Hooley proved that if the Dedekind zeta functions corresponding to these fields have all their zeros inside the critical strip actually on the line $\sigma = \frac{1}{2}$ (in the usual notation), then Artin's conjectured result indeed holds, namely that

$$P(x) = \sum_{\substack{p\leqslant x \\ 2 \text{ prim}\,(\mathrm{mod}\,p)}} 1 = (1+o(1))A\pi(x) \ (x\to\infty); \quad A = \prod_{p>3}\left(1-\frac{1}{p(p-1)}\right).$$

As is well known, if the integer 2 is replaced by another integer, then it is sometimes necessary to modify the definition of the constant A.

Let $\zeta_K(s)$ denote the Dedekind zeta function associated with the field $Q(\sqrt[k]{1}, \sqrt[k]{2})$, where k is a positive integer. Let $\zeta_k(s)$ denote the Dedekind zeta function associated with the cyclotomic field $Q(\sqrt[k]{1})$, and let $L_k(s,\chi)$ denote any L-series

$$\sum_{\mathfrak{a}} \chi(\mathfrak{a}) N\mathfrak{a}^{-s}$$

defined over $Q(\sqrt[k]{1})$ in terms of the character on the ideal class group $(\mathrm{mod}[2k^2])$ induced by the kth-power residue symbol evaluated at the point 2. Here the ideals $\mathfrak{a}$ run through the integral ideals of $Q(\sqrt[k]{1})$, and all of the norms are absolute. We maintain that the following relations hold:

(i) *If $8\nmid k$ then there exist ideals $\mathfrak{f}_\nu$ $(\nu = 1,\dots,k-1)$ dividing $[2k^2]$ in $Q(\sqrt[k]{1})$, and for each $\mathfrak{f}_\nu$ a character χ_ν, so that the characters $\chi_\nu(\mathrm{mod}\,\mathfrak{f}_\nu)$, and $\chi^\nu(\mathrm{mod}\,\mathfrak{f})$ are equivalent, and furthermore*

$$\zeta_K(s) = \zeta_k(s)\prod_{\nu=1}^{k-1} L_k(s,\chi_\nu).$$

(ii) *If $8\mid k$ then we can find similar moduli $\mathfrak{f}_\nu$, $\nu = 1,\dots,k/2-1$, and characters χ_ν so that*

$$\zeta_K(s) = \zeta_k(s)\prod_{\nu=1}^{k/2-1} L_k(s,\chi_\nu).$$

In either case each character χ_ν is primitive $(\mathrm{mod}\,\mathfrak{f}_\nu)$, and $\mathfrak{f}$ denotes the principal ideal $[2k^2]$.

These relations can be set in a more suggestive form; for example if $8\nmid k$, then there is a finite set of primes $\mathfrak{p}$ of $Q(\sqrt[k]{1})$, dividing $[2k^2]$, so that

$$\zeta_K(s) = \prod_{\mathfrak{p}\mid[2k^2]}{}'\left(1-\frac{\varepsilon}{N\mathfrak{p}^s}\right)^a \zeta_k(s)\prod_{\nu=1}^{k-1} L_k(s,\chi^\nu)$$

($|\varepsilon| = |\varepsilon(\mathfrak{p})| = 1$, each $a = a(\mathfrak{p})$ a rational integer satisfying $0\leqslant a\leqslant k-1$), and similarly if $8\mid k$. For

$$L_k(s,\chi^\nu) = L_k(s,\chi_\nu)\prod_{\substack{\mathfrak{p}\mid[2k^2] \\ \mathfrak{p}\nmid\mathfrak{f}_\nu}}\left(1-\frac{\chi_\nu(\mathfrak{p})}{N\mathfrak{p}^s}\right) \quad (\nu = 1,\dots,k-1),$$

and we may take

$$\prod_{\mathfrak{p}\mid[2k^2]}{}'\left(1-\frac{\varepsilon}{N\mathfrak{p}^s}\right)^a = \prod_{\nu=1}^{k-1}\prod_{\substack{\mathfrak{p}\mid[2k^2] \\ \mathfrak{p}\nmid\mathfrak{f}_\nu}}\left(1-\frac{\chi_\nu(\mathfrak{p})}{N\mathfrak{p}^s}\right).$$

We shall now justify the assertions (i) and (ii).

Let $\mathfrak{p}$ be a prime ideal of the ring $Q(\sqrt[k]{1})$ which does not divide $[2k^2]$. It is an immediate corollary of Kummer's theorem (see for example Weiss [18], Theorem 4.9.1 and exercise 4.9.2, on p. 169) that $\mathfrak{p}$ splits into kf^{-1} conjugate prime ideals in $Q(\sqrt[k]{1}, \sqrt[k]{2})$ if and only if the polynomial x^k-2 splits into kf^{-1} irreducible factors $(\mathrm{mod}\,\mathfrak{p})$ in $Q(\sqrt[k]{1})$. Moreover, it is clear that this last eventuality arises if and only if the congruence $2^l \equiv y^k (\mathrm{mod}\,\mathfrak{p})$ is soluble in $Q(\sqrt[k]{1})$, whilst $2^m \equiv y^k(\mathrm{mod}\,\mathfrak{p})$ is insoluble whenever $m<f$. Consider the product

$$\prod_{\mathfrak{p}\nmid[2k^2]}\prod_{\nu=1}^{k}\left(1-\left(\frac{2}{\mathfrak{p}}\right)_k^\nu N\mathfrak{p}^{-s}\right)^{-1} = \exp\left(\sum_{\mathfrak{p}\nmid[2k^2]}\sum_{r=1}^{\infty}\sum_{\nu=1}^{k}\frac{1}{r}\left(\frac{2}{\mathfrak{p}}\right)_k^{\nu r}(N\mathfrak{p})^{-rs}\right)$$

where the prime ideals $\mathfrak{p}$ are taken in $Q(\sqrt[k]{1})$. For each prime ideal $\mathfrak{q}$ of $Q(\sqrt[k]{1}, \sqrt[k]{2})$ which lies over such a prime ideal $\mathfrak{p}$ we write $e(\mathfrak{q})$ for the residue class degree of $\mathfrak{q}$ over $\mathfrak{p}$. Let $f\mid k$ and consider the contribution

to the triple sum on the right hand side of the above equation which is made by those q with $e(\mathfrak{q}) = f$. Since $(2/\mathfrak{p})_k = 1$ if and only if the congruence $2^r \equiv y^k \pmod{\mathfrak{p}}$ is soluble, and since $e(\mathfrak{q}) = f$, this can only happen if $f \mid r$. Moreover, exactly kf^{-1} of these q lie over each appropriate prime ideal p of $Q(\sqrt[k]{1})$. Thus the contribution is

$$\sum_{\substack{e(\mathfrak{q})=f \\ \mathfrak{q}\nmid[2k^2]}} \left(\frac{1}{N\mathfrak{q}^s} + \frac{1}{2N\mathfrak{q}^s} + \ldots\right) = \sum_{\substack{e(\mathfrak{q})=f \\ \mathfrak{q}\nmid[2k^2]}} -\log\left(1 - \frac{1}{N\mathfrak{q}^s}\right).$$

Summing over the values of f dividing k shows that

$$\prod_{\mathfrak{p}\nmid[2k^2]} \prod_{\nu=1}^{k} \left(1 - \left(\frac{2}{\mathfrak{p}}\right)_k^{\nu} N\mathfrak{p}^{-s}\right)^{-1} = \prod_{\mathfrak{q}\nmid[2k^2]} \left(1 - \frac{1}{N\mathfrak{q}^s}\right)^{-1},$$

where the final product is taken over the prime ideals of $Q(\sqrt[k]{1}, \sqrt[k]{2})$.

Let χ_ν be the primitive character $(\bmod \mathfrak{f}_\nu)$ which is induced by the character $(2/\mathfrak{p})_k^\nu \pmod{[2k^2]}$, and consider the function $g(s)$ which is defined in the region $\sigma > 1$ by

$$g(s) = g(s, \chi) = \zeta_K(s)\left[\zeta_k(s) \prod_{\nu=1}^{k-1} L_k(s, \chi_\nu)\right]^{-1}.$$

In view of the above relation $g(s)$ can be analytically continued over the whole plane, and all of its zeros or poles lie on the imaginary axis.

Now (see for example Landau [13]), if F is any algebraic number field, and χ is a primitive character $(\bmod \mathfrak{f})$, for an integral ideal $\mathfrak{f}$ of F, then the L-series formed over F with this character can be analytically continued over the whole complex plane, and there satisfies the functional equation

$$L_F(s, \chi) = \varepsilon(\chi, F)(A(\mathfrak{f}))^{1-2s} \left(\frac{\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)}\right)^{q} \left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\right)^{r_1-q} \times \times \left(\frac{\Gamma(1-s)}{\Gamma(s)}\right)^{r_2} L_F(1-s, \bar{\chi}).$$

Here

$$A(\mathfrak{f}) = 2^{-r_2}\pi^{-n/2}(\Delta N\mathfrak{f})^{1/2},$$

$2r_2$ denotes the number of pairs of complex conjugate fields to F, Δ denotes the discriminant and $n = r_1 + 2r_2$ the degree of F. $\bar{\chi}$ is the complex conjugate of the character χ, q is a rational integer satisfying $0 \leqslant q \leqslant r_1$, and finally $|\varepsilon(\chi, F)| = 1$. In the present circumstances we are only considering fields K which contain a non-trivial cyclotomic field. Since

any such field $Q(\sqrt[k]{1})$ is normal, and so self-conjugate, it therefore has no real conjugate fields, and so neither does K. In any application of the above functional equation we can thus set $q = 0 = r_1$. However, we shall not need this fact. We note that if $g(s, \chi)$ is not a constant, then it has a zero or a pole on the imaginary axis, and not at the origin. It is immediate from the functional equation for the L-series that one of the L-series defining $g(s, \chi)$ has a zero or a pole on the line-segments $\sigma = 1, t \neq 0$. Since neither of these eventualities can occur (see for example Lemma 2) $g(s, \chi)$ must be a constant. Letting $s \to \infty$ through real values shows that the value of this constant is 1. This justifies the assertion (i).

If $8 \mid k$ a similar argument can be given. In this case $\sqrt{2}$ belongs to $Q(\sqrt[k]{1})$ and the degree of the extension $K = Q(\sqrt[k]{1}, \sqrt[k]{2})$ over $Q(\sqrt[k]{1})$ is $k/2$. We see that our previous statements still hold if k is replaced in them by $k/2$, and 2 by $\sqrt{2}$. Thus the polynomials $x^{k/2} \pm \sqrt{2}$ are irreducible over $Q(\sqrt[k]{1})$, and an ideal p which does not divide $[k]$ (or therefore $[2]$) splits into $k/2f$ ideals of degree f in K if and only if the congruence $(\sqrt{2})^f \equiv y^{k/2} \pmod{\mathfrak{p}}$ is soluble in $Q(\sqrt[k]{1})$, but any congruence $(\sqrt{2})^m \equiv y^k \pmod{\mathfrak{p}}$ with $m < f$ is not. Since $-1 = i^2$, and $i = \sqrt{-1}$ belongs to $Q(\sqrt[k]{1})$, the relations $(\sqrt{2})^f \equiv y^{k/2} \pmod{\mathfrak{p}}$ and $2^f \equiv y^k \pmod{\mathfrak{p}}$ are equivalent. Finally we note that

$$\left(\frac{2}{\mathfrak{p}}\right)_k^{k/2} \equiv (\sqrt{2})^{k \cdot \frac{1}{k}(N\mathfrak{p}-1)} \equiv 1 \pmod{\mathfrak{p}} \qquad (\mathfrak{p}\nmid[k]),$$

so that for such prime ideals

$$\frac{1}{k}\sum_{\nu=1}^{k}\left(\frac{2}{\mathfrak{p}}\right)_k^{\nu} = \frac{2}{k}\sum_{\nu=1}^{k/2}\left(\frac{2}{\mathfrak{p}}\right)_k^{\nu}.$$

This justifies the assertion (ii). The method of proof is of course classical.

In conjunction with these two representations we note also that for each value of k

$$\zeta_k(s) = \zeta(s) \prod_{\chi \neq \chi_0} L(s, \chi)$$

where $\zeta(s)$ denotes the Riemann zeta function, and the L-series run over all those defined over the rational number field by the non-principal Dirichlet characters $(\bmod k)$. Thus if k is an odd prime we can find a divisor $\mathfrak{f}$ of $[2k^2]$ in $Q(\sqrt[k]{1})$, so that

$$\zeta_K(s) = \zeta(s) \prod_{\chi \neq \chi_0} L(s, \chi) \prod_{\nu=1}^{k-1} L_k(s, \chi^\nu)$$

where all of the L-series are formed with primitive characters. This is clearly the most important case.

In his paper [12] Hooley assumes that the zeros of the Dedekind zeta functions $\zeta_K(s)$, where $K = Q(\sqrt[k]{1}, \sqrt[k]{2})$, $\mu^2(k) = 1$, which lie in the strip $0 < \sigma < 1$, all lie on the line $\sigma = \frac{1}{2}$. In order to prove a weaker form of Artin's conjecture, namely that there exists an infinity of primes p for which 2 is a primitive root, we need only assume such a conjecture for prime values of k. For then Hooley's argument can be trivially modified to prove that

$$\liminf_{x\to\infty} \pi(x)^{-1} \sum_{\substack{p\leqslant x \\ 2\,\mathrm{prim}\,(\mathrm{mod}\,p)}} 1 \geqslant 1 - \sum \frac{1}{p(p-1)}$$

$$\geqslant 1 + \frac{1}{4\cdot 3} - \sum_{m=2}^{\infty} \frac{1}{m(m-1)} = \frac{1}{12} > 0.$$

The validity of the Riemann hypothesis for the fields $Q(\sqrt[k]{1}, \sqrt[k]{2})$, $k = 1, 2, \ldots$, implies its validity for $\zeta(s)$ and all of the L-functions formed with Dirichlet characters. It would be interesting to know what weaker hypothesis might be sufficient to prove the weaker form of Artin's conjecture. It was shown by Ankeny [1] that if one assumes the Riemann hypothesis and its generalization to Dirichlet L-series to be true, then $g(p)$ the least (prime) primitive root $(\mathrm{mod}\,p)$ satisfies

$$g(p) < c\left(2^{\nu(p)-1}\log p \log\left(2^{\nu(p)-1}\log p\right)\right)^2.$$

Here $\nu(p-1)$ denotes the number of distinct prime divisors of $(p-1)$. This was improved by Wang [17] to

$$g(p) < c'\nu(p-1)^6 \log^2 p.$$

More recently it was proved without any hypothesis that $g(p) = O\left((\log p)^{2+\varepsilon}\right)$ infinitely often for any fixed $\varepsilon > 0$ ([4]), and later that $g(p) < 4.75(\log p)^{6/5}$ holds infinitely often ([7]). These results are poor in comparison with the weak form of Artin's conjecture, which would imply that $g(p) = 2$ infinitely often. It would be interesting to know the best result in this direction which could be obtained by using only L-series formed with Dirichlet characters. Finally we note that the following analogue of a theorem of Bombieri ([3]) if true would immediately yield Artin's full conjecture. For example, for each integer $k > 1$ let $\chi(\mathfrak{a})$ denote the character induced on the ideal class group $(\mathrm{mod}\,[2k^2])$ in $Q(\sqrt[k]{1})$ by the kth-power residue symbol evaluated at the point 2, then it seems reasonable to

CONJECTURE. *For each positive real number B there is a further real number $C = C(B)$ so that for all large values of x*

$$\sum_{k\leqslant x^{1/2}(\log x)^{-C}} \frac{1}{k} \sum_{\nu=1}^{k-1} \Big| \sum_{N\mathfrak{p}\leqslant x} \chi^{\nu}(\mathfrak{p}) \Big| = O\left(x(\log x)^{-B}\right).$$

Indeed it might even be possible to replace the summand by

$$\max_{\chi} \Big| \sum_{N\mathfrak{p}\leqslant x} \chi(\mathfrak{p}) \Big|$$

where χ runs over all non-principal ideal class characters $(\mathrm{mod}\,[2k^2])$ in $Q(\sqrt[k]{1})$. In either case the prime ideals $\mathfrak{p}$ are to run through those belonging to the appropriate cyclotomic field.

References

[1] N. C. Ankeny, *The least quadratic non-residue*, Ann. of Math. 55 (1952), pp. 65–71.

[2] — and C. A. Rogers, *On a conjecture of Chowla*, Ann. of Math. 53 (1951), pp. 541–550.

[3] E. Bombieri, *On the Large Sieve*, Mathematika 12 (1965), pp. 201–225.

[4] D. A. Burgess and P. D. T. A. Elliott, *On the average value of the least primitive root*, Mathematika 15 (1968), pp. 39–50.

[5] P. D. T. A. Elliott, *A problem of Erdős concerning power residue sums*, Acta Arith. 13 (1967), pp. 131–149. See also *Corrigendum*, ibid. 14 (1968), p. 437.

[6] — *Some notes on k-th power residues*, Acta Arith. 14 (1968), pp. 153–162.

[7] — *The distribution of primitive roots*, Canad. J. Math. 21 (1969), pp. 822–841.

[8] — *On the mean value of f(p)*, to appear in Proc. London Math. Soc., Davenport Memorial Volume.

[9] P. Erdös, *Számelméleti megjeggyzések I*, Mat. Lapok 12 (1961), pp. 10–17.

[10] E. Fogels, *On the zeros of Hecke's L-functions I*, Acta Arith. 7 (1962), pp. 87–106.

[11] G. Halász, *Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen*, Acta Math. Acad. Sci. Hung. 19 (1968), pp. 365–403.

[12] C. Hooley, *On Artin's conjecture*, J. Reine Angew. Math. 225 (1967), pp. 209–220.

[13] E. Landau, *Über Ideale und Primideale in Idealklassen*, Math. Zeitschr. 2 (1918), pp. 52–154.

[14] K. Prachar, *Primzahlverteilung*, Berlin 1957.

[15] А. В. Соколовский, *Теорема о нулях дзета-функции Дедекинда и расстояние между „соседними" простыми идеалами*, Acta Arith. 13 (1968), pp. 321–334.

[16] E. Trost, *Zur Theorie der Potenzreste*, Nieuw Arch. Wisk. 18 (2) (1934), pp. 58–61.

[17] Y. Wang, *On the least primitive root of a prime*, Sci. Sinica 10(1) (1961), pp. 1–44.

[18] E. Weiss, *Algebraic Number Theory*, New York–London 1963.

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