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On sums of four cubes of polynomials

by

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It is well known that all integers $n \not\equiv \pm 4 \pmod 9$ can be expressed as a sum of four integer cubes, and numerical evidence suggests that this is also true for integers $\equiv \pm 4 \pmod 9$. A method of trying to prove this is to find polynomials P, Q, R, S in x with integer coefficients and degree $\leqslant 4$, such that

$$P^3+Q^3+R^3+S^3 = 9x+4. \tag{1}$$

Schinzel [1] has recently proved the more general result that such a representation with polynomials not all constant cannot hold for

$$P^3+Q^3+R^3+S^3 = Lx+M, \tag{2}$$

where L and M are integer constants and $M \equiv 4 \pmod 9$. Let

$$P = ax^4+bx^3+cx^2+dx+e$$

and write (2) as say,

$$\sum (ax^4+bx^3+cx^2+dx+e)^3 = 3^a Lx+M, \quad a \geqslant 0, \tag{3}$$

where here and throughout, summations will refer to the four sets typified by a, b, c, d, e. Suppose a representation is taken where the product of the leading coefficients of P, Q, R, S, has its least absolute value. Schinzel's proof, which is really a 3-adic one, is rather complicated since it requires the expansion of P^3 in powers of x and so it is not easy to see what underlies his proof.

He shows that $a \equiv 0 \pmod{81}$, $b \equiv 0 \pmod{27}$, $c \equiv 0 \pmod 9$, $d \equiv 0 \pmod 3$. Since obviously $a \geqslant 1$, then on replacing x by $x/3$, we have a representation

$$\sum \left(\frac{a}{81}x^4+\frac{b}{27}x^3+\frac{c}{9}x^2+\frac{d}{3}x+e\right)^3 = 3^{a-1}Lx+M$$

with a smaller product for the leading coefficients.

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(1) J. London Math. Soc. 43 (1968), pp. 143–145.

I give a simpler presentation of his method based on 3^{λ}-adic ideas where $\lambda = 1/4$ and $1/3$. A great simplification arises in the calculation since if, for example, n is an integer and $n \equiv 0 \pmod{3^{1/4}}$, then $n \equiv 0 \pmod 3$. The successive stages in the proof are $a = 3a_1$, $b = 3b_1$, $c = 3c_1$, $d = 3d_1$; then $a_1 = 3a_2$, $b_1 = 3b_2$; then $a_2 = 3a_3$, $c_1 = 3c_2$ and finally $b_2 = 3b_3$, $a_3 = 3a_4$.

Both proofs depend upon the obvious results:

LEMMA 1. *The only integer solution of*

$$\sum e^3 \equiv 4 \pmod 9$$

is given by $e \equiv 1 \pmod 3$ *etc.*

LEMMA 2. *The only integer solution of*

$$\sum a^3 \equiv 0 \pmod 9, \qquad \sum a^2 \equiv 0 \pmod 3$$

is given by $a \equiv 0 \pmod 3$ *etc.*

From (3), on equating coefficients of x, we have

$$3\sum de^2 = 3^{\alpha} L, \tag{4}$$

and so $\alpha \geqslant 1$. Taking residues of (3) mod 3, we have

$$\sum (a^3x^{12} + b^3x^9 + c^3x^6 + d^3x^3) \equiv 0 \pmod 3,$$

and so to mod 3, since $a^3 \equiv a$, we have

$$\sum a \equiv 0, \quad \sum b \equiv 0, \quad \sum c \equiv 0, \quad \sum d \equiv 0.$$

Since $e \equiv 1 \pmod 3$, (4) gives $\alpha \geqslant 2$. We may now take $\alpha = 2$ on absorbing powers of 3 in L. From (3), it is obvious that for all integers x,

$$ax^4 + bx^3 + cx^2 + dx \equiv 0 \pmod 3.$$

From $x \equiv \pm 1 \pmod 3$, then to mod 3,

$$a + c \pm (b + d) \equiv 0, \quad c \equiv -a, \quad d \equiv -b.$$

Now (3) gives identically in x,

$$\sum \big(a(x^4 - x^2) + b(x^3 - x) + e\big)^3 \equiv 4 \pmod 9,$$

$$\sum \big((x^3 - x)(ax + b) + e\big)^3 \equiv 4 \pmod 9.$$

Expanding and noting that $\sum a \equiv \sum b \equiv 0 \pmod 3$, we have

$$(x^3 - x)^2 \Big\{3\sum (ax + b)^2\Big\} + (x^3 - x)^3 \Big\{\sum (ax + b)^3\Big\} \equiv 0 \pmod 9,$$

$$3\sum (ax + b)^2 + (x^3 - x)\sum (ax + b)^3 \equiv 0 \pmod 9.$$

Take this as a congruence polynomial in x mod 3. Then $\sum (ax + b)^3 \equiv 0 \pmod 3$ identically in x. Now take residues mod 9 for integers x. Since $x^3 - x \equiv 0 \pmod 3$,

$$\sum (ax + b)^2 \equiv 0 \pmod 3.$$

From $x = 0, \pm 1$,

$$\sum a^2 \equiv 0, \qquad \sum b^2 \equiv 0.$$

Since $\sum a^3 = 0$, Lemma 2 gives $a = 3a_1$, etc.

Now (3) becomes

$$\sum (bx^3 + cx^2 + dx + e)^3 \equiv 4 \pmod 9.$$

Hence $\sum b^3 \equiv 0 \pmod 9$, and so (5) gives $b = 3b_1$, and then $c = 3c_1$, $d = 3d_1$. We now write (3) as

$$\sum (3a_1x^4 + 3b_1x^3 + 3c_1x^2 + 3d_1x + e)^3 = 9L_1x + M.$$

Replace x by $x/3^{1/4}$. Then

$$\sum (a_1x^4 + 3^{1/4}b_1x^3 + 3^{2/4}c_1x^2 + 3^{3/4}d_1x + e)^3 = 3^{7/4}L_1x + M. \tag{6}$$

Take this to mod $3^{5/4}$. Then

$$\sum (a_1x^4 + e)^3 + \sum (3^{1/4}b_1x^3)^3 \equiv 4. \tag{7}$$

From the coefficients of x^{12} in (6) and of x^8 in (7),

$$\sum a_1^3 = 0, \qquad \sum a_1^2 \equiv 0 \pmod{3^{2/4}},$$

and so $a_1 = 3a_2$ etc. Now (6) becomes

$$\sum (3^{1/4}b_1x^3 + 3^{2/4}c_1x^2 + 3^{3/4}d_1x + e)^3 \equiv 3^{7/4}L_1x + 4 \pmod 9, \tag{8}$$

or

$$\sum \big\{(3^{1/4}b_1x^3 + e)^3 + 3(3^{1/4}b_1x^3 + e)^2(3^{2/4}c_1x^2 + 3^{3/4}d_1x) + (3^{2/4}c_1x^2)^3\big\} \equiv 4 \pmod{3^{7/4}}.$$

From the coefficient of x^6 here and of x^9 in (8)

$$\sum b_1^2 \equiv 0 \pmod{3^{2/4}}, \qquad \sum b_1^3 \equiv 0 \pmod{3^{5/4}},$$

and so $b_1 = 3b_2$ etc. Now (6) becomes

$$\sum (3a_2x^4 + 3^{5/4}b_2x^3 + 3^{2/4}c_1x^2 + 3^{3/4}d_1x + e_1)^3 \equiv 3^{7/4}L_1x + M \pmod 9.$$

With $x \to x/3^{1/4}$, this becomes

$$\sum (a_2x^4 + 3^{2/4}b_2x^3 + c_1x^2 + 3^{2/4}d_1x + e)^3 \equiv 3^{6/4}L_1x + 4 \pmod 9,$$

or

$$\sum (a_2 x^4 + c_1 x^2 + e)^3 \equiv 4 \pmod{3^{6/4}}.$$

Since this is an identity in x, we can put $x^2 = \pm 1$. Then to mod 3

$$a_2 \pm c_1 \equiv 0, \quad a_2 \equiv 0, \quad c_1 \equiv 0,$$

and so $a_2 = 3a_3$, $c_1 = 3c_2$. Then (3) becomes

$$\sum (27a_3 x^4 + 9b_2 x^3 + 9c_2 x^2 + 3d_1 x + e)^3 = 9L_1 x + M.$$

With $x \to x/3^{2/3}$, this becomes

$$\sum (3^{1/3} a_3 x^4 + b_2 x^3 + 3^{2/3} c_2 x^2 + 3^{1/3} d_1 x + e)^3 = 3^{4/3} L_1 x + M.$$

Hence

$$\sum \{(b_2 x^3 + e)^3 + 3(a_3 x^4 + d_1 x)^3\} \equiv 4 \pmod{3^{4/3}}.$$

Then from the coefficients of x^9, x^6,

$$\sum b_2^3 \equiv 0 \pmod{3^{4/3}}, \quad \sum b_2^2 \equiv 0 \pmod{3^{1/3}},$$

and so $b_2 = 3b_3$ etc.

Now (3) becomes

$$\sum (27a_3 x^4 + 27b_3 x^3 + 9c_2 x^2 + 3d_1 x + e)^3 = 9L_1 x + M.$$

Write $x \to x/3^{3/4}$, and so

$$\sum (a_3 x^4 + 3^{3/4} b_3 x^3 + 3^{2/4} c_2 x^2 + 3^{1/4} d_1 x + e)^3 = 3^{5/4} L_1 x + M.$$

Then

$$\sum (a_3 x^4 + 3^{1/4} d_1 x + e)^3 \equiv 4 \pmod{3^{5/4}},$$

and

$$\sum (a_3 x^4 + e)^3 + (3^{1/4} d_1 x)^3 \equiv 4 \pmod{3^{5/4}}.$$

Hence from the coefficients of x^{12} and x^8,

$$\sum a_3^3 \equiv 0 \pmod 9, \quad \sum a_3^2 \equiv 0 \pmod 3,$$

and so $a_3 = 3a_4$. Hence replacing x by $x/3$ in (3),

$$\sum \left(\frac{a}{81} x^4 + \frac{b}{27} x^3 + \frac{c}{9} x^2 + \frac{d}{3} x + e\right)^3 = 3^{\alpha-1} Lx + M.$$

This is an integral representation with a smaller product for the leading term. This contradiction establishes the main result.

The success of the method depends on the existence of the congruences,

$$c \equiv -a \pmod 3, \quad d \equiv -b \pmod 3.$$

If we had tried a polynomial of the fifth degree, say,

$$P = ax^5 + bx^4 + cx^3 + dx^2 + ex + f,$$

we have now

$$a + c + e \equiv 0 \pmod 3, \quad b + d \equiv 0 \pmod 3.$$

These do not seem helpful, and so the possibility of representation by fifth degree polynomials is suggested ([1]).

([1]) Note added April 5, 1970. Dr. J. H. E. Cohn has shown that this is impossible.

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Reducibility of lacunary polynomials II

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*To the memory of my teachers
Wacław Sierpiński and Harold Davenport*

This paper is based on the results of [6] and the notation of that paper is retained. In particular $|f|$ is the degree of a polynomial $f(x)$ and $\|f\|$ is the sum of squares of the coefficients of f, supposed rational.

The aim of the paper is to prove the following theorem.

THEOREM. *For any nonzero integers A, B, and any polynomial $f(x)$ with integral coefficients, such that $f(0) \neq 0$ and $f(1) \neq -A-B$, there exist infinitely many irreducible polynomials $Ax^m+Bx^n+f(x)$ with $m > n > |f|$. One of them satisfies*

$$m < \exp\big((5|f|+2\log|AB|+7)(\|f\|+A^2+B^2)\big).$$

COROLLARY. *For any polynomial $f(x)$ with integral coefficients there exist infinitely many irreducible polynomials $g(x)$ with integral coefficients such that*

$$\|f-g\| \leqslant \begin{cases} 2 & \text{if } f(0) \neq 0, \\ 3 & \text{always.} \end{cases}$$

One of them, g_0, satisfies $|g_0| < \exp\big((5|f|+7)(\|f\|+3)\big)$.

The example $A = 12$, $B = 0, f(x) = 3x^9+8x^8+6x^7+9x^6+8x^4+3x^3+ +6x+5$ taken from [4], p. 4, shows that in the theorem above it would not be enough to assume $A^2+B^2 > 0$. On the other hand, in the first assertion of Corollary the constant 2 can probably be replaced by 1, but this was deduced in [5] from a hypothetical property of covering systems of congruences. Corollary gives a partial answer to a problem of Turán (see [5]). The complete answer would require $|g_0| \leqslant \max\{|f|, 1\}$.

LEMMA 1. *If $\sum_{\nu=1}^{k} a_\nu \zeta_l^{\alpha_\nu} = 0$, where a_ν, α_ν are integers, then either the sum $\sum$ can be divided into two vanishing summands or for all $\mu \leqslant \nu \leqslant k$*

$$l \,|\, (\alpha_\mu - \alpha_\nu) \exp\vartheta(k).$$