Nun sei bei genügend großem $C = c_4$

$$\log x > \log^2\frac{x}{y}, \qquad \log\frac{x}{y} = C\frac{\log_2 k}{\theta},$$

(3.11)

$$\xi = \max(y^C, k^C), \qquad \Delta = \frac{\log(x/y)}{\log y}$$

und

(3.12)
$$X = \sqrt{x}.$$

Insbesondere ist dann wegen (1.4) die Bedingung (2.5) und für $u = 1-\lambda$, $0 \leqslant \lambda \leqslant \theta/8$, $|v| \leqslant k/4$ die Bedingung (2.26) erfüllt.

Wir dürfen daher die Siebungleichung von Hilfssatz 5 mit den Hilfssätzen 3, 7, 8 und 9 unter Verwendung des Residuensatzes umformen. Dabei wird (3.10) in Verbindung mit (3.7), (3.8) und (3.9) in den Fällen (3.1) und (3.2) benötigt. Der Vergleich der beiden Hilfssätze 10 und 11 führt nunmehr zu der Abschätzung

$$\int_0^{\theta/8}\Bigg(\sum_{n<X}\frac{1+\chi_1(n)}{n^{1-\lambda}}\Lambda(n)(\log n)x^{-\lambda} + O\Big(x^{-\lambda}\big(1+\delta(\log^2\xi)\log x\big)\Big) + $$
$$+ O\left(\delta(\log^2 x)\int_1^X\left(\frac{\eta}{x}\right)^{\lambda}\frac{d\eta}{\eta}\right)\Bigg)\,d\lambda$$
$$\ll \Delta^3\log x + \Delta + \frac{1}{\log x} + \delta\log^2 x,$$

wegen (3.11) und (3.12) erhält man also

$$\sum_{p<X}\frac{1+\chi_1(p)}{p}\log^2 p \ll \frac{\log_2 k}{\theta} + \delta(\log^3 x + \log^3 k)$$

und damit wegen (1.1) und (1.4) sofort (1.2) bei genügend großem c_2.

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On the Siegel formula for ternary skew-hermitian forms

by

S. Raghavan and S. S. Rangachari (Bombay)

§ 1. Introduction. Let $\mathscr{A}$ be a simple algebra over an algebraic number field k and let ι be an involution in $\mathscr{A}$. Then $\mathscr{A}$ is the total-matrix algebra $\mathfrak{M}_m(\mathfrak{K})$ of m-rowed matrices over a division-algebra $\mathfrak{K}$ with an involution $\xi \to \tilde{\xi}$ and the involution ι takes x in $\mathscr{A}$ to $h^{-1}\cdot{}^t\tilde{x}\cdot h$ for a fixed m-rowed nonsingular matrix h satisfying the condition ${}^t h = \eta h$, $\eta = \pm 1$. Let X be a left $\mathscr{A}$-module of rank n and let G be the group of elements u in $\mathscr{A}$ for which $u\cdot u^{\iota} = 1$. G is precisely the group of u in $\mathfrak{M}_m(\mathfrak{K})$ for which ${}^t\tilde{u}\cdot h\cdot u = h$. Let δ, δ' be the dimensions over k of $\mathfrak{K}$ and of the space of elements ξ in $\mathfrak{K}$ for which $\tilde{\xi}' = \eta\xi$ and let $\varepsilon = \delta'/\delta$. Then for $m > 2n+4\varepsilon-2$, Weil has proved in [10] that the tempered measure $E(\Phi)$ defined by means of the "Eisenstein–Siegel series" on the space $\mathscr{S}(X_A)$ of Schwartz–Bruhat functions Φ associated with the adele-space X_A attached to X, coincides with the tempered measure $I(\Phi)$ defined by means of the "theta series" associated with G.

For $m = 2n+4\varepsilon-2$, the Eisenstein–Siegel series does not make sense, since it does not in general converge absolutely. It has been proved in [4] that when $n = 1$, $m = 4$, $\varepsilon = 1$, $k = \boldsymbol{Q}$, the field of rational numbers, and G is the orthogonal group of a quadratic form of index not exceeding 1 and with rational integral coefficients, one can define by using a limiting process, an Eisenstein–Siegel series and identify it with the corresponding measure $I(\Phi)$ defined by means of theta series.

Here we take up the case when $\mathscr{A}$ is the total matrix-algebra $\mathfrak{M}_3(\boldsymbol{D})$ over an indefinite quaternion division algebra $\boldsymbol{D}$ with the rational number field $\boldsymbol{Q}$ as centre and with an involution $\sim$ (of the first kind). Let h be the matrix of a non-degenerate skew-hermitian form defined over X which is now a vector-space of dimension 3 over $\boldsymbol{D}$. As pointed out earlier, the main difficulty here is that the Eisenstein–Siegel series $E(\Phi)$ as defined by Weil [10] does not converge absolutely and we have to modify its definition by following an idea of Hecke and Siegel [5]. However the "theta series" $I(\Phi)$ makes sense even in this case as shown by Weil [10].

§ 2. Notation and definitions. $\boldsymbol{D}$ stands for an indefinite quaternion division algebra over the field $\boldsymbol{Q}$ of rational numbers and let $\alpha \to \tilde{\alpha}$ be the involution given in $\boldsymbol{D}$. Let $\boldsymbol{R}$ $(= \boldsymbol{Q}_\infty)$, $\boldsymbol{Q}_p$ denote respectively the field of real numbers and the field of p-adic numbers (for a prime p). We denote $\boldsymbol{D} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_v$ by $\boldsymbol{D}_v$ for a valuation v of $\boldsymbol{Q}$ and denote the *discriminant* of $\boldsymbol{D}$ by d. The involution $\alpha \to \tilde{\alpha}$ of $\boldsymbol{D}$ extends in an obvious way to $\boldsymbol{D}_v$. By $\sigma_0(\alpha)$ and $N_0(\alpha)$, we mean respectively the *reduced trace* and *norm* of $\alpha \in \boldsymbol{D}$. They also extend to $\boldsymbol{D}_v$. For elements $\alpha \in \boldsymbol{D}$, we take the representation as two-rowed square matrices with elements in a (real) quadratic extension K of $\boldsymbol{Q}$. If

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

then "$\alpha = -\tilde{\alpha}$" is equivalent to the fact that the two-rowed real matrix αJ is symmetric.

Let X be a left vector space of dimension 3 over $\boldsymbol{D}$ and $\mathfrak{O}$, a maximal order in $\boldsymbol{D}$. Let $f(x)$ be a non-degenerate skew-hermitian form given on X. Taking the standard lattice $\mathfrak{O}^3$ in X, let $S = (s_{ij})$ be the associated 3-rowed skew-hermitian matrix. We may assume that $s_{ij} \in \mathfrak{O}$. Regarding S as a 6-rowed square matrix with elements in K, we see that

$$S \begin{pmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}$$

is symmetric, where 0 denotes the 2-rowed zero matrix. We denote by $\delta(S)$ the *discriminant* of S.

Let $N_0(x)$ be the quaternary form derived from the reduced norm $N_0(x)$, taking a fixed base of $\mathfrak{O}$ over the ring $\boldsymbol{Z}$ of rational integers. We denote by $\boldsymbol{D}^-$ the set of $\alpha \in \boldsymbol{D}$ with $\alpha = -\tilde{\alpha}$. Similarly we define $\boldsymbol{D}_v^-$. The restriction of $N_0(x)$ to $\boldsymbol{D}^-$ (resp. $\boldsymbol{D}_v^-$) is denoted by $N_0^-(x)$. For any order $\mathfrak{O}_1$ in $\boldsymbol{D}$, we set $\mathfrak{O}_1^- = \boldsymbol{D}^- \cap \mathfrak{O}_1$.

We observe that since $\boldsymbol{D}$ is *indefinite*, $\boldsymbol{D} \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ is isomorphic to the algebra $\mathfrak{M}_2(\boldsymbol{R})$ of all 2-rowed real matrices. For almost all v, $\boldsymbol{D}_v$ is isomorphic to the algebra $\mathfrak{M}_2(\boldsymbol{Q}_v)$ of 2-rowed matrices over $\boldsymbol{Q}_v$. We reserve the letter p to denote a non-archimedean prime.

We denote by Ps, P and Mp, the pseudo-symplectic group $\mathrm{Ps}(X|\mathscr{A})$, the parabolic group $\mathrm{P}(X|\mathscr{A})$ and the metaplectic group $\mathrm{Mp}(X|\mathscr{A})$ as defined in [9], respectively. We denote by $\mathrm{Ps}_{\boldsymbol{Q}}$, $\mathrm{P}_{\boldsymbol{Q}}$ and $\mathrm{Mp}_{\boldsymbol{Q}}$, the $\boldsymbol{Q}$-rational points of these groups respectively. Further Ps_A, P_A and Mp_A will denote the 'adelizations' of these groups. We denote the 'adelization' of X by X_A and the space of Schwartz–Bruhat functions on X_A by $\mathscr{S}(X_A)$.

For a matrix Y, ${}^tY$ and $\sigma(Y)$ denote the *transpose* and *trace* respectively. The ring of integers in $\boldsymbol{Q}_p$ is denoted by $\boldsymbol{Z}_p$. For $\alpha \in \boldsymbol{Q}_p$, $|\alpha|_p$ is the p-adic value of α normalized suitably.

§ 3. Bessel potentials. Let s be a real variable and let $s > 0$. For $X \in \mathfrak{M}_2(\boldsymbol{R}) = \boldsymbol{D} \otimes_{\boldsymbol{Q}} \boldsymbol{R}$, we define the *Bessel potential* $G_{s,\infty}(X)$ as the function in $L_1(\mathfrak{M}_2(\boldsymbol{R}))$ whose Fourier transform $G^*_{s,\infty}(Y)$ is just the function $\big(\det(E + {}^tYY)\big)^{-s}$. We know from [1], [2] that

i) $G_{s,\infty}(X) \geqslant 0$ for $X \in \boldsymbol{D} \otimes_{\boldsymbol{Q}} \boldsymbol{R}$,

ii) $\int_{\mathfrak{M}_2(\boldsymbol{R})} G_{s,\infty}(X)\,dX = 1 =$ value of $G^*_{s,\infty}(Y)$ at $Y = 0$,

iii) $|G_{s,\infty}(X)| \leqslant c_1 e^{-c_2(|\lambda_1|+|\lambda_2|)} |\lambda_1 \lambda_2|^{-\nu}$, for suitable ν and constants c_1, c_2, where λ_1, λ_2 are the eigenvalues of X (using 7.3 and (5.7′) of [2]).

For the p-adic completions $\boldsymbol{D}_p = \boldsymbol{D} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p$ of $\boldsymbol{D}$, the Bessel potential $G_{s,p}(X)$ for real $s > 0$ is defined as follows. Let for $Y \in \boldsymbol{D}_p$, the first elementary divisor $\lambda(Y)$ be the greatest common divisor of the elements of (the two-rowed matrix representation of) Y when $\boldsymbol{D}$ is unramified at p and the exact power of a generator of the unique prime ideal in $\boldsymbol{D}_p$ when $\boldsymbol{D}$ is ramified at p. We see that $\lambda(Y)$ is well-defined and in the former case, we see indeed that any $Y \in \boldsymbol{D}_p$ may be written as $\lambda(Y)Y_1$ with Y_1 primitive and integral and $|\lambda(Y)|_p$ is a power of p. Now $G_{s,p}(X)$ is defined as that function in $L_1(\boldsymbol{D}_p)$ whose Fourier transform $G^*_{s,p}(Y)$ is $\big(\mathrm{Max}(1, |\lambda(Y)|_p)\big)^{-s}$. Then $G_{s,p}(x)$ has the following properties:

i) $G_{s,p}(X) \geqslant 0$ for $X \neq 0$ in $\boldsymbol{D}_p$,

ii) $\int_{\boldsymbol{D}_p} G_{s,p}(X)\,dX_p = 1$ for the "normalized measure" dX_p,

iii) $G_{s,p}(X)$ has support contained in $\mathfrak{M}_2(\boldsymbol{Z}_p)$ for $\boldsymbol{D}$ unramified at p and in the unique maximal order $\mathfrak{O}_p$ of $\boldsymbol{D}_p = \boldsymbol{D} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_p$ in the case when $\boldsymbol{D}$ is ramified at p.

Let $\boldsymbol{D}_A$ be the adele-space corresponding to the affine variety $\boldsymbol{D}$. Then, for real $s > 0$ and for $X \in \boldsymbol{D}_A$, the Bessel potential $G_s(X)$ is defined on $\boldsymbol{D}_A$ as the function in $L^1(\boldsymbol{D}_A)$ whose Fourier transform $G^*_s(Y)$ is defined as

$$G^*_s(Y) = \det(E + {}^tY_\infty \cdot Y_\infty)^{-\frac{s}{2}} \prod_p \big(\mathrm{Max}(1, |\lambda(Y_p)|_p)\big)^{-s}$$

for $Y = (Y_\infty, \ldots, Y_p, \ldots) \in \boldsymbol{D}_A$.

§ 4. The Eisenstein-Siegel series. Associated to a Schwartz–Bruhat function Φ on $\boldsymbol{D}_A^m$, the Eisenstein–Siegel series is defined for $m > 3$, as follows. Denote by $\mathscr{A}$ the algebra of all m-rowed matrices over $\boldsymbol{D}$. Let Ps and P denote respectively the pseudo-symplectic group and the parabolic group associated with $\boldsymbol{D}_{\boldsymbol{Q}}^m$ considered as an $\mathscr{A}$-module of rank 1.

(See [10].) For $\sigma \in \mathrm{P}_Q$, define $(r_Q(\sigma)\Phi)(0) = \Phi(0)$ and for $\sigma = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{Ps}_Q$ with $\gamma \neq 0$, define

$$(r_Q(\sigma)\Phi)(0) = \int_{D_A^m} \Phi(x^*\gamma)\chi\big(\tfrac{1}{2}\gamma\tilde{\delta}f(x^*)\big)\,dx_A^* \tag{1}$$

where $\chi = \prod_v \chi_v$ is the fixed character of D_A setting it in duality with itself. Then Weil defines the Eisenstein–Siegel series $E(\Phi)$ associated with Φ by

$$E(\Phi) = \sum_{\sigma \in \mathrm{P}_Q/\mathrm{Ps}_Q} (r_Q(\sigma)\Phi)(0) \tag{2}$$

where σ runs over a complete set of representatives of the left cosets of Ps_Q modulo P_Q. We may rewrite (2) as

$$E(\Phi) = \Phi(0) + \sum_{i^* \in I(D_Q^m)^* = D_Q^-} F_\Phi^*(i^*)$$

where i^* runs over all the elements of D_Q such that $i^* = -\tilde{i}^*$ and where

$$F_\Phi^*(i^*) = \int_{D_A^m} \Phi(x)\chi\big(i^* f(x)\big)\,dx_A. \tag{3}$$

In (1) and (3), dx_A^* and dx_A denote the Tamagawa measure on D_A^m. The series (2) is known to converge absolutely only for $m > 3$. In the case $m = 3$ which is our concern, we modify the definition of $E(\Phi)$ by introducing a parameter $s > 0$. For real $s > 0$, we introduce the series

$$E(\Phi, s) = \Phi(0) + \sum_{i^* \in D_Q^-} F_\Phi^*(i^*) G_s^*(i^*) \tag{4}$$

where for every Schwartz–Bruhat function Φ on D_A^3, the corresponding F_Φ^* is defined by (3) and i^* runs over all the skew-symmetric elements of D_Q. The convergence of the series (4) may be proved as follows just as in [10]. In fact, let Mp_A be the adele-group corresponding to the metaplectic group Mp associated with Ps. Let π be the canonical projection from Mp_A to the adele-group Ps_A associated with Ps. There exists then a function $f_{\Phi,s}$ on Ps_A defined by

$$f_{\Phi,s}(\pi(\mathbf{s})) = (\mathbf{s}\Phi)(0)\,\lambda_\infty^{-s}(\sigma_\infty)\prod_p |\eta(\gamma, \delta)|_p^{-s}$$

where $\mathbf{s}$ is an element of Mp_A with $\pi(\mathbf{s}) = (\sigma, f)$ and $\sigma = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = (\sigma_\infty, \sigma_p, \ldots)$ being the "first component" of $\pi(\mathbf{s})$,

$$|\eta(\gamma,\delta)|_p = \begin{cases} |N_0(\gamma)|_p \mathrm{Max}(1, |\lambda(\gamma^{-1}\delta)|_p) & \text{if } D_p \simeq \mathfrak{M}_2(Q_p),\\ |\gamma|_p \mathrm{Max}(1, |\lambda(\gamma^{-1}\delta)|_p) & \text{if } D \text{ is ramified at } p.\end{cases}$$

For non-archimedean primes p, $\lambda(\gamma^{-1}\delta)$ is as defined on page 329 and $\lambda_\infty(\sigma_\infty)$ is the product of the two simple roots $\lambda_1\lambda_2^{-1}$ and λ_2^2 of σ_∞ regarded as an element of $\mathrm{Sp}(4, \mathbf{R})$. For any $p \in \mathrm{P}_A$, we have

$$f_{\Phi,s}(p\pi(\mathbf{s})) = \psi(p)|\mu|_A^{-s}$$

where $\psi(p) = |\mu|_A^{-1/2}$ for p of the form $t(q)d(\mu)$ with $t(q)$, $d(\mu)$ having their "first components" $\begin{pmatrix}1 & q\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}\mu & 0\\ 0 & \mu^{-1}\end{pmatrix}$ respectively. We apply the Godement criterion for the convergence of Eisenstein series as in [10], in order to conclude that the series $E(\Phi, s)$ defined by (4) converges absolutely for real $s > 0$, uniformly for Φ lying in a compact subset of $\mathscr{S}(D_A^3)$.

It is now immediate that the series

$$\sum_{i^* \in D_Q^-} F_\Phi^*(g^* + i^*) G_s^*(g^* + i^*) \tag{5}$$

converges absolutely for $s > 0$ and uniformly for g^* and Φ lying in compact subsets of D_A and $\mathscr{S}(D_A^3)$ respectively. In fact, for all $i^* \in D_Q^-$, we have

$$c_3 \leqslant \frac{G_s^*(g^* + i^*)}{G_s^*(i^*)} \leqslant c_4 \tag{6}$$

for suitable constants c_3, c_4 depending only on s and the compact set to which g^* belongs. This results from the existence of constants c_5, c_6 such that for $X \in D_Q^-$,

$$c_5 \leqslant \det\big(E + (X+Y)^2\big)^{-s}\big(\det(E + X^2)\big)^s \leqslant c_6,$$

provided that Y lies in a compact set of D_A^-. Further

$$F_\Phi^*(g^* + i^*) = F_{\Phi_{g^*}}^*(i^*), \quad \text{where} \quad \Phi_{g^*}(x) = \Phi(x)\chi\big(g^* f(x)\big).$$

A similar result holds also for the Bessel potential $G_{s,p}^*$ and the inequalities (6) are immediate. Now when Φ and g^* lie in compact subsets of $\mathscr{S}(D_A^3)$ and D_A respectively, so does $\Phi_{g^*}^*$ and hence the convergence of the series is a consequence of the uniform convergence of $E(\Phi, s)$.

Remark 1. Denoting $F_\Phi^*(g^*)G_s^*(g^*)$ by $F_{\Phi,s}^*(g^*)$ for $g^* \in D_A$, we have just seen that $\sum_{\gamma^* \in D_Q^-} |F_{\Phi,s}^*(g^* + \gamma^*)|$ converges uniformly for Φ and g^* lying in compact subsets of $\mathscr{S}(D_A^3)$ and D_A respectively. By the same arguments as in [10], we see that $F_{\Phi,s}^*(g^*) \in L_1(D_A)$ for every $\Phi \in \mathscr{S}(D_A^3)$.

§ 5. A Poisson summation formula. Our object is to define the Eisenstein–Siegel series $E(\Phi)$ as $\lim_{s\to 0} E(\Phi, s)$ for every $\Phi \in \mathcal{S}(\boldsymbol{D}_A^3)$ if it exists. In order to study the behaviour of $E(\Phi, s)$ as s tends to 0 from above, we need to prove a Poisson summation formula, as in [10]. It suffices to assume that $0 < s < 1$. Let $\boldsymbol{D}_A^-$ denote the adele-space corresponding to $\boldsymbol{D}_{\boldsymbol{Q}}^-$ and $d^- x_A$ the canonical measure on $\boldsymbol{D}_A^-$.

Proposition 1. *Let W be a compact neighbourhood of 0 in $\boldsymbol{D}_A^-$ and let φ_W be a non-negative continuous function on $\boldsymbol{D}_A^-$ with support contained in W such that $\int_{\boldsymbol{D}_A^-} \varphi_W(x) d^- x_A = 1$. For $g, h \in \boldsymbol{D}_A^-$, define*

$$t_W(h) = (\tilde{\varphi}_W * G_s * \delta_{\{g\}} * \varphi_W)(h)$$

where $*$ denotes the convolution-product in $L_1(\boldsymbol{D}_A^-)$ and $\delta_{\{g\}}$ stands for the Dirac distribution with mass 1 at g. For $\Phi \in L_1\big((\boldsymbol{D}_A^-)^3\big)$, if the integral

$$S(\Phi, g) = \int_{\boldsymbol{D}_A^-} F_\Phi^*(-g^*) G_s^*(g^*) \chi(-gg^*) d^- g_A^*$$

converges absolutely, then

$$S(\Phi, g) = \lim_{W\to\{0\}} \int_{\boldsymbol{D}_A^3} \Phi(x) t_W\big(f(x)\big) dx_A \tag{7}$$

the limit being taken over a filter of neighbourhoods W of 0 in $\boldsymbol{D}_A^-$. If $S(\Phi, g)$ converges uniformly for Φ lying in a bounded subset of $L_1\big((\boldsymbol{D}_A)^3\big)$, then the limit taken over W is also uniform on that bounded subset.

Proof. The proposition is a consequence of Lemma 1 of Weil [10], by taking $\boldsymbol{D}_A^-$ and $(\boldsymbol{D}_A)^3$ for G and X respectively and $\tau(g^*) = G_s^*(g^*) \times \times \chi(-gg^*)$. In view of Remark 1, the conditions of the proposition are fulfilled for $\Phi \in \mathcal{S}\big((\boldsymbol{D}_A)^3\big)$ and we have therefore the expression (7) for $S(\Phi, g)$ as a limit with respect to W.

Denoting $S(\Phi, g)$ as $F_{\Phi,s}(g)$, this is, by definition, the value at g of the Fourier transform of $F_{\Phi,s}^*(g^*)$. Now clearly $F_{\Phi,s}(g)$ is continuous, bounded and non-negative for non-negative functions Φ in $\mathcal{S}\big((\boldsymbol{D}_A)^3\big)$. Hence $F_{\Phi,s}^*(g^*)$ is a continuous function of positive type and is the Fourier transform of a bounded positive measure, which is nothing but $F_{\Phi,s}(g)$. Hence $F_{\Phi,s}(g)$ is in $L_1(\boldsymbol{D}_A^-)$. Now any Φ in $\mathcal{S}\big((\boldsymbol{D}_A)^3\big)$ may be written as the difference of two non-negative functions in $\mathcal{S}\big((\boldsymbol{D}_A)^3\big)$ so that $F_{\Phi,s}(g)$ is in $L_1(\boldsymbol{D}_A^-)$ for every Φ in $\mathcal{S}(\boldsymbol{D}_A^3)$. Thus $F_{\Phi,s}^*(g^*)$ is the Fourier transform of $F_{\Phi,s}$ for every Φ in $\mathcal{S}(\boldsymbol{D}_A^3)$.

Remark 2. As s tends to 0, $F_{\Phi,s}^*$ tends to F_Φ^* so that $F_{\Phi,s}$ tends to F_Φ. From [10], we know however that $F_\Phi(\gamma) = \int \Phi \, d\mu_\gamma$ with support of the measure μ_γ being contained in $f^{-1}(\{\gamma\})$.

Remark 3. If $\Phi(x) = \prod_v \Phi_v(x_v)$ with $\Phi_v \in \mathcal{S}(\boldsymbol{D}_v^3)$, then we see that

$$F_{\Phi,s}(g) = \prod_v F_{\Phi_v,s}(g_v) \quad \text{for} \quad g = (g_v),$$

where

$$F_{\Phi_v,s}(g_v) = \int_{\boldsymbol{D}_v^-} F_{\Phi_v}^*(-g_v^*) G_{s,v}^*(g_v^*) \chi_v(-g_v g_v^*) dg_v^*.$$

Applying Lemma 1 of Weil [10], to

$$G = \boldsymbol{D}_v^-, \quad \chi = \chi_v \quad \text{and} \quad \tau(g_v^*) = G_{s,v}^*(g_v^*) \chi_v(-g_v g_v^*),$$

we see that

$$F_{\Phi_v,s}(g_v) = \lim_{W_v\to\{0\}} \int_{\boldsymbol{D}_v^3} \Phi_v(x_v) t_{W_v}\big(f(x_v)\big) dx_v$$

the limit being taken over compact neighbourhoods W_v tending to 0.

We shall compute $F_{\Phi_p,s}(g_p)$ explicitly, for a special function Φ_p, namely, the characteristic function φ_p of a lattice L_p in $\boldsymbol{D}_p^3$. If $\mathfrak{O}_p$ denotes the (standard) maximal order in $\boldsymbol{D}_p$, we assume that the given skew-hermitian form $f(x)$ has on L_p its values in $\mathfrak{O}_p$ and we denote $F_{\Phi_p,s}(g_p)$ by $b_p(s, g_p)$. We further suppose that $\chi_p(g_p) = e^{2\pi i \langle\sigma_0(g_p)\rangle}$ for $g_p \in \boldsymbol{D}_p$; here σ_0 denotes the reduced trace from $\boldsymbol{D}_p$ to $\boldsymbol{Q}_p$ and for any $a \in \boldsymbol{Q}_p$, $\langle a\rangle$ denotes its "principal part". For a lattice M, let M' denote its χ_p-dual.

We choose a special filter of neighbourhoods W_p, namely $W_p = p^{n_p}\mathfrak{M}_2(\boldsymbol{Z}_p)$ if $\boldsymbol{D}$ is unramified at p and $W_p = p^{n_p}\mathfrak{O}_p$ if $\boldsymbol{D}_p$ is a division algebra with $\mathfrak{O}_p$ as its unique maximal order. We may then set $\varphi_{W_p}(g_p) = p^{3n_p}$ or 0 according as g_p is in W_p or not. The Fourier transform $\varphi_{W_p}^*$ of φ_{W_p} has support contained in $p^{-n_p}\mathfrak{M}_2(\boldsymbol{Z}_p)$ or $p^{-n_p}\mathfrak{O}_p'$ according as $\boldsymbol{D}$ is unramified or ramified at p. We have therefore

$$\begin{aligned} t_{W_p}\big(f(x_p)+g_p\big) &= \int_{\boldsymbol{D}_p^-} \varphi_{W_p}^*(g_p^*) \chi_p\big(g_p^*(f(x_p)+g_p)\big)\big(\mathrm{Max}(1, |\lambda(g_p^*)|_p)\big)^{-s} d^- g_p^* \\ &= \int_{p^{-n_p}\hat{\mathfrak{O}}_p} \chi_p\big(g_p^*(f(x_p)+g_p)\big)\big(\mathrm{Max}(1, |\lambda(g_p^*)|_p)\big)^{-s} d^- g_p^*, \end{aligned}$$

where $\hat{\mathfrak{O}}_p = \boldsymbol{D}_p^- \cap \mathfrak{M}_2(\boldsymbol{Z}_p)$ or $\boldsymbol{D}_p^- \cap \mathfrak{O}_p'$ and in the sequel ϱ_p is the measure of $\hat{\mathfrak{O}}_p$.

Hence writing

$$p^{-n_p}\hat{\mathfrak{O}}_p = \hat{\mathfrak{O}}_p \bigcup_{\nu=1}^{n_p} p^{-\nu}(\hat{\mathfrak{O}}_p - p\hat{\mathfrak{O}}_p)$$

we see that

$$t_{W_p}(f(x_p)+g_p) = \sum_{\nu=1}^{n_p} p^{-3\nu s} \int\limits_{p^{-\nu}(\hat{\mathfrak{O}}_p - p\hat{\mathfrak{O}}_p)} + \ldots + \int\limits_{\hat{\mathfrak{O}}_p} \chi_p\big(g_p^*(f(x_p)+g_p)\big)\, d^- g_p^*$$

$$= \sum_{\nu=1}^{n_p} p^{-3\nu s}\Big(\int\limits_{p^{-\nu}\hat{\mathfrak{O}}_p} - \int\limits_{p^{-(\nu-1)}\hat{\mathfrak{O}}_p}\Big) + \begin{cases} \varrho_p, & \text{if } f(x_p)+g_p \in \mathfrak{O} \\ 0, & \text{otherwise}\end{cases}$$

$$= \sum_{\nu=1}^{n_p} p^{-3\nu s}\Big[p^{3\nu} \int\limits_{\hat{\mathfrak{O}}_p} \chi_p\Big(\frac{g_p^*}{p^\nu}(f(x_p)+g_p)\Big)\, d^- g_p^* -$$

$$- p^{3(\nu-1)} \int\limits_{\hat{\mathfrak{O}}_p} \chi_p\big(p^{-(\nu-1)} g_p^*(f(x_p)+g_p)\big)\, d^- g_p^*\Big] + \begin{cases}\varrho_p \\ 0.\end{cases}$$

If $f(x_p)+g_p \notin \hat{\mathfrak{O}}'_p$, then $t_{W_p}(f(x_p)+g_p) = 0$. We may therefore ass that $f(L_p)+g_p \subset \hat{\mathfrak{O}}'_p$, i.e. $g_p \in \hat{\mathfrak{O}}'_p$. Denoting $\int\limits_{L_p} dx_p$ by $m(L_p)$, we

$$\int\limits_{D_p^3} \Phi_p(x_p)\, t_{W_p}(f(x_p)+g_p)\, dx_p$$

$$= \int\limits_{L_p}\Big[1+\sum_{\nu=1}^{n_p} p^{-3\nu s}\Big\{p^{3\nu}\int\limits_{\hat{\mathfrak{O}}_p}\chi_p\Big(\frac{g_p^*}{p^\nu}(f(x_p)+g_p)\Big)\, d^- g_p^* -$$

$$- p^{3(\nu-1)} \int\limits_{\hat{\mathfrak{O}}_p} \chi_p\Big(\frac{g_p^*}{p^{\nu-1}}(f(x_p)+y_p)\Big)\, d^- g_p^*\Big\}$$

$$= m(L_p)\Big[1+\sum_{\nu=1}^{n_p} p^{-3\nu s}\Big\{p^{3\nu} p^{-12\nu} \sum_{x_p \in L_p \bmod p^\nu L_p} \int\limits_{\hat{\mathfrak{O}}_p} \chi_p\big(p^{-\nu} g_p^*(f(x_p)+g_p)\big)\, d^-$$

$$- p^{3(\nu-1)} p^{-12(\nu-1)} \sum_{x_p \in L_p \bmod p^{\nu-1} L_p} \int\limits_{\hat{\mathfrak{O}}_p} \chi_p\big(p^{-(\nu-1)} g_p^*(f(x_p)+g_p)\big)\, d^-$$

For x_p in L_p modulo $p^\nu L_p$, we have

$$\int\limits_{\hat{\mathfrak{O}}_p} \chi_p\big(p^{-\nu} g_p^*(f(x_p)+g_p)\big)\, d^- g_p^* = \begin{cases}\varrho_p, & \text{if } f(x_p)+g_p \in p^\nu \hat{\mathfrak{O}}'_p, \\ 0, & \text{otherwise,}\end{cases}$$

since χ_p is a non-trivial character of $\boldsymbol{D}_p$.

Denote by $A_{p^\nu; L_p}(f(x), \mu)$ the number of distinct x_p in L_p mc $p^\nu L_p$ for which $f(x) - \mu \in p^\nu \hat{\mathfrak{O}}'_p$. Then

$$\varrho_p^{-1} \int\limits_{D_p^3} \Phi_p(x_p)\, t_{W_p}(f(x_p)+g_p)\, dx_p$$

$$= m(L_p)\Big[1+\sum_{\nu=1}^{n_p} p^{-3\nu s}\{D_{p^\nu, L_p}(f(x_p), -g_p) - D_{p^{\nu-1}, L_p}(f(x_p), -g_p)$$

where, by definition, for $\nu \geqslant 0$

(8) $$D_{p^\nu, L_p}(f(x_p), -g_p) = p^{-9\nu} A_{p^\nu, L_p}(f(x_p), -g_p).$$

Setting

$$B_{p^\nu, L_p} = B_{p^\nu, L_p}(f(x_p), -g_p) = D_{p^\nu, L_p}(f(x_p), -g_p) - D_{p^{\nu-1}, L_p}(f(x_p), -g_p),$$

we have

$$\frac{1}{\varrho_p m(L_p)} \int\limits_{D_p^3} \Phi_p(x_p)\, t_{W_p}(f(x_p)+g_p)\, dx_p = 1+\sum_{\nu=1}^{n_p} p^{-3\nu s} B_{p^\nu, L_p}(f(x_p), -g_p).$$

We see then that

$$\lim_{W_p \to \{0\}} \int\limits_{D_p^3} \Phi_p(x_p)\, t_{W_p}(f(x_p)+g_p)\, dx_p = \lim_{n_p \to \infty} \int\limits_{D_p^3} \Phi_p(x_p)\, t_{p^{n_p}\mathfrak{O}_p^-}(f(x_p)+g_p)\, dx_p$$

$$= \varrho_p m(L_p)\, b_{L_p}(s, -g_p)$$

where $b_{L_p}(s, -g_p) = 1+\sum_{\nu=1}^{\infty} p^{-3\nu s} B_{p^\nu, L_p}(f(x_p), -g_p)$.

Referred to a base of $\mathfrak{O}^3$ in $\boldsymbol{D}^3$, let the skew-hermitian form $f(x)$ be represented by the matrix S with elements in a maximal order $\mathfrak{O}$ of D. Denote by $D = D(\Phi, f)$ the product of 2, odd primes p dividing $\delta(S)$, odd primes p for which Φ_p is not the characteristic function of the standard lattice $\mathfrak{O}_p^3$ in $\boldsymbol{D}_p^3$ and the odd primes p over which $\boldsymbol{D}$ is ramified. Then for a prime p not dividing $D_1 = N_0(g)D$, we can show that $B_{p^\nu, \mathfrak{O}_p^3}(f(x_p), -g) = 0$ for $\nu > 1$ and further that

$$B_{p, \mathfrak{O}_p^3}(f(x_p), -g) = -\Big(\frac{N_0(g)}{p}\Big) p^{-5} - \Big(\frac{\delta(S)}{p}\Big)\Big(1+\Big(\frac{N_0(g)}{p}\Big)p\Big) p^{-3}.$$

Further, we observe that there exists an integer D_2 divisible by D_1 such that for a prime p not dividing D_2, we have

(9) $$b_{L_p}(s, -g) = 1 - p^{-3s}\Big[\Big(\frac{N_0(g)}{p}\Big) p^{-5} + \Big(\frac{\delta(S)}{p}\Big) p^{-3}\Big(1+\Big(\frac{N_0(g)}{p}\Big)p\Big)\Big]$$

$$= 1 - p^{-5-3s}\Big(\frac{N_0(g)}{p}\Big) - \Big(\frac{\delta(S)}{p}\Big) p^{-3-3s} - \Big(\frac{N_0(g)\,\delta(S)}{p}\Big) p^{-2-3s}.$$

For functions $\Phi = \prod_v \Phi_v(x_v)$ in $\mathscr{S}(\boldsymbol{D}_A^3)$ with Φ_p equal to the characteristic function of $\mathfrak{O}_p^3$, we have then for $g \neq 0$ in $\mathfrak{O}^-$ that,

(10) $$F_{\Phi, s}(g) = \prod_{p \mid D_1} b_{L_p}(s, -g) \prod_{p \nmid D_1}\Big(1 - \Big(\frac{\delta(S) N_0(g)}{p}\Big) p^{-2-3s} + O(p^{-3})\Big)$$

by choosing $W_p = p^{n_p}\mathfrak{O}_p$ and $n_p = -\log(|n!|_p)/\log p$ and taking the limit as n tends to infinity.

If $g = 0$ and if p is a prime not dividing a suitable multiple D_3 of D we can show that

(11) $$|B_{p^\nu, \mathfrak{D}_p^3}(f(x), 0)| \leqslant p^{-(\nu-1/4)}, \quad \nu \geqslant 2$$

and that

$$B_{p,\mathfrak{D}_p^3}(f(x), 0) = p^{-6}(p^3 - p^2) + \left(\frac{-\delta(S)}{p}\right) p^{-3}(p^2 - 1).$$

Now

(12) $$b_{L_p}(s, 0) = 1 + p^{-(1+s)}\left(\frac{-\delta(S)}{p}\right) + p^{-(3+s)} - p^{-(4+s)} - \left(\frac{-\delta(S)}{p}\right) p^{-(3+s)} + M_p(s)$$

where $|M_p(s)| \leqslant \sum_{\nu=2}^{\infty} p^{-(\nu-1/4)+\nu|s|} \leqslant p^{-3/2}$ for small s. Thus, from (12) we see that $\prod_{p\nmid D_3} b_{L_p}(s, 0)$ converges absolutely for $s > 0$, provided tha $-\delta(S)$ is not a square.

It remains then to consider the case when $-\delta(S)$ is a square In this case $f(x)$ cannot be a zero-form (i.e. $f(x)$ cannot represent 0 non trivially); for, if it did, there would exist $\lambda = -\bar{\lambda} \neq 0$ such tha $\delta(S) \in N_0(\lambda) \boldsymbol{Q}^{*2}$, i.e. $N_0(\lambda) = -a^2$ for some a in $\boldsymbol{Q}^*$ since $-\delta(S) \in \boldsymbol{Q}^*$ i.e. $\lambda^2 = -N_0(\lambda) = a^2$, i.e. $\lambda = \pm a \in \boldsymbol{Q}^*$ contradicting the fact tha $\lambda = -\bar{\lambda} \neq 0$. We now observe that $f(x)$ represents 0 non-trivially i $\boldsymbol{D}_p^3$ if and only if there exists $\lambda = -\bar{\lambda} \neq 0$ in $\boldsymbol{D}_p$ such that $N_0(\lambda) = \delta(S$ Hence the set of primes p for which $f(x)$ fails to represent zero in $\boldsymbol{L}$ is precisely the set of primes at which the norm-form $N_0(x)$ of $\boldsymbol{D}$ fai to represent zero in $\boldsymbol{D}_p$. (In view of $-\delta(S)$ being a square, the norn form $N_0(x)$ is just $N_0^-(y) - \delta(S)x_0^2$, writing $x = x_0 + y$ with $x_0 \in \boldsymbol{Q}$.) Thu the set of primes p for which $f(x)$ does not represent 0 over $\boldsymbol{D}_p^3$ is eve in number. (In the special case when $-\delta(S)$ is a square, the Hasse Theorei for $f(x)$ is just the Hasse–Brauer theorem for $\boldsymbol{D}$.) Now $\boldsymbol{D}$ is a divisio algebra so that the number of primes p for which $f(x)$ fails to represent in $\boldsymbol{D}_p^3$ is at least two. Let p_0, p_1 be two primes such that $f(x)$ does nc represent 0 in $\boldsymbol{D}_{p_0}^3$ and $\boldsymbol{D}_{p_1}^3$. Then, in view of Remark 2, we see that

$$\operatorname*{Lt}_{s\to 0} b_{p_i}(s, 0) = \operatorname*{Lt}_{s\to 0} F_{\Phi_{p_i}, s}(0) = F_{\Phi_{p_i}}(0) = 0, \quad i = 0, 1.$$

Now we see that

(13) $$F_{\Phi,s}(0) = b_\infty(s, 0)\zeta(s+1) \prod_{p|D_3} \{b_{L_p}(s, 0)(1 - p^{-(1+s)})\} \times \prod_{p\nmid D_3} (1 - p^{-(1+s)})\big(1 + p^{-(1+s)} - p^{-(4+s)} + M_p(s)\big)$$
$$= b_\infty(s, 0)\zeta(s+1) \prod_{p|D_3} \{b_{L_p}(s, 0)(1 - p^{-(1+s)})\} \prod_{p\nmid D_3} (1 - p^{-(2+2s)} - p^{-(4+s)} + \ldots$$

where $\zeta(s)$ denotes the Riemann zeta function.

Since $\boldsymbol{D}$ is an indefinite quaternion algebra, $N_0(x)$ represents 0 in $\boldsymbol{R}^3$ and hence by our remarks above, $f(x)$ represents 0 in $\boldsymbol{D}_\infty^3$. Hence we may assume that p_0 and p_1 are non-archimedean primes. Now, by (13)

$$F_{\Phi,s}(0) = b_\infty(s, 0) b_{L_{p_0}}(s, 0)\zeta(s+1)(1 - p_0^{-(1+s)}) \times \times \Big\{\prod_{p\nmid D_3} (1 - p^{-(2+2s)} - \ldots) \prod_{\substack{p\neq p_0 \\ p|D_3}} \{(1 - p^{-(1+s)}) b_{L_p}(s, 0)\Big\}.$$

Since $b_{L_{p_0}}(s, 0)$ vanishes at $s = 0$, $F_{\Phi,s}(0)$ is regular at $s = 0$. Furthermore,

$$F_{\Phi,s}(0) = b_\infty(s, 0)(1 - p_0^{-3s})(1 + D_{p_0} p^{-3s} + \ldots)(1/s + \gamma + \ldots) \times \times (c_0 + c_1 s + \ldots) \prod_{\substack{p|D_3 \\ p\neq p_0}} \{(1 - p^{-(1+s)}) b_{L_p}(s, 0)\}$$

has at $s = 0$, the constant term $c \prod_{\substack{p|D_3 \\ p\neq p_0}} b_{L_p}(0, 0)$ and since $b_{L_{p_1}}(0, 0) = 0$, it follows that

$$\lim_{s\to 0} F_{\Phi,s}(0) = 0.$$

Proposition 2. *For any $\Phi \in \mathscr{S}(\boldsymbol{D}_A^3)$ and for $s > 0$, let*

$$\dot{S}(\Phi) = \int_{(\boldsymbol{D}_A^-/\boldsymbol{D}_{\boldsymbol{Q}}^-)^* = \boldsymbol{D}_{\boldsymbol{Q}}^-} F_\Phi^*(-\gamma^*) G_s^*(\gamma^*) d\gamma^* = \sum_{\gamma^* \in \boldsymbol{D}_{\boldsymbol{Q}}^-} F_\Phi^*(-\gamma^*) G_s^*(\gamma^*).$$

(This series converges absolutely, uniformly on compact subsets of $\mathscr{S}(\boldsymbol{D}_A^3)$.) Then $\dot{S}$ defines a tempered measure on $\mathscr{S}(\boldsymbol{D}_A^3)$ and further

$$\dot{S}(\Phi) = \sum_{\gamma \in \boldsymbol{D}_{\boldsymbol{Q}}^-} F_{\Phi,s}(\gamma).$$

Before we prove Proposition 2, we need to introduce some notation. For x in $\boldsymbol{D}_A^-$, denote by $\dot{x}$ the image of x under the homomorphism $\boldsymbol{D}_A^- \to \boldsymbol{D}_A^-/\boldsymbol{D}_{\boldsymbol{Q}}^-$ (which may be identified with $(A_{\boldsymbol{Q}}/\boldsymbol{Q})^3$). Corresponding to the Bessel potential G_s on $\boldsymbol{D}_A^-$, we define its "periodization" $\dot{G}_s$ on $\boldsymbol{D}_A^-/\boldsymbol{D}_{\boldsymbol{Q}}^-$ as a distribution by the "scalar product formula",

(14) $$\dot{G}_s(\dot{\varphi}) = (\dot{G}_s^*, \dot{\varphi}^*)_{\boldsymbol{D}_{\boldsymbol{Q}}^-} = \sum_{\gamma \in \boldsymbol{D}_{\boldsymbol{Q}}^-} G_s^*(\gamma)\varphi^*(\gamma)$$

where φ is any function in $\mathscr{S}(\boldsymbol{D}_A^-)$ with compact support and $\dot{\varphi}(\dot{g}) = \sum_{\gamma \in \boldsymbol{D}_{\boldsymbol{Q}}^-} \varphi(g + \gamma)$ for any $g \in \boldsymbol{D}_A^-$. Since φ_p is the characteristic function of $\mathfrak{D}_p^-$ for almost all primes p, the summation over γ on the right hand side of (14) is carried out essentially over the elements of $\mathfrak{D}_1 \cap \boldsymbol{D}^-$ for

an order $\mathfrak{O}_1$ in $\boldsymbol{D}_{\boldsymbol{Q}}$. The convergence of the series on the right hand sid
may be verified as follows:

$$\Big|\sum_{\gamma\in \boldsymbol{D}_{\boldsymbol{Q}}^-} G_s^*(\gamma)\varphi^*(\gamma)\Big| = \Big|\sum_{\gamma\in\mathfrak{O}_1\cap \boldsymbol{D}^-} G_{s,\infty}^*(\gamma)\Big(\prod_p G_{s,p}^*(\gamma)\varphi_p^*(\gamma)\Big)\varphi_\infty^*(\gamma)\Big|$$
$$= a'\sum_{\gamma\in\mathfrak{O}_1\cap \boldsymbol{D}^-} G_{s,\infty}^*(\gamma)\varphi_\infty^*(\gamma),$$

since $G_{s,p}^*(\gamma)\leqslant 1$ and $G_{s,p}^*(\gamma)=1$ for almost all p. Now since $\varphi_\infty\in\mathscr{S}(\boldsymbol{D}_\infty$ we know that, for any $\beta>0$, $|\varphi_\infty^*(\gamma)|\leqslant c(\beta)(x_1^2+x_2^2+x_3^2)^{-\beta}$ wher $\gamma = x_1\omega_1+x_2\omega_2+x_3\omega_3$. Here $\omega_1,\omega_2,\omega_3$ is a $\boldsymbol{Q}$-base of $\boldsymbol{D}^-$, x_1, x_2, x being rational numbers with bounded denominator d (uniform for a $\gamma\in\mathfrak{O}_1\cap\boldsymbol{D}^-$). Since

$$\sum_{\gamma\in\mathfrak{O}_1\cap \boldsymbol{D}^-}(x_1^2+x_2^2+x_3^2)^{-\beta}\big(1+\lambda(x_1^2+x_2^2+x_3^2)\big)^{-s} < c_7 d^{-\beta}\zeta^3(2\beta) < \infty,$$

our assertion is true.

For any $\dot g\in \boldsymbol{D}_{\boldsymbol{A}}^-/\boldsymbol{D}_{\boldsymbol{Q}}^-$, we define the distribution $t_{\dot W}(\dot g) = (\tilde{\dot\varphi}_{\dot W}*\dot G_s*\dot\varphi_{\dot W})($ where W is a compact neighbourhood of 0 in $\boldsymbol{D}_{\boldsymbol{A}}^-$ such that for n two distinct x, y in W, $x-y$ lies in $\boldsymbol{D}_{\boldsymbol{Q}}^-$. We take $\dot W = W+\boldsymbol{D}_{\boldsymbol{Q}}$. Let φ be a non-negative continuous function with support contained in W an $\int_{\boldsymbol{D}_{\boldsymbol{A}}^-}\varphi_W d^- x_A = 1$. Then $\dot\varphi_{\dot W}(\dot g) = \sum_{\gamma\in\boldsymbol{D}_{\boldsymbol{Q}}^-}\varphi_W(g+\gamma)$. Now $t_{\dot W}(\dot g)$ is a contir uous function on $\boldsymbol{D}_{\boldsymbol{A}}^-/\boldsymbol{D}_{\boldsymbol{Q}}^-$ with the Fourier coefficient

$$c_\gamma = \int_{\boldsymbol{D}_{\boldsymbol{A}}^-/\boldsymbol{D}_{\boldsymbol{Q}}^-} t_{\dot W}(\dot g)\,\chi(g\gamma)\,d\dot g_A^- = |\varphi_{\dot W}^*(\gamma)|^2 G_s^*(\gamma).$$

For every non-archimedean prime p, we know that $\tilde\varphi_{W_p}*G_{s,p}*\varphi_{W_p}$ ha support contained in $\mathfrak{O}_p^-$ (choosing $W_p = p^{n_p}\mathfrak{O}_p$ as before). Hence i the series $\sum_{\gamma\in\boldsymbol{D}_{\boldsymbol{Q}}^-}(\tilde\varphi_W*G_s*\varphi_W)(g+\gamma)$, the summation is over the elemen of $\mathfrak{O}_1\cap\boldsymbol{D}^-$ for an order $\mathfrak{O}_1$ in $\boldsymbol{D}$. Further

$$(\tilde\varphi_{W_\infty}*G_{s,\infty}*\varphi_{W_\infty})(x) = \int_{\boldsymbol{R}^3}(\tilde\varphi_{W_\infty}*\varphi_{W_\infty})(y)G_{s,\infty}(x-y)\,dy$$
$$= \int_{W_\infty^{(1)}}(\tilde\varphi_{W_\infty}*\varphi_{W_\infty})(y)G_{s,\infty}(x-y)\,dy$$

where $W_\infty^{(1)}$ = support of $\tilde\varphi_{W_\infty}*\varphi_{W_\infty}$. For $|x|\geqslant 2c_7$ (depending only $W_\infty^{(1)}$), we know that $G_{s,\infty}(x)\leqslant c_8 e^{-c_9|x|_\infty}$ so that

$$(15)\qquad |(\tilde\varphi_{W_\infty}*G_{s,\infty}*\varphi_{W_\infty})(x)|\leqslant c_{10}e^{-c_{11}|x|}\quad\text{for}\quad |x|\geqslant 2c_7.$$

(If $x = x_1\omega_1+x_2\omega_2+x_3\omega_3$, then $|x|_\infty^2 = x_1^2+x_2^2+x_3^2$ and $\frac{1}{2}\sigma({}^t xx) = \lambda_1^2+\lambda_2^2$ where λ_1,λ_2 are the eigenvalues of xJ^{-1}). Hence the series

$$\sum_{\gamma\in\boldsymbol{D}_{\boldsymbol{Q}}^-}(\tilde\varphi_W*G_s*\varphi_W)(g+\gamma)$$

is majorized by

$$c_{12}\sum_{n_1,n_2,n_3\in d^{-1}\boldsymbol{Z}} e^{-c_{13}(n_1^2+n_2^2+n_3^2)}$$

for a suitable integer $d = d(\mathfrak{O}^-)$ and uniformly for g lying in a compact subset of $\boldsymbol{D}_{\boldsymbol{A}}^-$. It converges to a continuous function on $\boldsymbol{D}_{\boldsymbol{A}}^-/\boldsymbol{D}_{\boldsymbol{Q}}^-$ whose Fourier coefficients are given by

$$c'_\gamma = \int_{\boldsymbol{D}_{\boldsymbol{A}}^-/\boldsymbol{D}_{\boldsymbol{Q}}^-}\sum_{\gamma_1\in\boldsymbol{D}_{\boldsymbol{Q}}^-}(\tilde\varphi_W*G_s*\varphi_W)(g+\gamma_1)\,\chi(g\gamma_1)\,d^-g_A$$
$$= \int_{\boldsymbol{D}_{\boldsymbol{A}}^-}(\tilde\varphi_W*G_s*\varphi_W)(g)\,\chi(g\gamma)\,d^-g_A$$

in view of the uniform convergence of the series $\sum_{\gamma_1}$, i.e.

$$c'_\gamma = |\varphi_W^*(\gamma)|^2 G_s^*(\gamma) = c_\gamma.$$

Thus we obtain

$$t_{\dot W}(\dot g) = \sum_{\gamma\in\boldsymbol{D}_{\boldsymbol{Q}}^-}(\tilde\varphi_W*G_s*\varphi_W)(g+\gamma).$$

Proof of Proposition 2. Applying Lemma 1 of Weil [10] to the case where $G = \boldsymbol{D}_{\boldsymbol{A}}^-/\boldsymbol{D}_{\boldsymbol{Q}}^-$, $X = \boldsymbol{D}_{\boldsymbol{A}}^3$ and $\tau(g^*) = G_s^*(g^*)$, we get

$$\dot S(\Phi) = \lim_{\dot W\to\{0\}}\int_{\boldsymbol{D}_{\boldsymbol{A}}^3}\Phi(x)\,t_{\dot W}\big(\dot{\overline{f(x)}}\big)\,dx_A,$$

the limit being taken over compact neighbourhoods W of 0 in $\boldsymbol{D}_{\boldsymbol{A}}^-/\boldsymbol{D}_{\boldsymbol{Q}}^-$. We now assume Φ to have compact support C in $\boldsymbol{D}_{\boldsymbol{A}}^3$ so that $f(C)$ is again compact. Since the series

$$E(\Phi, s) = \Phi(0)+\sum_{i^*\in\boldsymbol{D}_{\boldsymbol{Q}}^-}F_\Phi^*(i^*)G_s^*(i^*)$$

converges uniformly on $f(C)$, we have

$$(16)\qquad \dot S(\Phi) = \lim_{\dot W\to\{0\}}\sum_{\gamma\in\boldsymbol{D}_{\boldsymbol{Q}}^-}\int_{\boldsymbol{D}_{\boldsymbol{A}}^3}\Phi(x)\,t_W\big(f(x)+\gamma\big)\,dx_A.$$

Since Φ_p has compact support C_p, only those γ for which $f(C_p)+\gamma \subset \boldsymbol{D}_p^- \cap \mathfrak{O}_p$ for every p would give a non-zero contribution to $\dot{S}(\Phi)$ sc that the summation in the series (16) over γ is just over the elements o $\mathfrak{O}_1 \cap \boldsymbol{D}^-$ for an order $\mathfrak{O}_1$ in $\boldsymbol{D}_{\boldsymbol{Q}}$. Using the estimate (15) for $t_{W\infty}(f(x)+\gamma)$ we see that

$$\Big|\sum_{\gamma \in \boldsymbol{D}_{\boldsymbol{Q}}^-} \int_C \Phi(x) t_W\big(f(x)+\gamma\big)\, dx_A\Big| \leqslant c_{14}\|\Phi\|_1 \sum_{\gamma \in \mathfrak{O}_1 \cap \boldsymbol{D}^-} e^{-c_{15}|\gamma|_\infty},$$

where $\|\Phi\|_1$ is the norm of Φ in $L_1(\boldsymbol{D}_A^3)$. Thus the series

$$\sum_{\gamma \in \boldsymbol{D}_{\boldsymbol{Q}}} \int_{\boldsymbol{D}_A^3} \Phi(x) t_W\big(f(x)+\gamma\big)\, dx_A$$

converges uniformly with respect to W and we may interchange in th series above for $\dot{S}(\Phi)$, the order of the summation over γ and the proces of taking the limit over $\dot{W}$ so that

$$\dot{S}(\Phi) = \sum_{\gamma \in \boldsymbol{D}_{\boldsymbol{Q}}^-} \lim_{W \to \{0\}} \int_{\boldsymbol{D}_A^3} \Phi(x) t_W\big(f(x)+\gamma\big)\, dx_A = \sum_{\gamma \in \boldsymbol{D}_{\boldsymbol{Q}}^-} F_{\Phi,s}(\gamma).$$

Hence, for functions Φ in $\mathscr{S}(\boldsymbol{D}_A^3)$ with compact support,

$$\dot{S}(\Phi) = \sum_{\gamma \in \boldsymbol{D}_{\boldsymbol{Q}}^-} F_{\Phi,s}(\gamma).$$

In other words, they coincide as measures. But $\dot{S}$ is a tempered distr bution and for $\Phi \geqslant 0$, we know that $\dot{S}(\Phi) \geqslant 0$ so that for all non-negative in $\mathscr{S}(\boldsymbol{D}_A^3)$, we may conclude the absolute convergence of the serie $\sum_{\gamma \in \boldsymbol{D}_{\boldsymbol{Q}}^-} F_{\Phi,s}(\gamma)$ and hence for all Φ in $\mathscr{S}(\boldsymbol{D}_A^3)$.

PROPOSITION 3. *For Φ in $\mathscr{S}(\boldsymbol{D}_A^3)$ the series $\sum_{0 \neq \gamma \in \boldsymbol{D}_{\boldsymbol{Q}}^-} F_{\Phi,s}(\gamma)$ co verges uniformly for s lying in the interval $[0,1]$ and also for Φ lying i a compact subset of $\mathscr{S}(\boldsymbol{D}_A^3)$.*

Proof. Let $\Phi = \prod_v \Phi_v(x_v)$ where Φ_p is the characteristic functio of $\mathfrak{O}_p^3$ for almost all primes p, $\mathfrak{O}$ being an order in $\boldsymbol{D}$. We take D_2 d visible by D and $N_0(\gamma)$ as on p. 335. Then using formula (9) for $F_{\Phi_p,s}($ $= b_{L_p}(s, -\gamma)$ for $p \nmid D_2$, we have

$$\Big|\prod_{p \nmid D_2} F_{\Phi_p,s}(\gamma)\Big| \leqslant \sum_{\substack{n=1\\(n,D_2)=1}}^{\infty} n^{-2} < c_{16}.$$

We now take primes q dividing D_2. In this case

$$F_{\Phi_q,s}(\gamma) = b_{L_q}(s,\gamma) = 1 + \sum_{\nu=1}^{\infty} B_{q^\nu,L_q}\big(f(x), -\gamma\big),$$

$$\begin{aligned} B_{q^\nu,L_q}\big(f(x), -\gamma\big)\big(&= D_{q^\nu} - D_{q^{\nu-1}} = q^{-9\nu} A_{q^\nu,L_q} - q^{-9(\nu-1)} A_{q^{\nu-1},L_q}\big) \\ &= q^{-12\nu} \sum_{\substack{\omega = -\tilde{\omega}\\ \omega \in \mathfrak{O}_q' \bmod q^\nu \mathfrak{O}_q'}} \sum_{\substack{x \bmod q^\nu \mathfrak{O}_q^3\\ x \in \mathfrak{O}_q^3}} e^{2\pi i q^{-\nu}\sigma_0((f(x)-\gamma)\omega)} - \\ &\quad - q^{-12(\nu-1)} \sum_{\substack{\omega_1 = -\tilde{\omega}_1\\ \omega_1 \bmod q^{\nu-1}}} \sum_{x \in \mathfrak{O}_q^3 \bmod q^{\nu-1}\mathfrak{O}_q^3} e^{2\pi i q^{-(\nu-1)}\sigma_0((f(x)-\gamma)\omega_1)} \\ &= \sum_{\substack{q^\nu \omega \in \mathfrak{O}_q'\\ \omega \bmod \mathfrak{O}_q'\\ q^{\nu-1}\omega \notin \mathfrak{O}_q'}} q^{-12\nu} \sum_{\substack{x \bmod q^\nu \mathfrak{O}_q^3\\ x \in \mathfrak{O}_q^3}} e^{2\pi i \sigma_0((f(x)-\gamma)\omega)}. \end{aligned}$$

Thus using the easily proved estimate for Gauss sums, namely

$$\Big|\sum_{\mathfrak{X} \bmod \mathfrak{O}\tilde{\gamma}} e^{2\pi i \sigma_0(\tilde{\mathfrak{X}} S \mathfrak{X} a \gamma^{-1})}\Big| \leqslant 8 N_0\big(\delta(S)\big)|d|^3 N_0^3(\gamma),$$

where d is the discriminant of $\boldsymbol{D}$, and where $a\mathfrak{O} + \mathfrak{O}\tilde{\gamma} = \mathfrak{O}$, we have

$$|b_{L_q}(s,\gamma)| \leqslant 1 + c_{17} \sum_{\nu=1}^{\infty} q^{-3\nu s} q^{-12\nu} q^{6\nu} q^{3\nu} \leqslant c_{18}(1-q^{-3})^{-1}.$$

Thus for primes q dividing D_2 $|F_{\Phi_q,s}(\gamma)| \leqslant c_{19}\, m(\mathfrak{O}_q)(1-q^{-3})^{-1}$. Hence

$$\Big|\prod_{q|D_2} F_{\Phi_q,s}(\gamma)\Big| \leqslant c_{20} \prod_{q|D_2} (1-q^{-3})^{-1} \leqslant c_{21} \log\log |N_0(\gamma)|_\infty.$$

Thus

$$|F_{\Phi,s}(\gamma)| \leqslant c_{21}\big(\log\log |N_0(\gamma)|_\infty\big) F_{\Phi_\infty,s}(\gamma).$$

We proceed to estimate $F_{\Phi_{\infty},s}(\gamma)$. Let us observe that $F_{\Phi_\infty,s}(x) = (G_{s,\infty} * F_{\Phi\infty})(x)$. (The equality holds first in the sense of distributions but since $G_{s,\infty}$ is tempered and since F_Φ is bounded, they are equal as functions.) We have $F_{\Phi\infty}(\gamma) = O(|\gamma|_\infty^{-\nu})$ for every $\nu > 0$. (The constants in the O-estimate may depend on ν.) Further, for $\gamma \neq 0$ in $\boldsymbol{D}$,

$$\begin{aligned} |F_{\Phi_\infty,s}(\gamma)| &= \Big|\int_{\boldsymbol{R}^3} G_{s,\infty}(\gamma) F_{\Phi_\infty}(\gamma - x)\, d^- x\Big| \leqslant \Big|\int_{|x|_\infty \leqslant \frac{1}{2}|\gamma|_\infty}\Big| + \Big|\int_{|x|_\infty > \frac{1}{2}|\gamma|_\infty}\Big| \\ &\leqslant c_{22}|\gamma|_\infty^{-\nu} \int_{|x|_\infty \leqslant \frac{1}{2}|\gamma|_\infty} G_{s,\infty}(x)\, d^- x + c_{23} e^{-c_{24}|\gamma|_\infty^{1/2}} \int_{|x|_\infty > \frac{1}{2}|\gamma|_\infty} |F_{\Phi_\infty}(\gamma - x)|\, d^- x \\ &\leqslant c_{25}|\gamma|_\infty^{-\nu} + c_{23} e^{-c_{24}|\gamma|_\infty^{1/2}} \|F_{\Phi_\infty}\|_1, \quad \text{since } \int_{\boldsymbol{R}^3} G_{s,\infty}\, d^- x = 1 \\ &\leqslant c_{26}|\gamma|_\infty^{-\nu} \end{aligned}$$

this being valid uniformly in s, for $0 \leqslant s \leqslant 1$. Here $\|F_{\Phi_\infty}\|_1$ is the norr of F_{Φ_∞} in $L_1(D_\infty^-)$. Choosing $\nu = 2$, we derive the uniform convergenc of $\sum_{\gamma\neq 0} F_{\Phi,s}(\gamma)$.

From Proposition 3, it is clear that $\sum_{0\neq\gamma\in D_Q^-} F_{\Phi,s}(\gamma)$ is a continuou function of s for $0 \leqslant s \leqslant 1$ and

$$\lim_{s\to 0}\sum_{\gamma\neq 0} F_{\Phi,s}(\gamma) = \sum_{\gamma\neq 0}\lim_{s\to 0} F_{\Phi,s}(\gamma) = \sum_{\gamma\neq 0} F_{\Phi}(\gamma).$$

We have still to consider the constant term $F_{\Phi,s}(0)$. Now

$$F_{\Phi,s}(0) = F_{\Phi_\infty,s}(0)\prod_{q|D_2} F_{\Phi_q,s}(0)\prod_{p\nmid D_2} F_{\Phi_p,s}(0)$$
$$= F_{\Phi_\infty,s}(0)\prod_{q|D_2} F_{\Phi_q,s}(0)\times$$
$$\times\prod_{p\nmid D_2}\Big\{(1+p^{-(1+s)})\Big(\frac{-\delta(S)}{p}\Big)+p^{-(3+s)}-p^{-(4+s)}-p^{-(3+s)}\Big(\frac{-\delta(S)}{p}\Big)+M_p(s$$

where $M_p(s) = O(p^{-3/2})$ for small s. If $-\delta(S)$ is not a square, the produ $\prod_{p\nmid D_2}$ converges absolutely, uniformly for $s \geqslant 0$. Hence, in this ca $F_\Phi(0) = \lim_{s\to 0} F_{\Phi,s}(0)$ exists. Moreover, if $f(x)$ does not represent 0 no trivially in D^3, then $U(0)_Q = \emptyset$ and hence by a result of Springer [6 $U(0)_A = \emptyset$ so that at least one factor in $F_\Phi(0) = \prod_p F_{\Phi_p}(0)$ vanish and consequently $F_\Phi(0) = 0$. If $-\delta(S)$ is a square, then we know fro page 336 that $f(x)$ does not represent 0 in D^3 nontrivially and as we ha seen already on p. 337, we have in this case as well that $\lim_{s\to 0} F_{\Phi,s}(0) =$

§ 6. The Siegel formula. For $i\in D_Q^-$, denote by $U(i)_Q$ the s $\{x\in D_Q^3 \mid f(x) = i, x\neq 0\}$. Let, for $i\in D_Q^-$, $U(i)_Q \neq \emptyset$ and let $\xi_i \in U(i)$ Denote by H_i, the isotropy subgroup of ξ_i in G_Q where G is the (specia orthogonal group of $f(x)$ (being defined over Q). Let $(dg)_A$ be an invaria gauge-form on G_A and let $\lambda(dh_i)_A = \prod_v \lambda_v(dh_i)_v$ be an invariant gau form on $(H_i)_A$ with suitable convergence factors λ_p. Then

$$\prod_v \lambda_v^{-1}\Big(\frac{dg}{dh_i}\Big)_v = \prod_v \lambda_v^{-1}(\vartheta_i)_v,$$

where $(\vartheta_i)_v = \Big(\frac{dg}{dh_i}\Big)_v$, is a gauge-form on $U(i)_A$ with convergence facto λ_p^{-1}. By [8], we know that

(17) $$\tau_\lambda(H_i)\int_{U(i)_A}\Phi(g(\xi))\prod_p \lambda_p^{-1}(\vartheta_i)_p = \int_{G_A/G_Q}\sum_{\xi\in U(i)_Q}\Phi(g(\xi))dg_A$$

for any $\Phi\in\mathcal{S}(D_A^3)$. Here $\tau_\lambda(H_i)$ is the volume of $(H_i)_A/(H_i)_Q$ with respect to the measure $\lambda(dh_i)_A$. Formula (17) is valid even if $U(i)_Q = \emptyset$, since then by the theorem of Springer [6] on the Hasse principle for skew-hermitian forms over D, $U(i)_A = \emptyset$. Thus both sides of (17) are zero and hence equal.

When $i\neq 0$, we know from [8] that H_i is the orthogonal group of a non-degenerate binary skew-hermitian form and choosing $\lambda_p = 1$ for all p, we have by the classical isomorphism theorems ([10], p. 82) that $\tau_\lambda(H_i) = 2$. But $F_\Phi(i) = \prod F_{\Phi_v}(i) = \int_{U(i)_v}\Phi_v\,d\mu_v(i)$. Now, for each v,

$$\int_{D_v^3}\Phi_v\,dx_v = \int_{D_v}|di|_v\int_{U(i)_v}\Phi_v\,d\mu(i)$$

and

$$\int_{D_v^3-\{*\}}\Phi_v\,dx_v = \int_{D_v}|di|_v\int_{U(i)_v}\Phi_v\Big(\frac{dx}{di}\Big)_v = \int_{D_v}|di|_v\int_{U(i)_v}\Phi_v\,|\vartheta_i|_v$$

where $\{*\}$ denotes the set of points of D_v^3 of rank < 2. By Proposition 1 of Weil [10], $\mu_v = (\vartheta_i)_v$ for every v and for $i\neq 0$ in D_Q^-. Then, for every $i\neq 0$ in D_Q^-, we have by the foregoing

(18) $$F_\Phi(i) = \prod_v\int_{U(i)_v}\Phi_v\,d\mu_v(i) = \int_{U(i)_A}\Phi(\vartheta_i)_A = \tfrac{1}{2}\int_{G_A/G_Q}\sum_{\xi\in U(i)_Q}\Phi(g\xi)\,|dg|_A.$$

We now consider the case when $i = 0$ and $U(i)_Q\neq\emptyset$. First, let $-\delta(S)$ be not a square. Then H_0 is the semi-direct product of the special orthogonal group U of a skew-hermitian form in one variable, i.e $\tilde{\alpha}\alpha t$ with $\alpha = -\tilde{\alpha}$ and $N_0(\alpha) = -\delta(S)$ and the additive group $S_1 = G_a$. Hence U is an anisotropic torus defined over Q and contained in the group of elements of D, of norm equal to ± 1. It splits over the quadratic field $Q(\sqrt{-\delta(S)})$. For U, we choose the convergence factors $\lambda_p = (1-\chi_0(p)p^{-1})$ where $\chi_0(p) = \Big(\frac{-\delta(S)}{p}\Big)$, $\lambda_\infty = 1$ and for S_1, we choose the convergence factors $\lambda_v = 1$. Hence, by [8], (λ_p) is a set of convergence factors also for H_0. From [3], we have

$$\tau_\lambda(H_0) = \tau_\lambda(U)\tau(G_a) = \tau_\lambda(U) = 2\prod_p(1-\chi_0(p)p^{-1})^{-1} = 2L(1,\chi_0)$$

since $\hat{U}_Q = 1$. Moreover,

$$\int_{U(0)_A}\Phi(g\xi)\prod_p\lambda_p^{-1}(\vartheta_0)_p = \prod_{q|D_2}\lambda_q^{-1}F_{\Phi_q}(0)\times$$
$$\times\prod_{p\nmid D_2}(1-\chi_0(p)p^{-1})(1+\chi_0(p)p^{-1}+p^{-3}-p^{-4}-p^{3}\chi_0(p)+M_p(0))$$
$$= \prod_{q|D_2}\lambda_q^{-1}F_{\Phi_q}(0)\prod_{p\nmid D_2}(1-\beta_p(0))$$

where $\beta_p(s) = p^{-2-2s} - (1-\chi_0(p)p^{-(1+s)})(p^{-3-s} - p^{-4-s} - \chi_0(p)p^{-3-s} + M_p($
and $\beta_p(0) = O(p^{-2})$. Thus we have

$$(19)\quad F_\Phi(0) = \lim_{s\to 0} F_{\Phi,s}(0) = \prod_{q|D_2} F_{\Phi_q}(0) \times \lim_{s\to 0} \prod_{p\nmid D_2} F_{\Phi_p,s}(0)$$

$$= \prod_{q|D_2} F_{\Phi_q}(0) \times$$

$$\times \lim_{s\to 0} \prod_{q|D_2} (1-\chi_0(q)q^{-1-s}) L(1+s, \chi_0) \prod_{p\nmid D_2} (1-\beta_p(s))$$

$$= \prod_{q|D_2} \lambda_q^{-1} F_{\Phi_q}(0) L(1, \chi_0) \prod_{p\nmid D_2} (1-\beta_p(0))$$

$$= L(1, \chi_0) \int_{U(0)_A} \Phi(g\xi) \prod_p \lambda_p^{-1}(\vartheta_0)_p$$

$$= \tfrac{1}{2}\tau_\lambda(H_0) \int_{U(0)_A} \Phi(g\xi) \prod_p \lambda_p^{-1}(\vartheta_0)_p$$

$$= \tfrac{1}{2} \int_{G_A/G_{\mathbf{Q}}} \sum_{\xi\in U(0)_{\mathbf{Q}}} \Phi(g\xi)|dg|_A.$$

The relation above is valid even if $-\delta(S)$ is not a square and if $f(x)$ d
not represent 0 non-trivially over $\boldsymbol{D}^3$, since then $F_\Phi(0) = 0$ and t
right hand side is zero, $U(0)_{\mathbf{Q}}$ being empty. In view of our remarks
p. 336, $f(x)$ cannot be a zero-form when $-\delta(S)$ is a square and here aga
we know that $F_\Phi(0) = 0$ from p. 343 and further $U(0)_{\mathbf{Q}} = \emptyset$ so that t
expression on the right hand side of (19) is zero, again. Thus (19) is va
in this case as well.

We now define $E(\Phi) = \lim_{s\to 0} E(\Phi, s)$. Then E is a positive tempe
measure on $\mathscr{S}(\boldsymbol{D}_A^3)$ and

$$E(\Phi) = \Phi(0) + \lim_{s\to 0} \sum_{i^*\in \boldsymbol{D}_{\mathbf{Q}}} F_\Phi^*(i^*) G_s^*(i^*)$$

$$= \Phi(0) + \lim_{s\to 0} \sum_{i\in \boldsymbol{D}_{\mathbf{Q}}} F_{\Phi,s}(i) \quad \text{(by Proposition 2)}$$

$$= \Phi(0) + F_\Phi(0) + \sum_{0\neq i\in \boldsymbol{D}_{\mathbf{Q}}} F_\Phi(i)$$

$$= \tfrac{1}{2}\Phi(0) \int_{G_A/G_{\mathbf{Q}}} |dg|_A + \tfrac{1}{2} \sum_{i\in \boldsymbol{D}_{\mathbf{Q}}} \int_{G_A/G_{\mathbf{Q}}} \sum_{\xi\in U(i)_{\mathbf{Q}}} \Phi(g\xi)|dg|_A,$$

(in view of (18), (19) and in view of $\tau(G)$ being equal to 2), i.e.

$$E(\Phi) = \tfrac{1}{2} \int_{G_A/G_{\mathbf{Q}}} \sum_{\xi\in \boldsymbol{D}_{\mathbf{Q}}^3} \Phi(g\xi)|dg|_A = I_\nu(\Phi)$$

where ν is a normalized measure on G_A with $\nu(G_A/G_{\mathbf{Q}}) = 1$. Thus
have proved the following

THEOREM. *For $\Phi \in \mathscr{S}(\boldsymbol{D}_A^3)$,*

$$E(\Phi) = I_\nu(\Phi),$$

where $E(\Phi)$ is defined as

$$\lim_{s\to 0} \Big\{\Phi(0) + \sum_{i^*\in \boldsymbol{D}_{\mathbf{Q}}^-} F_\Phi^*(i^*) G_s^*(i^*)\Big\}$$

and

$$I_\nu(\Phi) = \int_{G_A/G_{\mathbf{Q}}} \sum_{\xi\in \boldsymbol{D}_{\mathbf{Q}}^3} \Phi(g\xi)\,d\nu(g).$$

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TATA INSTITUTE OF FUNDAMENTAL RESEARCH
and
CENTRE FOR ADVANCED STUDY IN MATHEMATICS, UNIVERSITY OF BOMBAY

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