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An asymptotic formula in the theory of numbers

by

YOICHI MOTOHASHI (Tokyo)

§ 1. Introduction. Asymptotic formulae for the sums

$$\sum_{n\leqslant N}\tau(n)\tau_k(n+a)$$

have been considered by Ingham [4], Hooley [3], and Linnik [5] for the case $k=2$, $k=3$ and all k, respectively, where $\tau_k(n)$ is the number of the representation of n as the product of k factors and $\tau(n)=\tau_2(n)$.

The purpose of this paper is to prove an asymptotic formula for the sum

$$\sum_{n\leqslant N}\tau^2(n)\tau(n+1).$$

Our method depends on the recently obtained result in the theory of the large sieve. It may be interesting to remark that the method of Hooley and Linnik largely depends on the very deep A. Weil's estimate of Kloosterman's sum, but our proof does not make any use of it.

Our result is as follows:

THEOREM.

$$\sum_{n\leqslant N}\tau^2(n)\tau(n+1) = \mathfrak{S}N(\log N)^4+O\big(N(\log N)^3\log\log N\big).$$

Here the constant $\mathfrak{S}$ is defined by

$$\mathfrak{S} = \pi^{-2}\prod_p\left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^2\left(1+\frac{1}{p}\right)^{-1}\right).$$

Notations. Let p be generally a prime, χ a Dirichlet character and $\varphi(m)$ the Euler φ-function. The notation "$\ll$" is the usual Vinogradov's symbol.

§ 2. Preliminary lemmas.

LEMMA 1 (E. Bombieri, 1965). *Let a_n be arbitrary complex numbers. Then, if $\sum^*$ denotes a summation over all the primitive characters* $\chi \bmod q$, *we have*

$$\sum_{q\leqslant M}\ \sum_{\chi \bmod q}^{*}\Big|\sum_{n=Y+1}^{Y+U} a_n\chi(n)\Big|^2 \leqslant 4.4\max(U, M^2)\sum_{n=Y+1}^{Y+U}\tau(n)|a_n|^2.$$

Proof. See Davenport [1], Chapter 23.

LEMMA 2. *Let $L(s,\chi)$, $s=\sigma+it$, be Dirichlet's L-series with character χ. Then we have for $1/2\leqslant\sigma<1$*

$$\sum_{q\leqslant M}\frac{1}{\varphi(q)}\sum_{\chi \bmod q}|L(s,\chi)|^4 \ll M|s|^2\log^{26}(M|s|).$$

Proof. Let $L(M)$ be the left side, and further let $\chi^* \bmod q^*$ be the primitive character corresponding to $\chi \bmod q$, then we have

$$L(s,\chi)=\prod_{p\mid \frac{q}{q^*}}\Big(1-\frac{\chi^*(p)}{p^s}\Big)L(s,\chi^*).$$

Since

$$\Big|\prod_{p\mid \frac{q}{q^*}}\Big(1-\frac{\chi^*(p)}{p^s}\Big)\Big|\leqslant\prod_{p\mid \frac{q}{q^*}}2\leqslant\tau\Big(\frac{q}{q^*}\Big),$$

we have

$$L(M)\leqslant\sum_{q^*\leqslant M}\ \sum_{\chi^* \bmod q^*}^{*}|L(s,\chi^*)|^4\sum_{\substack{rq^*=q\\ q\leqslant M}}\frac{\tau^4(r)}{\varphi(r)\varphi(q^*)}$$

$$\leqslant \log^2 M\sum_{q\leqslant M}\frac{1}{q}\sum_{\chi \bmod q}^{*}|L(s,\chi)|^4\sum_{r\leqslant M}\frac{\tau^4(r)}{r}$$

$$\ll \log^{18}M\sum_{q\leqslant M}\frac{1}{q}\sum_{\chi \bmod q}^{*}|L(s,\chi)|^4.$$

Now let us consider the sum

$$L^*(V)=\sum_{q\leqslant V}\ \sum_{\chi \bmod q}^{*}|L(s,\chi)|^4.$$

By the well-known estimate of character sum we have

$$L(s,\chi)=\sum_{n\leqslant U}\chi(n)n^{-s}+O\big(U^{-1/2}\sqrt{q}|s|\log(q+1)\big).$$

Hence we have

$$L^*(V)\ll\sum_{q\leqslant V}\ \sum_{\chi \bmod q}^{*}\Big|\sum_{n\leqslant U}\chi(n)n^{-s}\Big|^4+U^{-2}V^4|s|^4\log^4 V.$$

Now from Lemma 1 we can estimate this sum as follows:

$$\sum_{q\leqslant V}\ \sum_{\chi \bmod q}^{*}\Big|\sum_{n\leqslant U}\chi(n)n^{-s}\Big|^4=\sum_{q\leqslant V}\ \sum_{\chi \bmod q}^{*}\Big|\sum_{n\leqslant U^2}e_n\chi(n)n^{-s}\Big|^2,$$

where $e_n=\sum_{d\mid n,\, d\leqslant U}1\leqslant\tau(n)$, and so this sum is

$$\ll\max(V^2, U^2)\sum_{n\leqslant U^2}\tau^3(n)n^{-1}\ll\max(V^2, U^2)\log^8 U.$$

Taking $U=V|s|$, we obtain $L^*(V)\ll V^2|s^2|\log^8(V|s|)$. This gives

$$L(M)\ll\log^{18}M\Big\{M^{-1}L^*(M)+\int_1^M V^{-2}L^*(V)\,dV\Big\}\ll M|s|^2\log^{26}(M|s|).$$

Hence the lemma is proved.

LEMMA 3. *We have for* $\sigma>1$

$$\sum_{n=1}^{\infty}\chi(n)\tau^2(n)n^{-s}=\frac{L^4(s,\chi)}{L(2s,\chi^2)}.$$

Proof. This can be proved analogously as the well-known Ramanujan's identity.

LEMMA 4. *Let χ_0 be the principal character* $\bmod q$, *and let $f_q(s)$ be the function of q and s defined by*

$$f_q(s)=\prod_{p\mid q}\Big(1-\frac{1}{p^s}\Big).$$

Then we have

$$\operatorname*{Res}_{s=1}\frac{L^4(s,\chi_0)}{L(2s,\chi_0^2)}\cdot\frac{N^s}{s^4}$$
$$=\sigma_0(q)N\log^3 N+\sigma_1(q)N\log^2 N+\sigma_2(q)N\log N+\sigma_3(q)N,$$

where

$$\sigma_0(q)=\frac{f^4(1)}{6\zeta(2)f_q(2)},$$
$$\sigma_1(q)=a_1(q)f_q^3(1)f_q^{(1)}(1)+a_2(q)f_q^4(1),$$
$$\sigma_2(q)=a_3(q)f_q^2(1)f_q^{(1)2}(1)+a_4(q)f_q^3(1)f_q^{(2)}(1)+a_5(q)f_q^3(1)f_q^{(1)}(1)+a_6(q)f_q^4(1),$$

and

$$\sigma_3(q)=a_7(q)f_q^3(1)f_q^{(3)}(1)+a_8(q)f_q^2(1)f_q^{(1)}(1)f_q^{(2)}(1)+$$
$$+a_9(q)f_q(1)f_q^{(1)3}(1)+a_{10}(q)f_q^2(1)f_q^{(1)2}(1)+a_{11}(q)f_q^3(1)f_q^{(2)}(1)+$$
$$+a_{12}(q)f_q^3(1)f_q^{(1)}(1)+a_{13}(q)f_q^4(1).$$

Here $|a_j(q)|\ll 1$, $j=1,2,\dots,13$.

Proof. We have $L(s,\chi_0)=f_q(s)\zeta(s)$. Hence we get

$$\frac{L^4(s,\chi_0)}{L(2s,\chi_0^2)}\cdot\frac{N^s}{s^4}=\frac{\zeta^4(s)}{f_q(2s)\zeta(2s)s^4}f_q^4(s)N^s.$$

Also it is easy to see that

$$\frac{\zeta^4(s)}{f_q(2s)\zeta(2s)s^4}$$

$$=(s-1)^{-4}\left\{\frac{1}{f_q(2)\zeta(2)}+b_1(q)(s-1)+b_2(q)(s-1)^2+b_3(q)(s-1)^3+\ldots\right\},$$

where $|b_j(q)|\ll 1$. Further we have

$$f_q^4(s)=f_q^4(1)+4f_q^3(1)f_q^{(1)}(1)(s-1)+\left(6f_q^2(1)f_q^{(1)2}(1)+2f_q^3(1)f_q^{2}(1)\right)(s-1)^2+$$
$$+\left(\tfrac{2}{3}f_q^3(1)f_q^{(3)}(1)+6f_q^2(1)f_q^{(1)}(1)f_q^{(2)}(1)+4f_q(1)f_q^{(1)3}(1)\right)(s-1)^3+\ldots,$$

and

$$N^s=N+N\log N(s-1)+\frac{N}{2}\log^2N(s-1)^2+\frac{N}{6}\log^3N(s-1)^3+\ldots$$

Collecting these expansions we get the lemma.

Lemma 5. *Let $\omega(n)$ be the number of different prime divisors of n. Then we have*

$$\sigma_j(q)\ll f_q^4(1)\,\omega^j(q)\quad \textit{for}\quad j=1,2,3.$$

Proof. We have

$$f_q^{(1)}(1)=f_q(1)\sum_{p|q}\frac{\log p}{p-1}\ll f_q(1)\,\omega(q),$$

$$f_q^{(2)}(1)=-f_q(1)\sum_{p|q}\left(\frac{\log p}{p-1}\right)^2+\frac{f_q^{(1)2}(1)}{f_q(1)}$$
$$=-f_q(1)\sum_{p|q}\left(\frac{\log p}{p-1}\right)^2+f_q(1)\left(\sum_{p|q}\frac{\log p}{p-1}\right)^2\ll f_q(1)\,\omega^2(q)$$

and

$$f_q^{(3)}(1)=f_q(1)\sum_{p|q}\left(\frac{\log p}{p-1}\right)^3+3\,\frac{f_q^{(1)}(1)f_q^{(2)}(1)}{f_q(1)}-2\,\frac{f_q^{(1)3}(1)}{f_q^2(1)}$$
$$=f_q(1)\sum_{p|q}\left(\frac{\log p}{p-1}\right)^3+3f_q(1)\sum_{p|q}\frac{\log p}{p-1}\left\{-\sum_{p|q}\left(\frac{\log p}{p-1}\right)^2+\right.$$
$$\left.+\left(\sum_{p|q}\frac{\log p}{p-1}\right)^2\right\}+2f_q(1)\left(\sum_{p|q}\frac{\log p}{p-1}\right)^3$$
$$\ll f_q(1)\,\omega^3(q).$$

From these estimates of $f_q^{(j)}(1)$, $j=1,2,3$, the lemma follows at once.

§ 3. Further lemmas.

Lemma 6. *Let N be a large number, $q\leqslant N^{1-a}$, where a is a positive constant, $0<a<1/2$; $(q,l)=1$. Then the following estimate holds*

$$\sum_{\substack{n\equiv l \bmod q\\ n\leqslant N}}\tau^k(n)\leqslant c_a\frac{N}{q}\left\{\prod_{p|q}\left(1-\frac{1}{p}\right)\log\frac{N}{q}\right\}^{2^k-1},$$

where c_a is a positive constant depending only on a and $k=1,2,\ldots$

Proof. See Vinogradov and Linnik [6]. Actually the case $k=1$ has been treated in their work, but the general case is a trivial generalization.

Lemma 7. *Let $D_k(y;q,l)$ be the series*

$$\frac{1}{k!}\sum_{\substack{n\equiv l \bmod q\\ n\leqslant y}}\tau^2(n)\left(\log\frac{y}{n}\right)^k.$$

Then we have

$$\sum_{q\leqslant M}\max_{y\leqslant N}\max_{(q,l)=1}\left|D_3(y;q,l)-\varphi(q)^{-1}\operatorname*{Res}_{s=1}\frac{L^4(s,\chi_0)}{L(2s,\chi_0^2)}\cdot\frac{y^s}{s^4}\right|$$
$$\ll N^{1/2}M\log N\log^{26}M.$$

Proof. From Lemma 3 we get

$$D_3(y;q,l)=\frac{1}{\varphi(q)}\cdot\frac{1}{2\pi i}\int_{(2)}\sum_{\chi \bmod q}\bar\chi(l)\frac{L^4(s,\chi)}{L(2s,\chi^2)}\cdot\frac{y^s}{s^4}ds.$$

Here we remark that $|L(s,\chi)|^4\leqslant C(q)|s|^2$ for $1/2\leqslant\sigma$ and $|t|\geqslant 1$, and so we have

$$D_3(y;q,l)-\frac{1}{\varphi(q)}\operatorname*{Res}_{s=1}\frac{L^4(s,\chi_0)}{L(2s,\chi_0^2)}\cdot\frac{y^s}{s^4}$$
$$=\frac{1}{\varphi(q)}\cdot\frac{1}{2\pi i}\int_{(\beta)}\sum_{\chi \bmod q}\bar\chi(l)\frac{L^4(s,\chi)}{L(2s,\chi^2)}\cdot\frac{y^s}{s^4}ds,$$

where $\beta=1/2+(\log N)^{-1}$. For this β we have $L(2s,\chi^2)^{-1}\ll\log N$. Hence we obtain from Lemma 2

$$\sum_{q\leqslant M}\max_{y\leqslant N}\max_{(q,l)=1}\left|D_3(y;q,l)-\frac{1}{\varphi(q)}\operatorname*{Res}_{s=1}\frac{L^4(s,\chi)}{L(2s,\chi^2)}\cdot\frac{y^s}{s^4}\right|$$
$$\ll N^{1/2}\log N\int_{(\beta)}\sum_{q\leqslant M}\frac{1}{\varphi(q)}\sum_{\chi \bmod q}|L(s,\chi)|^4\frac{|ds|}{|s|^4}\ll N^{1/2}M\log N\log^{26}M.$$

LEMMA 8. *We have*

$$\sum_{q\leqslant N^{1/2}(\log N)^{-B}} \max_{y\leqslant N} \max_{(q,l)=1} \Big| \sum_{\substack{n\equiv l \bmod q\\ n\leqslant y}} \tau^2(n) - \varphi(q)^{-1}A_0(y, q)\Big| \ll N(\log N)^{-A},$$

where

$$A_0(y, q) = \sigma_0(q)y\log^3 y + \big(9\sigma_0(q)+\sigma_1(q)\big)y\log^2 y + \\ +\big(18\sigma_0(q)+\sigma_1(q)+\sigma_2(q)\big)y\log y + \big(6\sigma_0(q)+6\sigma_1(q)+3\sigma_2(q)+\sigma_3(q)\big)y$$

and $B = 6A+62$. *Here the numbers* $\sigma_j(q)$ *are defined in Lemma* 4.

Proof. We will deduce this lemma from Lemma 7 by the assertion of Gallagher ([2], pp. 5–6).

Since $D_k(y; q, l)$ is an increasing function of y, we have, for $0<\lambda\leqslant 1$,

$$\frac{1}{\lambda}\int_{e^{-\lambda}y}^{y} D_2(\xi; q, l)\frac{d\xi}{\xi} \leqslant D_2(y; q, l) \leqslant \frac{1}{\lambda}\int_{y}^{e^{\lambda}y} D_2(\xi; q, l)\frac{d\xi}{\xi}.$$

The integrals are

$$D_3(y; q, l) - D_3(ye^{-\lambda}; q, l) \quad \text{and} \quad D_3(e^{\lambda}y; q, l) - D_3(y; q, l).$$

From Lemma 4 these are both equal to

$$\lambda\{\sigma_0(q)y\log^3 y + (3\sigma_0(q)+\sigma_1(q))y\log^2 y + \\ +(2\sigma_1(q)+\sigma_2(q))\,y\log y + (\sigma_2(q)+\sigma_3(q))\,y\} + \\ +O\{\lambda^2(\sigma_0(q)y\log^3 y+\sigma_1(q)y\log^2 y+\sigma_2(q)y\log y+\sigma_3(q)y)\} + \\ +O\Big\{\max_{\xi\leqslant y}\Big|D_3(\xi; q, l) - \varphi(q)^{-1}\operatorname{Res}_{s=1}\frac{L^4(s,\chi_0)}{L(2s,\chi_0^2)}\cdot\frac{\xi^s}{s^4}\Big|\Big\}.$$

Hence from Lemma 5 and 7 we get, putting

$$A_2(y, q) = \sigma_0(q)y\log^3 y + (3\sigma_0(q)+\sigma_1(q))y\log^2 y + \\ +(2\sigma_1(q)+\sigma_2(q))y\log y + (\sigma_2(q)+\sigma_3(q))y,$$

$$\sum_{q\leqslant M}\max_{y\leqslant N}\max_{(q,l)=1}|D_2(y; q, l) - \varphi(q)^{-1}A_2(y, q)|$$

$$\ll \lambda N\log^3 N\sum_{q\leqslant M}\frac{f_q^4(1)}{\varphi(q)}\omega^3(q) + \lambda^{-1}N^{1/2}M\log N\log^{26}M$$

$$\ll \lambda N\log^3 N\log M(\log\log M)^3 + \lambda^{-1}N^{1/2}M\log N\log^{26}M,$$

since

$$\frac{f_q^4(1)}{\varphi(q)} = \frac{\varphi(q)^3}{q^4} \leqslant \frac{1}{q} \quad \text{and} \quad \sum_{q\leqslant M}\omega^3(q) \ll M(\log\log M)^3.$$

By the same assertion taking $\lambda^{1/2}$ instead of λ, we have

$$\sum_{q\leqslant M}\max_{y\leqslant N}\max_{(q,l)=1}|D_1(y; q, l) - \varphi(q)^{-1}A_1(y, q)| \\ \ll \lambda^{1/2}N\log^3 N\log M(\log\log M)^3 + \lambda^{-3/2}N^{1/2}M\log N\log^{26}M,$$

where

$$A_1(y, q) = \sigma_0(q)y\log y + (6\sigma_0(q)+\sigma_1(q)) + \\ +(6\sigma_0(q)+4\sigma_1(q)+\sigma_2(q))y\log y + (2\sigma_1(q)+2\sigma_2(q)+\sigma_3(q))y.$$

Finally taking $\lambda^{1/4}$ instead of λ, we have

$$\sum_{q\leqslant M}\max_{y\leqslant N}\max_{(q,l)=1}|D_0(y; q, l) - \varphi(q)^{-1}A_0(y, q)| \\ \ll \lambda^{1/4}N\log^3 N\log M(\log\log M)^3 + \lambda^{-7/4}N^{1/2}M\log M\log^{26}M.$$

If we take

$$\lambda = (\log N)^{-4(A+5)} \quad \text{and} \quad M = N(\log N)^{-(6A+62)},$$

then the lemma follows.

LEMMA 9. *We have for* $M>1$

$$\sum_{q\leqslant M}\frac{\sigma_0(q)}{\varphi(q)} = \pi^{-2}\prod_p\Big(1-\frac{1}{p}+\frac{1}{p}\Big(1-\frac{1}{p}\Big)^2\Big(1+\frac{1}{p}\Big)^{-1}\Big) + c_0 + O(M^{-1/5}\log M),$$

where c_0 *is a constant.*

Proof. Let us consider the series

$$\Delta(s) = \sum_{q=1}^{\infty}\frac{\sigma_0(q)}{q^s\varphi(q)}.$$

Obviously this series converges absolutely for $\sigma>0$. Also we have

$$\Delta(s) = \pi^{-2}\sum_{q=1}^{\infty}\frac{1}{q^{s+1}}\prod_{p|q}\Big(1-\frac{1}{p}\Big)^2\Big(1+\frac{1}{p}\Big)^{-1} \\ = \pi^{-2}\prod_p\Big\{1-\frac{1}{p^{s+1}}+\frac{1}{p^{s+1}}\Big(1-\frac{1}{p}\Big)^2\Big(1+\frac{1}{p}\Big)^{-1}\Big\}\zeta(s+1).$$

The last infinite product converges absolutely for $\sigma>-1$.

Hence we have

$$\sum_{q\leqslant M}\frac{\sigma_0(q)}{\varphi(q)}=\frac{1}{2\pi i}\int\limits_{1+\frac{1}{\log M}-iT}^{1+\frac{1}{\log M}+iT}\Delta(s)\frac{M^s}{s}\,ds+O\left(\frac{M}{T}\right)$$

$$=\operatorname*{Res}_{s=0}\Delta(s)\frac{M^s}{s}+O\{T^{-1}M+(T^{-1}+M^{-1/2}\log T)\max_{|t|\leqslant T}|\zeta(\tfrac{1}{2}+it)|\}.$$

Now we have

$$\operatorname*{Res}_{s=0}\Delta(s)M^s s^{-1}=\pi^{-2}\prod_p\left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^2\left(1+\frac{1}{p}\right)^{-1}\right)\log M+c_0$$

and

$$\max_{|t|\leqslant T}|\zeta(1/2+it)|\ll T^{1/4}.$$

Hence if we put $T=M^{6/5}$, then the lemma follows.

§ 4. Proof of the theorem. We devide the sum

$$\sum_{n\leqslant N}\tau^2(n)\tau(n+1)=\sum_{n\leqslant N}\tau^2(n)\sum_{q|n+1}1$$

as follows:

$$\sum_{n\leqslant N}\tau^2(n)\sum_{q|n+1}1$$

$$=\sum_{n\leqslant N}\tau^2(n)\Big\{\sum_{\substack{q|n+1\\ q\leqslant N^{1/2}(\log N)^{-62}}}1+\sum_{\substack{q|n+1\\ N^{1/2}(\log N)^{-62}<q\leqslant N^{1/2}(\log N)^{62}}}1+\sum_{\substack{N^{1/2}(\log N)^{62}<q\\ q|n+1}}1\Big\}$$

$$=S_1+S_2+S_3.$$

From Lemma 6 taking $k=2$ we have

$$S_2=\sum_{N^{1/2}(\log N)^{-62}<q\leqslant N^{1/2}(\log N)^{62}}\ \sum_{\substack{n\equiv-1\bmod q\\ n\leqslant N}}\tau^2(n)$$

$$\ll N\log^3 N\sum_{N^{1/2}(\log N)^{-62}<q\leqslant N^{1/2}(\log N)^{62}}\frac{1}{q}$$

$$\ll N\log^3 N\log\log N.$$

Also from Lemma 8 taking $A=0$ we get

$$S_1=S_2=\sum_{q\leqslant N^{1/2}(\log N)^{-62}}\ \sum_{\substack{n\equiv-1\bmod q\\ n\leqslant N}}\tau^2(n)$$

$$=\sum_{q\leqslant N^{1/2}(\log N)^{-62}}\varphi(q)^{-1}A_0(N,q)+$$

$$+O\Big\{\sum_{q\leqslant N^{1/2}(\log N)^{-62}}\max_{(q,l)=1}\Big|\sum_{\substack{n\equiv l\bmod q\\ n\leqslant N}}\tau^2(n)-\varphi(q)^{-1}A_0(N,q)\Big|\Big\}$$

$$=\sum_{q\leqslant N^{1/2}(\log N)^{-62}}\varphi(q)^{-1}A_0(N,q)+O(N).$$

Now by Lemma 5 and 9, and by the definition $A_0(y,q)$ in Lemma 8 we have

$$\sum_{q\leqslant N^{1/2}(\log N)^{-62}}\varphi(q)^{-1}A_0(N,q)$$

$$=(2\pi^2)^{-1}\prod_p\left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^2\left(1+\frac{1}{p}\right)^{-1}\right)N\log^4 N+$$

$$+O(N\log^3 N\log\log N)+O\Big\{N\log^2 N\sum_{q\leqslant N^{1/2}}\frac{f_1^4(q)}{\varphi(q)}\omega(q)+$$

$$+N\log N\sum_{q\leqslant N^{1/2}}\frac{f_1^4(q)}{\varphi(q)}\omega^2(q)+N\sum_{q\leqslant N^{1/2}}\frac{f_1^4(q)}{\varphi(q)}\omega^3(q)\Big\}.$$

The last O-term can be easily estimated to be

$$O(N\log^3 N\log\log N),$$

since

$$\sum_{q\leqslant M}\omega(q)^j\ll M(\log\log M)^j,\quad j=1,2,3,\ldots$$

Collecting these results we obtain the theorem at once.

Concluding remarks: Almost same method is applicable for the sums $\sum\tau(n)\tau_j(n+1)$, $j=3$ and 4, with error terms of the form $(\log N)^{j-1}\log\log N$. These are the improvements of the results of Hooley and Linnik.

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DEPARTMENT OF MATHEMATICS
COLLEGE OF SCIENCE AND ENGINEERING
NIHON UNIVERSITY
Tokyo, Japan

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Généralisation des nombres de Salem aux adèles

par

ANNETTE DECOMPS-GUILLOUX (Paris)

INTRODUCTION; RESUMÉ

I

Rappelons la définition et quelques propriétés essentielles des ensembles S (nombres de Pisot–Vijayaraghavan) et T (nombres de Salem) qui sont à la base de ce travail.

S désigne l'*ensemble des entiers algébriques réels* $\theta > 1$ *dont tous les conjugués (différents de* θ*) ont une valeur absolue inférieure à* 1 *strictement.* (C. Pisot [7].)

T désigne *l'ensemble des entiers algébriques réels* $\tau > 1$, *dont tous les conjugués (différents de* τ*) ont une valeur absolue inférieure ou égale à* 1, *l'un au moins ayant une valeur absolue égale à* 1. (R. Salem [13].)

Cette définition entraîne qu'un élément τ de T est racine d'un polynôme à coéfficients entiers rationnels, réciproque, de degré pair, ayant un zéro τ extérieur au cercle unité, un zéro $1/\tau$ intérieur au cercle unité, tous les autres zéros appartenant au cercle unité.

1.1. Dans tout corps de nombres algébriques réels, il existe des nombres de l'ensemble S ayant le degré du corps, au contraire il n'existe des éléments de l'ensemble T que dans certaines extensions quadratiques des corps totalement réels.

1.2. Nombres de Pisot et de Salem peuvent être caractérisés par des propriétés de répartition modulo 1. Soit θ un réel >1; l'étude de la décomposition, pour un élément λ convenable non nul,

$$\lambda\theta^n = u_n + \varepsilon_n$$

où u_n est un entier rationnel et où ε_n vérifie $-1/2 \leqslant \varepsilon_n < 1/2$, permet les caractérisations suivantes: