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A general arithmetic construction of transcendental non-Liouville normal numbers from rational fractions

by

R. G. STONEHAM (New York, N. Y.)

1. Introduction. In this paper, we derive a general arithmetic construction of an extensive class of transcendental non-Liouville normal numbers based on any given rational fraction $Z/m < 1$ in lowest terms. The construction is founded on the results in [14] wherein we proved that certain broad classes of rational fractions are (j, ε)-normal.

In [14], (1.1), we extended the original definition of (j, ε)-normality due to Besicovitch ([15], p. 201) so as to apply to appropriate rational fractions $Z/m < 1$ in lowest terms. Essentially, we showed that the definition of (j, ε)-normality which Besicovitch defined for finite sets of digits could be applied to the infinite periodic sequences which represent certain broad classes of rational fractions. Therefore, we can consider whether some given rational fraction Z/m when represented in appropriate bases g is (j, ε)-normal or not in this sense.

Consider the real number $x = .x_1x_2\ldots$ represented in the scale g and let $N(B_j, X_\lambda)$ denote the number of occurrences of the block B_j consisting of any combination of j digits chosen from $0, 1, \ldots, g-1$ in the first λ digits $x_1x_2\ldots x_\lambda$ of x. We have the following definition ([7], p. 95, 104) equivalent to that given by Borel in 1908. Unless otherwise indicated, lower case letters will represent positive integers.

DEFINITION. *Normal number*. The number x is *normal* in the scale g if

$$\lim_{\lambda\to\infty} N(B_j, X_\lambda)/\lambda = 1/g^j \tag{1.0}$$

for all $j = 1, 2, 3, \ldots$

If x is any real number, x is said to be normal to the base g if $\{x\}$ $\{x\} = x-[x]$ is normal to the base g where $\{x\}$ is the fractional part of x and $[x]$ is the greatest integer not exceeding x. Furthermore, if some x is to satisfy (1.0), i.e. be a normal number, then it is, necessarily, an irrational.

Borel proved ([7], p. 103) by means of probabilistic arguments that almost all real numbers are absolutely normal, i.e. normal to every positive integer base and the set of non-normal numbers has measure zero.

In spite of this broad conclusion, there are only a few simple constructive methods in the literature today which give normal numbers by elementary arithmetic procedures and there are no results to date which show that a given construction produces a transcendental or algebraic irrational. Furthermore, no methods have been found to prove that a given irrational like π, e, $\sqrt{2}$, etc. is normal to any base.

In 1959, Mark Kac ([5], p. 18) remarks "As is often the case, it is much easier to prove that an overwhelming majority of objects possess a certain property than to *exhibit* (italics his) even one such object. The present case is no exception. It is quite difficult to exhibit a 'normal' number! The simplest example is the number (written in decimal notation) $x = .1234567891011\ldots$ where after the decimal point we write the positive integers in succession. The proof that this number is normal is by no means trivial." The normality to the base 10 of this example was proved by Champernowne ([1], and [7], p. 112) in 1934.

Based on the results of H. Weyl [16], Pólya and Szegö in 1925 showed ([8], p. 71) that the same $x = .1234567891011\ldots$ was such that $\{10^j x\}$ for $j = 0, 1, \ldots$ is uniformly distributed on $[0, 1]$. In 1937, K. Mahler ([6], p. 6; 9] showed that the irrational $\alpha = .p(1)p(2)\ldots$ where $p(x)$ is an integral-valued polynomial which is positive for $x \geqslant 1$, as a special case of theorems of the Siegel–Schneider type, is a transcendental of the non-Liouville type. It can easily be shown that the Liouville number $\beta = \sum_{j=1}^{\infty} 1/g^{j!}$ is not normal to the base g.

Now the Champernowne example of a normal number is precisely Mahler's construction ([3], p. 6) if $p(x) = x$ and in 1952 Davenport and Erdös ([2]) generalized Champernowne's example to the arbitrary integral-valued polynomial case of Mahler, i.e they proved such a construction normal to appropriate bases.

However, to date it has not been pointed out that these normal numbers of Champernowne, Davenport and Erdös are transcendentals of the non-Liouville type by the results of Mahler ([6]) in 1937.

The first specific construction of an absolutely normal number after the results of Borel was given by Sierpiński ([12]) in 1917 and later, a method was presented by Schmidt ([11]) in 1962. These results were obtained by complex constructive devices usually the upper or lower bound of certain iterative schemes which were quite specialized for the purpose. They are not simple arithmetic constructions like Champernowne's example. Indeed, Sierpiński says in his recent 1964 book ([13]) on number theory, "Therefore, though according to the theorem of Borel, almost all numbers are absolutely normal, it was by no means easy to construct an example of an absolutely normal number. Examples of such numbers are, indeed, fairly complicated." The existential measure-theoretic results fairly frequent in the literature do not lead to elementary constructions. A good recent survey of constructive methods for normal numbers is given by A. G. Postnikov ([9], p. 64) in 1967.

In 1954, Hanson ([4], Th. (2.2)) studied the sufficient conditions under which the number $x = .a_1 a_2 \ldots$ formed by adjoining an increasing sequence $\{a_n\}$ of (j, ε)-normal sets of digits a_n to the base g was normal to the base g where almost all of the sets a_n were (j, ε)-normal. Hanson did not give examples of specific normal numbers due to the lack of a ready source of (j, ε)-normal numbers.

By means of the results in ([14], Theorem 6), we are able to prove a general construction of normal numbers in Theorem 1 of this paper based on the (j, ε)-normality of a broad class of rational fractions Z/m of type A. The construction has considerable flexibility in the choice of quantities that enter the construction. The resulting normal number can also be written as a closed algebraic sum. For example, we will prove (in Corollary 2 to Theorem 1) that given any odd prime p that

$$x(g, p) = (p-1) \sum_{n=0}^{\infty} 1/p^{n+1} g^{(np^{n+1}-(n+1)p^n+1)/(p-1)} \tag{1.1}$$

is a transcendental of the non-Liouville type normal to any base g^t where t is any positive integer and g is a primitive root $\bmod p^2$.

In Theorem 2, we will prove that the normal numbers constructed in Theorem 1 are transcendentals of the non-Liouville type by an argument similar to that used by Mahler in [6] based on a well-known theorem of Schneider concerning transcendentals.

The normal numbers are constructed by adjoining in juxtaposition a_n repetitions of the sets of digits contained in the recurring period of each rational fraction Z_n/m^n expanded in a base g such that $(g, m) = 1$, m is any given positive integer, the Z_n are any positive integers such that for every $n = 1, 2, \ldots, 1 \leqslant Z_n < m^n$, $(Z_n, m) = 1$, and the a_n are any increasing sequence of positive integers such that $a_n \to \infty$ as $n \to \infty$. The resulting irrational number which we denote by $x(g, m)$ given by this construction is normal to every base g relatively prime to the given positive integer m. It has been shown by Schmidt [10] that if a number is normal to the base g that it is also normal to any positive integral power of that base. We make use of this result in extending the set of bases to g^t for any positive integer t.

2. The normal number construction. We prove the following theorem from which we may derive a number of corollaries giving specific forms for normal numbers based on various types of rational fractions.

THEOREM 1. *Let g and m denote a pair of relatively prime integers $\geqslant 2$. For each $y \geqslant 2$ and relatively prime to g, let $\omega(y)$ denote the order of g to the modulus y. We let a_n, Z_n for $n = 1, 2, \ldots$ denote any two sequences of positive integers such that $a_0 = 0 = Z_0$, and also the further conditions $a_n \to \infty$ as $n \to \infty$, as well as $1 \leqslant Z_n < m^n$ and $(Z_n, m) = 1$ for $n \geqslant 1$. If we let $S(n, m) = \sum_{i=1}^{n} a_i \omega(m^i)$ where $S(0, m) = 0$ and define*

$$x(g, m) = \sum_{n=0}^{\infty} (Z_{n+1} - mZ_n)/m^{n+1} g^{S(n,m)} \tag{2.0}$$

then the real number $x(g, m)$ is normal to the base g^t for each integer $t > 0$.

Proof. Let the rational fractions $Z_1/m, Z_2/m^2, \ldots, Z_n/m^n, \ldots$ in lowest terms be defined for every positive integer n where the integers Z_n are chosen as specified in Theorem 1 and $m = 2^a \prod_{i=1}^{r} p_i^{a_i}$ is some positive integer.

Consider the set E_n consisting of $\omega(m^n)$ digits contained in the recurring periods of Z_n/m^n defined for every $n = 1, 2, \ldots$ represented in a scale g such that $(g, m) = 1$. Let $N(B_j, E_i)$ denote the number of occurrences of the block B_j in the set of digits E_i contained in one recurring period of the ith rational fraction Z_i/m^i where the block B_j commences in one period and extends at most $j-1$ places into the next.

Let $E_i(a_i)E_i$ denote the set of digits $E_iE_i \ldots E_i$ placed in juxtaposition $a_i > 0$ times, and write the number $x(g, m, n)$ as

$$x(g, m, n) = .E_1(a_1)E_1E_2(a_2)E_2 \ldots E_{n-1}(a_{n-1})E_{n-1}E_n(k)E_nB_r \tag{2.1}$$

where B_r consists of the first r digits into the $(k+1)$-st repetition of the E_n-th set such that $0 \leqslant r < \omega(m^n)$. We distinguish two cases for the positive integer k in the construction, Case 1: $1 \leqslant k < a_n$ and Case 2: $k = a_n$. The construction for Case 1 is that given in (2.1). For Case 2, if $k = a_n$, then the block of digits $B_r = b_1b_2 \ldots b_r$ at the end of $x(g, m, n)$ consists of the first r digits of the first repetition of the periodic set E_{n+1} from Z_{n+1}/m^{n+1} where we have adjoined complete sets of $E_1, \ldots, E_n$ repeated $a_1, a_2, \ldots, a_n$ times, respectively, and $0 \leqslant r < \omega(m^{n+1})$.

Let $N(t, x, B_j)$ denote the number of occurrences of the block B_j in the first t digits of $x(g, m, n)$ where, for Case 1, the sequence of t digits terminates r digits into the $(k+1)$-st repetition of the $\omega(m^n)$ digits contained in E_n, and for Case 2, terminates at the rth digit of the first repetition of the set E_{n+1}.

Let the quantity $N(B_j, E_n)$ essential to determining $N(t, x, B_j)$ designate the number of occurrences of independent blocks B_j commencing in one period of Z_n/m^n and terminating in at most $j-1$ places of the next period. Since the initial digit b_i of any B_j corresponds to any residue r_i which belongs to the complete periodic set of reduced power residues $r_i/m^n = \{Z_n g^i/m^n\}$ for $i = 0, 1, \ldots, \omega(m^n)-1$ contained in $B_j/g^j < r_i/m^n < (B_j+1)/g^j$, we must consider the fact that there is included in $N(B_j, E_n)$ the count of what we called "anomalous blocks" that were defined in ([15], p. 204). These were defined as independent blocks B_j whose initial digit corresponded to a residue near the end of the period and whose length was such that the final digits corresponded to residues in at most $j-1$ residues of the next period repetition.

In order to determine bounds on $N(t, x, B_j)/t$, the relative frequency of the B_j contained in $x(g, m, n)$, we define the ratio I for Cases 1 and 2 by

$$I = \Big(\sum_{i=1}^{n-1} a_i N(B_j, E_i) + kN(B_j, E_n) + N(B_j, r)\Big)/t \tag{2.2}$$

where the total number of digits in $x(g, m, n)$ is given by

$$t = \sum_{i=1}^{n-1} a_i\omega(m^i) + k\omega(m^n) + r. \tag{2.3}$$

To complete the bounds, it is necessary to account for the anomalous blocks in the count of the B_j which may occur across the $n-1$ junctures of E_iE_{i+1} for $i = 1, 2, \ldots, n-1$ and possibly an additional $j-1$ between E_n and B_r for both Case 1 and 2. From such considerations and using I in (2.2), we obtain

$$I - n(j-1)/t \leqslant N(t, x, B_j)/t \leqslant I + n(j-1)/t \quad \text{or} \tag{2.4}$$

$$|N(t, x, B_j)/t - I| \leqslant R_n \tag{2.5}$$

where $R_n = n(j-1)/t$ accounts for either the anomalous blocks in excess or deficiency in the count $N(t, x, B_j)$.

Since $t \to \infty$ as $n \to \infty$ according to (2.3), we prove in Lemma 1 that $\lim_{n\to\infty} R_n = 0$ and hence from (2.5), we have

$$\lim_{n\to\infty} N(t, x, B_j)/t = \lim_{n\to\infty} I. \tag{2.6}$$

In Lemma 2, we evaluate $\lim_{n\to\infty} I$ and then proceed with the argument.

LEMMA 1. *If t is defined by (2.3), then for any fixed $j = 1, 2, \ldots$, $\lim_{n\to\infty} R_n = \lim_{n\to\infty} n(j-1)/t = 0$.*

Proof. Consider Case 1 where $1 \leqslant k < a_n$, then we have

$$R_n = n(j-1)/t < n(j-1)/\sum_{i=1}^{n-1} a_i \tag{2.7}$$

since $\sum_{i=1}^{n-1} a_i\omega(m^i) + k\omega(m^n) + r > \sum_{i=1}^{n-1} a_i$.

By hypothesis, $a_i \to \infty$ as $i \to \infty$ with $a_i > 0$, therefore for every $\varepsilon > 0$, there exists an N such that for all $k > N$, $a_k > 1/\varepsilon$. Hence

$$\sum_{i=N+1}^{n-1} a_i > (n-N-1)/\varepsilon,$$

and it follows that

$$R_n = n(j-1)/t < n(j-1)\varepsilon/(n-N-1) < 2\varepsilon(j-1) \tag{2.8}$$

for all $n > 2N+2$. Consequently, under the hypotheses in Theorem 1 on a_i, we have for any fixed choice of $j \geqslant 1$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} n(j-1)/t = 0. \tag{2.9}$$

A similar line of reasoning shows that (2.9) also holds for Case 2. The proof of Lemma 1 is now complete.

Lemma 2. *If I is defined by (2.2), then*

$$\lim_{n\to\infty} I = \lim_{n\to\infty} N(B_j, E_n)/\omega(m^n).$$

Proof. For Case 1, I can be written

$$I = \Big(\Big(\sum_{i=1}^{n-1} a_i N(B_j, E_i) + kN(B_j, E_n)\Big)\Big/P_n + N(B_j, r)/P_n\Big)\Big/(1+r/P_n) \tag{2.10}$$

where $P_n = \sum_{i=1}^{n-1} a_i\omega(m^i) + k\omega(m^n)$ for $1 \leqslant k < a_n$. For Case 2, we have $k = a_n$ and

$$I' = \Big(\Big(\sum_{i=1}^{n} a_i N(B_j, E_i)\Big)\Big/P'_n + N(B_j, r)/P'_n\Big)\Big/(1+r/P'_n) \tag{2.11}$$

where $P'_n = \sum_{i=1}^{n} a_i\omega(m^i)$.

Since the least exponent $\omega(m^n)$ is such that $g^{\omega(m^n)} \equiv 1 \bmod m^n$ for $n > z_M$ where z_M is the maximum integer in the set of positive integers z_i such that $p_i^{z_i} \| (g^{d_i}-1)$, for each prime factor p_i contained in m, it follows inductively that

$$g^{m\omega(m^{n-1})} \equiv g^{\omega(m^n)} \equiv 1 \bmod m^n$$

or

$$\omega(m^n) = m\omega(m^{n-1}) \tag{2.12}$$

for all $n > z_M$, i.e. n sufficiently large.

From the definitions of $N(B_j, r)$, r, and the period length of the set E_n in Case 1, we have the inequalities

$$0 \leqslant \big(N(B, r) \text{ or } r\big) \leqslant \omega(m^n) \tag{2.13}$$

and using (2.12), we obtain

$$0 \leqslant \big(N(B_j, r) \text{ or } r\big) \leqslant m\omega(m^{n-1}) \tag{2.14}$$

for n sufficiently large. The upper bound in (2.14) can be written as in

$$m\omega(m^{n-1})/P_n = m\Big/\Big(\sum_{i=1}^{n-2} a_i\omega(m^i)/\omega(m^{n-1}) + a_{n-1} + km\Big). \tag{2.15}$$

Since m is fixed and by hypothesis $a_n \to \infty$ as $n \to \infty$, it follows from (2.15) that $\lim_{n\to\infty} m\omega(m^{n-1})/P_n = 0$. Therefore, using this result in (2.14), we have $\lim_{n\to\infty} N(B_j, r)/P_n = 0$ and $\lim_{n\to\infty} r/P_n = 0$ which are the limits for such ratios in (2.10). By using (2.12) again in similar inequalities for Case 2, the same results hold for the ratios $N(B_j, r)/P'_n$ and r/P'_n.

If we apply Cauchy's generalized limit theorem to the remaining ratios in (2.10) and (2.11), we find for Case 1 from (2.10) replacing P_n that

$$\lim_{n\to\infty} I = \lim_{n\to\infty}\Big(\sum_{i=1}^{n-1} a_i N(B_j, E_i) + kN(B_j, E_n)\Big)\Big/\Big(\sum_{i=1}^{n-1} a_i\omega(m^i) + k\omega(m_n)\Big) \tag{2.16}$$

or

$$\lim_{n\to\infty} I = \lim_{n\to\infty} N(B_j, E_n)/\omega(m^n). \tag{2.17}$$

By the same argument, we find for Case 2 that $\lim_{n\to\infty} I' = \lim_{n\to\infty} I$. Lemma 2 is now complete.

Therefore, for both Case 1 and 2, we have from (2.6)

$$\lim_{n\to\infty} N(t, x, B_j)/t = \lim_{n\to\infty} N(B_j, E_n)/\omega(m^n). \tag{2.18}$$

Since the rational fractions $Z_n/m^n < 1$ in lowest terms are rationals of Type A for n sufficiently large, we have from ([14], (1.1), and Theorem 6), for a given integer m, the (j, ε)-normal property, so that we may write

$$\lim_{n\to\infty} |N(B_j, E_n)/\omega(m^n) - 1/g^j| < \lim_{n\to\infty} \varepsilon \tag{2.19}$$

where $\lim_{n\to\infty} \varepsilon = \lim_{n\to\infty} 1/\prod_{i=1}^{r} p^{a_i n - t_i}$ according to ([14], Theorem 6), for $m^n = 2^{an}\prod_{i=1}^{r} p_i^{a_i n}$ and all j such that

$$j \leqslant \lim_{n\to\infty} [\log_g 1/\varepsilon] = \lim_{n\to\infty}\Big[\log_g \prod_{i=1}^{r} p_i^{a_i n - t_i}\Big]. \tag{2.20}$$

Since $t_i = \min(a_i n, z_i + s_i)$ and the z_i and s_i are fixed for a given set of a_i and prime factors p_i in any m, it follows that $t_i = z_i + s_i$ is also fixed for n sufficiently large. Therefore, $\lim_{n\to\infty} \varepsilon = 0$, thus (2.19) and (2.20) yield

$$\lim_{n\to\infty} N(B_j, E_n)/\omega(m^n) = 1/g^j \tag{2.21}$$

for all j and $g \geqslant 2$ such that $(g, m) = 1$ where the upper bound on the bases g according to ([14], Theorem 6), is $\lim_{n\to\infty} 1/\varepsilon \to \infty$.

Finally, if we consider (2.5) and apply Lemmas 1 and 2 in combination with (2.17) and (2.21), we have

$$\lim_{t\to\infty} N(t, x, B_j)/t = 1/g^j \tag{2.22}$$

for all j. If we define $x(g, m) = \lim_{n\to\infty} x(g, m, n)$, then we have shown that $x(g, m)$ is normal in any scale $g \geqslant 2$ relatively prime to m.

Finally, by appropriate shifts in the place position of each set $E_i(a_i)E_i$ in the symbolic representation of $x(g, m) = \lim_{n\to\infty} x(g, m, n)$ in (2.1) and summing, we may obtain the closed form in (2.0). For example, it is clear that

$$E_1(a_1)E_1 = Z_1/m - Z_1/mg^{a_1\omega(m)}. \tag{2.23}$$

If we shift the set $E_2(a_2)E_2$ to its proper position $a_1\omega(m)$ places to the right, then we have

$$E_1(a_1)E_1E_2(a_2)E_2 = Z_1/m - Z_1/mg^{a_1\omega(m)} + Z_2/m^2 g^{a_1\omega(m)} - Z_2/m^2 g^{a_1\omega(m)+a_2\omega(m^2)} \tag{2.24}$$

or

$$E_1(a_1)E_1E_2(a_2)E_2 = Z_1/m + (Z_2/m^2 - Z_1/m)g^{a_1\omega(m)} - Z_2/m^2 g^{a_1\omega(m)+a_2\omega(m^2)}. \tag{2.25}$$

If we continue in this way, we will obtain

$$x(g, m) = Z_1/m + (Z_2/m^2 - Z_1/m)/g^{a_1\omega(m)} + (Z_3/m^3 - Z_2/m^2)/g^{a_1\omega(m)+a_2\omega(m^2)} + \ldots + (Z_{n+1}/m^{n+1} - Z_n/m^n)/g^{S(n,m)} + \ldots \tag{2.26}$$

where the positive integer

$$S(n, m) = \sum_{i=1}^{n} a_i\omega(m^i). \tag{2.27}$$

From (2.26), we obtain the form given for $x(g, m)$ in (2.0) where we assume $a_0 = 0$ and $Z_0 = 0$ in order to include the first term Z_1/m in the sum for $n = 0$. From (2.27) and the fact that $a_0 = 0$, we have $S(0, m) = 0$. The proof of Theorem 1 is now complete.

THEOREM 2. *If there exists 2 positive constants independent of n such that*

$$\delta < a_{n+1}\omega(m^{n+1})/S(n, m) < \beta \quad \textit{for} \quad n = 1, 2, \ldots$$

then $x(g, m)$ in (2.0) is a transcendental of the non-Liouville type.

Proof. From (2.0), we have

$$|x(g, m) - p_s/q_s| = |R_s| = \Big|\sum_{n=s+1}^{\infty} (Z_{n+1} - mZ_n)/m^{n+1}g^{S(n,m)}\Big| \tag{2.28}$$

where $\sum_{n=0}^{s} (Z_{n+1} - mZ_n)/m^{n+1}g^{S(n,m)} = p_s/q_s$ with p_s and q_s positive integers for each s. Since by hypothesis, $S(i, m) \leqslant S(s, m)$ for every $i = 0, 1, \ldots, s$, it follows that $m^{i+1}g^{S(i,m)} | m^{s+1}g^{S(s,m)}$ and $q_s = m^{s+1}g^{S(s,m)}$ is the L.C.D.

In Mahler ([6], p. 427 (top)), we identify $q_s = q_s' q_s'' = m^{s+1}g^{S(s,m)}$, i.e. $q_s' = m^{s+1}$ and $q_s'' = g^{S(s,m)}$ where $q = g$ as required in the special case of Schneider's transcendence theorem which Mahler ([6], p. 427 (footnote)) proved in 1936. We are required to show two preliminary conditions, $\lim_{s\to\infty} \log q_s'/\log q_s = 0$ and $\lim_{s\to\infty} \sup \log q_{s+1}/\log q_s < \infty$.

Since it is possible that $\omega(m) = \omega(m^2) = \ldots = \omega(m^k)$ for some fixed positive integer k and some base g, we have using (2.12) for s sufficiently large that

$$S(s, m) = (a_1 + a_2 + \ldots + a_k)\omega(m^k) + a_{k+1}m\omega(m^k) + \ldots + a_s m^{s-k}\omega(m^k). \tag{2.29}$$

From the hypothesis in Theorem 1 which states that the $a_i \geqslant 1$ for all i, we obtain from (2.29) the fact that

$$S(s, m) \geqslant \big(k + m(m^{s-k} - 1)/(m-1)\big)\omega(m^k) \tag{2.30}$$

or

$$S(s, m) \geqslant C_0 + C_1 m^{s-k} \tag{2.31}$$

where $C_0 = \big(k - m/(m-1)\big)\omega(m^k)$ and $C_1 = m\omega(m^k)/(m-1)$ for some fixed k. Hence (2.31) implies that

$$\lim_{s\to\infty} (s+1)/S(s, m) = 0. \tag{2.32}$$

From (2.32), it follows that

$$\lim_{s\to\infty} \log q_s'/\log q_s = \lim_{s\to\infty} \log m^{s+1}/\log m^{s+1}g^{S(s,m)} = 0. \tag{2.33}$$

For the second preliminary condition, if we require that

$$a_{s+1}\omega(m^{s+1})/S(s,m) < \beta \tag{2.34}$$

where β is some constant independent of s, then

$$\limsup_{s\to\infty} \log q_{s+1}/\log q_s < \infty. \tag{2.35}$$

This follows since

$$\log q_{s+1}/\log q_s = \log m^{s+2} g^{S(s,m)}/\log m^{s+1} g^{S(s,m)} \tag{2.36}$$

and using the identity

$$S(s+1,m) = S(s,m) + a_{s+1}\omega(m^{s+1}) \tag{2.37}$$

for sufficiently large s, we obtain

$$\log q_{s+1}/\log q_s < \big((s+2)\log m/S(s,m) + (1+\beta)\log g\big)/\big((s+1)\log m/S(s,m) + \log g\big) \tag{2.38}$$

where we have introduced (2.34), (2.37), and arranged. Clearly then,

$$\limsup_{s\to\infty} \log q_{s+1}/\log q_s < \infty \tag{2.39}$$

where we can make use of (2.31) as we did in (2.32).

Next, we must show that there exists a constant $x > 1$, independent of s, such that from (2.28)

$$|x(g,m) - p_s/q_s| = |R_s| \leqslant (q_s)^{-x}. \tag{2.40}$$

The hypothesis $1 \leqslant Z_n < m^n$ implies that it is sufficient to consider

$$|R_s| \leqslant \sum_{n=s+1}^{\infty} 1/g^{S(n,m)} \leqslant 2/g^{S(s+1,m)} \tag{2.41}$$

since $(Z_{n+1} - mZ_n)/m^{n+1} g^{S(n,m)} < 1/g^{S(n,m)}$ and we have rapid convergence. Therefore, we will show that there exists an $x > 1$ independent of s such that

$$2/g^{S(s+1,m)} = 2/g^{S(s,m)+a_{s+1}\omega(m^{s+1})} < (1/m^{s+1} g^{S(s,m)})^{x}. \tag{2.42}$$

From (2.42), taking logarithms to the base g, we may obtain

$$x < \big(1 + a_{s+1}\omega(m^{s+1})/S(s,m) - \log_g 2/S(s,m)\big)/\big(1 + (s+1)\log_g m/S(s,m)\big). \tag{2.43}$$

Let ε_1 and ε_2 be arbitrarily small fixed positive constants independent of s, then we clearly have for s sufficiently large

$$\log_g 2/S(s,m) < \varepsilon_1 \quad \text{and} \quad (s+1)\log_g m/S(s,m) < \varepsilon_2. \tag{2.44}$$

Since $1 + (s+1)\log_g m/S(s,m) > 1/(1+\varepsilon_2)$ and $-\log_g 2/S(s,m) > -\varepsilon_1$, if we assume that $a_{s+1}\omega(m^{s+1})/S(s,m) > \delta$ where $\delta > 0$ is some positive constant independent of s, then for s sufficiently large, we have

$$x < (1+\delta-\varepsilon_1)/(1+\varepsilon_2). \tag{2.45}$$

Let $x = 1+\delta/2$, then we obtain

$$0 < 2(\varepsilon_1+\varepsilon_2)/(1-\varepsilon_2) < \delta < a_{s+1}\omega(m^{s+1})/S(s,m) < \beta \tag{2.46}$$

in combination with (2.34) where the lower bound on δ can be chosen arbitrarily small for s sufficiently large. Therefore, we have shown as required that there exists a positive constant $x > 1$ independent of s such that (2.40) holds.

For the non-Liouville part, Mahler ([6], p. 427) shows that if there exists an s sufficiently large for some large denominator Q in a sequence of successive approximations P/Q to $x(g,m)$ such that

$$|Q_s R_s| < 1/2Q \leqslant |Q_{s-1} R_{s-1}| \tag{2.47}$$

then

$$|x(g,m) - P/Q| \geqslant \tfrac{1}{2}|R_s|. \tag{2.48}$$

Now, we have

$$|Q_s R_s| = \Big| m^{s+1} g^{S(s,m)} \sum_{n=s+1}^{\infty} (Z_{n+1} - mZ_n)/m^{n+1} g^{S(n,m)} \Big| < 2m^{s+1} g^{S(s,m)}/g^{S(s+1,m)} = 2m^{s+1}/g^{a_{s+1}\omega(m^{s+1})} \to 0 \tag{2.49}$$

as $s \to \infty$ since

$$2m^{s+1}/g^{a_{s+1}\omega(m^{s+1})} = 2/g^{a_{s+1}m^{s+1-k}\omega(m^k) - (s+1)\log_g m} \to 0$$

easily for some fixed k. For some arbitrarily chosen large Q, choose the first s such that

$$|Q_s R_s| < 1/2Q \tag{2.50}$$

then clearly (2.47) is satisfied. Since

$$|x(g,m) - P/Q| \geqslant \tfrac{1}{2}|R_s| > \tfrac{1}{2}\big(1/m^{s+2} g^{S(s+1,m)} - 2/g^{S(s+2,m)}\big), \tag{2.51}$$

it follows that for s sufficiently large, we have

$$|x(g,m) - P/Q| > 1/m^{s+2} g^{S(s+1,m)}\big(\tfrac{1}{2} - 1/(m^{s+2})^{a_{s+2}-1}\big) \geqslant 1/3Q_{s+1} \tag{2.52}$$

where we have used the fact that $g^{\omega(m^{s+2})} > m^{s+2} \Rightarrow 1/g^{a_{s+2}\omega(m^{s+2})} < 1/(m^{s+2})^{a_{s+2}}$.

Also, we have

$$1/2Q \leqslant |Q_{s-1} R_{s-1}| < 2m^s g^{S(s-1,m)}/g^{S(s,m)} = 2m^s/g^{a_s\omega(m^s)} = 1/Q_s^t \tag{2.53}$$

where

(2.54) $$t = \left(a_s\omega(m^s) - s\log_g m - \log_g 2\right)/\log_g Q_s.$$

Since $Q_{s+1} = mg^{a_{s+1}\omega(m^{s+1})}Q_s = Q_s^y$, then $1/2Q < 1/Q_s^t \Rightarrow$

(2.55) $$|x(g, m) - P/Q| > 1/3(2Q)^{y/t}$$

and the non-Liouville character follows if y/t is bounded for s sufficiently large.

We find that

(2.56) $$\frac{y}{t} = \frac{1 + a_{s+1}\omega(m^{s+1})/S(s, m) + (s+2)\log_g m/S(s, m)}{a_s\omega(m^s)/S(s, m) - s\log_g m/S(s, m) - \log_g 2/S(s, m)}.$$

If we require again as in (2.34) that $a_{s+1}\omega(m^{s+1})/S(s, m) < \beta$ then it also follows that

$$a_s\omega(m^s)/S(s, m) \leqslant a_{s+1}\omega(m^{s+1})/S(s, m) < \beta$$

and y/t is bounded for s sufficiently large. Theorem 2 is now complete.

An interesting g-adic form for the normal numbers $x(g, m)$ in (2.0) can be given. Of the many possible corollaries to Theorem 1, we present the following:

COROLLARY 1.

$$x(g, m) = \sum_{n=1}^{\infty} A_n/g^{S(n,m)}$$

where the positive integers A_n are given by $A_n = Z_n(g^{a_n\omega(m^n)} - 1)/m^n$.

Proof. By rearrangement of (2.26), we may obtain

(2.57)
$$x(g, m) = (Z_1/m)(1 - 1/g^{a_1\omega(m)}) + (Z_2/m^2)(1/g^{a_1\omega(m)} - 1/g^{a_1\omega(m)+a_2\omega(m^2)}) + \\ + (Z_3/m^3)(1/g^{S(2,m)} - 1/g^{S(3,m)}) + \ldots + (Z_n/m^n)(1/g^{S(n-1,m)} - 1/g^{S(n,m)}) + \ldots$$

Now

$$Z_n(1/g^{S(n-1,m)} - 1/g^{S(n,m)})/m^n = Z_n(g^{a_n\omega(m^n)} - 1)/m^n g^{S(n,m)}$$

and since

$$g^{\omega(m^n)} \equiv 1 \bmod m^n \Rightarrow m^n | (g^{a_n\omega(m^n)} - 1),$$

we have

$$(Z_n/m^n)(1/g^{S(n-1,m)} - 1/g^{S(n,m)}) = A_n/g^{S(n,m)}$$

where $A_n = Z_n(g^{a_n\omega(m^n)} - 1)/m^n$ is some positive integer. Q.E.D.

Since the positive integers in the increasing sequence a_n can be freely chosen, and the Z_n such that $1 \leqslant Z_n < m^n$ and $(Z_n, m) = 1$ are any

positive integers which can be chosen independently for each n, we may derive a great variety of specific examples. As an illustration, let $m = p$, g be a primitive root $\bmod p^2$, $Z_n/m^n = (p^n - 1)/p^n$, and $a_n = n$, then we have the following corollary.

COROLLARY 2. *The real number*

$$x(g, p) = (p-1)\sum_{n=0}^{\infty} 1/p^{n+1}g^{(np^{n+1}-(n+1)p^n+1)/(p-1)}$$

is a transcendental normal number of the non-Liouville type where p is any odd prime and g is a primitive root $\bmod p^2$.

Proof. From (2.0), we have

$$x(g, p) = \sum_{n=0}^{\infty} \left(p^{n+1} - 1 - p(p^n - 1)\right)/p^{n+1}g^{S(n,p)}$$

where

$$S(n, p) = \sum_{i=1}^{n} i\omega(p^i) = (p-1)\sum_{i=1}^{n} ip^{i-1} = \left(np^{n+1} - (n+1)p^n + 1\right)/(p-1).$$

For the transcendence condition, we have

$$a_{n+1}\omega(m^{n+1})/S(n, m) = (n+1)\omega(p^{n+1})/S(n, p) \\ = (n+1)p^n(p-1)^2/\left(np^{n+1} - (n+1)p^n + 1\right) \sim p - 1$$

for large n and

$$a_{n+1}\omega(m^{n+1})/S(n, m) = 2p$$

for $n = 1$. Therefore, we may write

$$\delta = p - 2 < a_{n+1}\omega(m^{n+1})/S(n, m) < 3p = \beta$$

and we have satisfied the transcendence conditions in Theorem 2 for $n = 1, 2, 3, \ldots$

Finally, consider the application of these results to Diophantine approximations. Let $\theta = (.b_1b_2 \ldots b_r)b_{r+1}\ldots$ be some irrational < 1. Consider a normal number $x(g, m)$ constructed from some rational approximation to θ given by

$$x(g, m) = (.b_1b_2 \ldots b_r)a_{r+1}\ldots a_{\omega(m)}E_1(a_1 - 1)E_1E_2(a_2)E_2 \ldots$$

which agrees with θ to r figures where $Z_1/m = (.b_1b_2 \ldots b_r)a_{r+1}\ldots \ldots a_{\omega(m)}b_1b_2\ldots$

For example, the approximations Z_1/m to θ could be the convergents $h_n/k_n = Z_1/m$ in a continued fraction representation of θ. We may, therefore, construct a normal number arbitrarily close to a given θ. In fact, we can show

$$|\theta - x(g, k_n)| < 2/g^{a_1\omega(k_n)} + 2/k_n^2 \tag{2.58}$$

where $x(g, k_n)$ is normal to some base g such that $(g, k_n) = 1$.

This follows from (2.0) by noting that

$$|x(g, m) - Z_1/m|$$
$$< |Z_2/m^2 g^{S(1,m)}| + |Z_1/mg^{S(1,m)}| + \Big|\sum_{n=2}^{\infty} (Z_{n+1} - mZ_n)/m^{n+1} g^{S(n,m)}\Big|.$$

Since $Z_1/m < 1$ and the constants Z_n can be chosen independent of n, we may set $Z_n = 1$ for $n \geqslant 2$ and obtain

$$|x(g, m) - Z_1/m| < 1/m^2 g^{S(1,m)} + 1/g^{S(1,m)} + \sum_{n=2}^{\infty} 1/m^{n+1} + \sum_{n=2}^{\infty} 1/m^n$$
$$< 2/g^{S(1,m)} + (m+1)/m^2(m-1) < 2/g^{S(1,m)} + 1/m^2$$

where $1/m^{n+1} g^{S(n,m)} < 1/m^{n+1}$ and $(m+1)/m^2(m-1) < 1/m^2$ holds for $m > 3$. Therefore, from the well-known fact that for convergents h_n/k_n, we have

$$|\theta - h_n/k_n| < 1/k_n k_{n+1} < 1/k_n^2,$$

and from the above $|x(g, m) - Z_1/m| < 2/g^{a_1\omega(m)} + 1/m^2$, we obtain (2.58).

Clearly then, we may exhibit a normal number as close to a given $\theta < 1$ as we please by this construction. This is a constructive result in contrast to the same derivable existential conclusion based on Borel's theorem that almost all real numbers are absolutely normal ([3], Th. 8. 11, p. 103]) which, of course, as most measure-theoretic conclusions proceed, does not indicate any method for the construction of the conclusion.

However, at present, there is still an open question which may not be trivial. Can we decide in a constructive sense, the base to which $x(g, k_n)$ constructed from the h_n/k_n is normal? The Theorem 1 requires that g be such that $(g, k_n) = 1$ and, of course, for a given k_n, we know such a g exists.

On the other hand, can one determine another sequence of rational approximations to a given irrational for which all denominators k_n are relatively prime to some fixed positive integer g?

In some future studies of these results, we will show that we can relax the requirement on the g in Theorem 1 to those g such that $(g, m) \geqslant 1$. Also, recently, in a private communication which will be published, E. Wirsing has shown that these transcendental non-Liouville normal numbers have measure zero.

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THE CITY COLLEGE OF THE CITY UNIVERSITY OF NEW YORK

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An asymptotic formula in the theory of numbers

by

YOICHI MOTOHASHI (Tokyo)

§ 1. Introduction. Asymptotic formulae for the sums

$$\sum_{n\leqslant N} \tau(n)\tau_k(n+a)$$

have been considered by Ingham [4], Hooley [3], and Linnik [5] for the case $k=2$, $k=3$ and all k, respectively, where $\tau_k(n)$ is the number of the representation of n as the product of k factors and $\tau(n)=\tau_2(n)$.

The purpose of this paper is to prove an asymptotic formula for the sum

$$\sum_{n\leqslant N} \tau^2(n)\tau(n+1).$$

Our method depends on the recently obtained result in the theory of the large sieve. It may be interesting to remark that the method of Hooley and Linnik largely depends on the very deep A. Weil's estimate of Kloosterman's sum, but our proof does not make any use of it.

Our result is as follows:

THEOREM.

$$\sum_{n\leqslant N} \tau^2(n)\tau(n+1) = \mathfrak{S}N(\log N)^4 + O\big(N(\log N)^3\log\log N\big).$$

Here the constant $\mathfrak{S}$ is defined by

$$\mathfrak{S} = \pi^{-2}\prod_p \left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^2\left(1+\frac{1}{p}\right)^{-1}\right).$$

Notations. Let p be generally a prime, χ a Dirichlet character and $\varphi(m)$ the Euler φ-function. The notation "$\ll$" is the usual Vinogradov's symbol.