On \((j, \varepsilon)\)-normality in the rational fractions

by

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1. Introduction. In 1964, we proved [6] that the distribution of the digits in the recurring period of the reciprocal of an integral power of an odd prime when represented in a scale \(g\) which is a primitive root \(\mod p^2\) is \((j, \varepsilon)\)-normal in the sense of Besicovitch ([6], p. 201). The computer studies we have carried out show that the \((j, \varepsilon)\)-normal phenomenon is quite extensive in the rational fractions.

In this paper, we generalize the results in [6] in several directions and show that broad classes of rational fractions \(\mathbb{Z}/m < 1\) in lowest terms, when represented in a base \(g\) such that \(2 < g < C(m)\) and \((g, m) = 1\) where \(C(m)\) is a constant that depends on \(m\), are \((j, \varepsilon)\)-normal.

In a sequel to the present paper, we will give a fairly elementary arithmetic construction of normal numbers which can be written in closed arithmetic forms based on any given rational fraction \(\mathbb{Z}/m < 1\). They are of such generality that we have been able to prove that they are transcendental of the non-Liouville type. These are, apparently, the first known general class of normal numbers whose irrational character has been demonstrated.

We have found it convenient to extend the Besicovitch definition ([6], p. 201) so as to apply to an infinite periodic representation of \(\mathbb{Z}/m\) which may or may not have a non-periodic part. Let \(N(B_j, g)\) denote the number of occurrences of the block of \(j\) digits \(B_j\) chosen from \(0, 1, \ldots, g-1\) commencing in any period of the representation of \(\mathbb{Z}/m\) in the scale \(g\) and terminating in at most \(j-1\) digits of the next period. Let \(x = x_0 x_1 x_2 \ldots\) be the representation of \(x\) in the scale \(g\) and let \(X_1\) denote the block of the first \(j\) digits in \(\bar{x}\) where \(N(B_j, X_1)\) denotes the number of occurrences of the block \(B_j\) in \(X_1\).

For convenience in notation, let us define the base dependent number-theoretic function \(\sigma(m) = \text{ord}_m g\) which will denote the number of digits in one period of \(\mathbb{Z}/m\) when represented in the scale \(g\). Without much difficulty, one can prove the following result. Unless otherwise indicated, lower case letters are positive integers.
**Lemma.** Let $Z/m < 1$ in lowest terms be represented in some scale $g$ such that $2 < g < m$, then if the representation is periodic, we have

$$\lim_{\lambda \to \infty} N(B_1, X_1)/\lambda = N(B_1, g)/\omega(m).$$

Essentially, this lemma states that the limiting relative frequency over the periodic infinite set is the same as the relative frequency over one period.

If $g$ contains all the prime factors of $m$, then $Z/m$ is terminating, in which case, we can define $(j, \varepsilon)$-normality for the finite set of digits by ([6], p. 201). However, if $g$ contains some but not all prime factors of $m$, then the expansion of $Z/m$ in the scale $g$ is periodic and may or may not have a periodic part. If this is so, we will use the following definition of $(j, \varepsilon)$-normality.

**Definition.** $(j, \varepsilon)$-normal rational fractions. Let $Z/m < 1$ in lowest terms have a periodic representation that may or may not have a non-periodic part in a scale $g$ such that $2 < g < m$. If for a given $j$ and $\varepsilon > 0$, every $j$ digit sequence $B_j$ which occurs in the expansion is such that

$$\lim_{\lambda \to \infty} |N(B_1, X_1)/\lambda - 1/g_j| = |N(B_1, g)/\omega(m) - 1/g_j| < \varepsilon$$

then $Z/m$ is $(j, \varepsilon)$-normal in the scale $g$.

Let $[a]$ denote the greatest integer not exceeding $a$ and $\{x\}$, the fractional part. If $(g, m) = 1$, then $Z/m$ has no non-periodic part, in which case, we have the periodic set of digits $E$ given by

$$Z/m = b_1b_2\ldots b_kb_{k+1}\ldots b_{k+n} = E$$

where $B_k = b_kb_{k+1}\ldots b_{k+n-1}$ is any block of $j$ digits chosen from $0, 1, \ldots, g-1$ whose first digit commences anywhere in $E$ and may extend at most $j-1$ digits into the next period. The digits $b_i$ are given by $b_i = \{gr_i/m\}$ and $B_1 = \{gr_1/m\}$ with the initial digit $b_1$ where the power residues are generated by $Zg = r_i \mod m$ for $i = 0, 1, \ldots, \omega(m) - 1$. Since there is a 1-1 correspondence for some bounded consecutive sequence of $j$ values between every $B_k$ for a given $j > 1$ and the power residues $r_i/m$ for $i = 0, 1, 2, \ldots, \omega(m) - 1$, the fundamental issue in order to prove the $(j, \varepsilon)$-normality of $Z/m$ is to show that the fractional parts $\{Zg/m\}$ are approximately “uniformly” distributed in some sense on $[0, 1]$.

We find most appropriate for the description of this finite discrete approximately uniform distribution the notion of what we shall call a “uniform distribution” which can be defined in terms of the discrepancy of the finite set $\{Zg/m\}$ on $[0, 1]$ as utilized by Weyl [7] and others.

In 1965, LeVeque ([3], p. 23) obtained a precise upper bound on the discrepancy $D_n$ ([3], p. 22) for finite sets in terms of the associated exponential sums by means of the characteristic function of probability theory.

**Definition.** Uniform $\varepsilon$-distribution. A sequence of real numbers $0 = a_0 < a_1 < \ldots < a_n < x_{n+1} = 1$ has a uniform $\varepsilon$-distribution on $[0, 1]$ if for a given $n$ sufficiently large there is an $\varepsilon > 0$ depending on $n$ and a $\delta$ such that $\max(x_{i+1} - x_i) \leq \delta < 1/4$ for $i = 0, 1, \ldots, n$ such that

$$D_n := \sup_{0 < \varepsilon < \delta < 1} |N(I)|/(\varepsilon(\beta - \alpha)) < \varepsilon$$

for all choices of $\beta - \alpha > \delta$ on $[0, 1]$ where $N(I)$ denotes the number of $a_k$ for $k \leq n$ contained in $[\alpha, \beta]$.

For the bound on $\delta$, we have made the minimal requirement on the distribution such that $\max(x_{i+1} - x_i) < 1/4$ for $i = 0, 1, \ldots, n$ on $[0, 1]$. The need for this condition will appear near the end of Theorem 2.

As an example to illustrate a uniform $\varepsilon$-distribution, we may prove the following theorem based on the results in ([6], p. 201).

**Theorem 1.** The rational fraction $Zg/p$ in lowest terms for $r \geq 1$ has a uniform $\varepsilon$-distribution of fractional parts $\{Zg/p\}$ for $i = 0, 1, \ldots, \varphi(p^-)/\varphi(p) - 1$, where $p$ is an odd prime, $g$ is a primitive root mod $p^1$, such that $2 < \varphi(p^-)/\varphi(p) < 1$, and if $e = 2/(p-1)$, then $\delta = 1/p$ for $r = 1$.

**Proof.** Following the procedure in ([6], p. 203), we determine number $N(I)$ of residues $r_i/p^e$ contained in an arbitrary interval $[a, b] \subseteq [0, 1]$, i.e., we have $\{Zg/p\} = r_i/p^e$ with

$$\alpha < r_i/p^e < \beta$$

where $0 \leq \alpha < \beta < 1$. Removing the number of residues in (1.4) not relatively prime to $p$, we have the number $N(I)$ of $\{Zg/p\} \subseteq [a, \beta]$

$$N(I) = \lfloor p^\beta \rfloor - \lfloor p^\alpha \rfloor - (\lfloor p^\beta - 1 \rfloor - \lfloor p^\alpha - 1 \rfloor)$$

similar to ([6], (3.12)). It is clear that (1.5) implies the following for $n = \varphi(p^-)

$$|\{Zg/p\} = \lfloor p^\beta \rfloor - \lfloor p^\alpha \rfloor < 2/\varphi(p^e)$$

where $\varphi(p^e) = (p-1)p^{e-1}$. Since we delete the residues not relatively prime to $p$, then at most 2 adjacent residues in the complete set differ by $2/p^e$. Therefore, for any choice of $[a, \beta] \subseteq [0, 1]$, $\delta = 1/p^e$ will insure that $[a, \beta]$ contains at least one $r_i/p^e$. For $r = 1$, we find that $\delta = 1/p$ and since

$$N(I) = \lfloor p^\beta \rfloor - \lfloor p^\alpha \rfloor = |\{Zg/p\} = (\beta - \alpha) < 1 + \beta - \alpha)/(p-1),$$

we have

$$\sup_{a < \beta < \delta, \varepsilon < \delta} (\beta - \alpha)/(p-1) = \varepsilon = 2/(p-1).$$
We see that the above $e$ and $\delta$ satisfy the requirements for a uniform $e$-distribution. Q.E.D.

In Theorem 1, note that the set of primitive roots is now confined to $2 \leq g < g'/2$ whereas in ([5], p. 201), we have $2 \leq g < g'$. The reason for this change will be discussed after Theorem 3.

In 1949, D. D. Wall ([5], p. 110) proved that a number $x$ is normal to the base $g$ if and only if $\{eg^i\}$ for $i = 0, 1, 2, \ldots$ are uniformly distributed on $[0, 1]$. There is a considerable literature to date which studies the relations between irrationals, normal numbers, Diophantine approximations, uniform distributions, etc. since the definitive paper of H. Weyl [7] in 1916.

The results which we present here show that there exists analogous properties between certain broad classes of rational fractions, $(j, e)$-normality, and uniform $e$-distributions. The next theorem is analogous to Wall's theorem in that it shows that the uniform $e$-distribution is a necessary and sufficient condition for $(j, e)$-normality.

2. Uniform $e$-distributions and $(j, e)$-normality.

THEOREM 2. The rational fraction $Z/m$ in lowest terms is $(j, e)$-normal in a scale $g$ such that $(g, m) = 1$ and $2 < g < 3/\delta$ for some $e > 0$ and $j = 1, 2, \ldots$, $\log_{10}/g$ for $i = 0, 1, \ldots, \omega(m) - 1$ has a uniform $e$-distribution, and conversely, if $Z/m$ is $(j, e)$-normal, then there exists an $e_1$ such that $\{Zg^i/m\}$ has a uniform $e_1$-distribution.

Proof. Consider (1.2) and the associated description below (1.2). If the $\hat{w}(m)$ residues $\hat{r}/m = \{Zg^i/m\}$ for $i = 0, 1, \ldots, \hat{w}(m) - 1$ have a uniform $e$-distribution then according to the definition (1.3), for each $n = \hat{w}(m)$, there is an $e$ and $\delta < \frac{1}{2}$ such that

$$D(\hat{w}(m)) = \sup_{0 < \alpha < 1} |N(I)/\hat{w}(m) - (\beta - \alpha)| < e$$

for all $\beta - \alpha > \delta$. Let us assume equal sub-intervals on $[0, 1]$ of width $1/g$ and choose $\beta = (B_1 + 1)/g$, $\alpha = B_1/g$ and $\beta - \alpha = 1/g$ for some $j$ value. Thus it follows from (2.0) that

$$|N(B_1, g)/\hat{w}(m) - 1/g| < \delta$$

and all $j$ such that $1/g > \delta$ for some $j$ where $N(I) = N(B_1, g)$, the number of $\{Zg^i/m\}$ contained in the sub-interval $[B_1/g, (B_1 + 1)/g]$ for some choice of $j$ digits $B_1$ which is, therefore, the count $N(B_1, g)$ of those $B_1$ whose initial digit $b_1$ is some digit in the recurring period. The block $B_1$ may extend into $j - 1$ digits of the next period. Furthermore, (2.1) holds for all $j > 1$ such that $j < \log_{10}/\delta$. Consequently, we require that $0 < \delta < 1/g < \frac{1}{2}$ in the given uniform $e$-distribution so that (2.1) holds for at least $j = 1$ for some given $g > 2$. Therefore, $Z/m$ is $(j, e)$-normal according to (1.1).

Conversely, assume that $[0, 1]$ has been divided into $g'$ equal sub-intervals $[0, 1/g'], [1/g', 2/g'], \ldots, [1 - 1/g', 1]$ for some choice of $Z/m$ in lowest terms which is $(j, e)$-normal for some given $e$ and $\delta > 0$. Let an arbitrary interval $I = (\alpha - e, \alpha) g'$ such that $0 < \alpha < \beta \leq 1$ contain some integral number $t$ of sub-intervals $1/g'$ such that $\beta - \alpha > t/g'$ or more precisely

$$\beta - \alpha = (t + \theta)/g'$$

where $0 < \theta < 2$. Since $N(B_1, g)$ denotes the number of $\hat{r}/m = \{Zg^i/m\}$ contained in the interval $[B_1/g', (B_1 + 1)/g']$ of width $1/g'$ for some subset of $\hat{w}$ values, and assuming $j$ so that $1/g > \max(\hat{r}/m - \hat{r}/m)$ for adjacent residues and $N(B_1, g) > 0$, we have

$$tN(B_1, g) < N(I) < (t + 2)N(B_1, g)$$

as bounds on the number of $\{Zg^i/m\}$ contained in the arbitrary sub-interval $I = \beta - \alpha$. Now $(j, e)$-normality implies bounds on $N(B_1, g)$ such that

$$1/g - e < N(B_1, g)/\hat{w}(m) < e + 1/g$$

for some given $j$ and $e > 0$. We write (2.3) as

$$tN(B_1, g)/\hat{w}(m) < N(I)/\hat{w}(m) < (t + 2)N(B_1, g)/\hat{w}(m)$$

and combining (2.4), (2.5) and (2.2), we obtain

$$\theta g - 2t < N(I)/\hat{w}(m) - (\beta - \alpha) < (2 - \theta) g + e(t + 2)$$

Since $\theta$ is an absolute constant, and $t$ is fixed for some choice of $[a, b]$ of appropriate width on $[0, 1]$, we assume the $(j, e)$-normality so that there exists an $e_1$ and a $\delta$ for $e$ sufficiently small and $j$ sufficiently large such that

$$D(\hat{w}(m)) = \sup_{0 < \alpha < 1} |N(I)/\hat{w}(m) - (\beta - \alpha)| < e_1$$

for all choices of $\beta - \alpha > t/g' > \delta$ with $t \geq 1$ for some fixed $g$ and $\delta < 1/g$. A suitable choice of $e_1 > 0$ which can be arbitrarily small for a given $t \geq 1$, $0 < \theta < 2$, and $g \geq 2$ for appropriate $j$ and $e$ is

$$e_1 = (t + 2)e + (2 - \theta) g$$

where $-e_1 < -t \theta - \theta g/g'$ and $e_1 > (t + 2)e + (2 - \theta) g'/g'$ yields (2.7) from (2.6). Therefore, with some $t \geq 1$ fixed for a particular choice of $\beta - \alpha$ on $[0, 1]$ not too small, i.e. $\max(\hat{r}/m - \hat{r}/m) < \delta < t/g' < \beta - \alpha < 1/2$, we have satisfied the requirements for a uniform $e$-distribution on $[0, 1]$ by appropriate $(j, e)$-normality. This completes the proof of Theorem 2.
Therefore, if we can establish the uniform $\varepsilon$-distribution of the fractional parts $\{Z^i\rho\}$ for $i = 0, 1, \ldots, \omega(m) - 1$ on $[0, 1]$, Theorem 2 shows that $Z^i | m$ in lowest terms is $(j, \varepsilon)$-normal in the sense of definition (1.1). The $\varepsilon$ for the $(j, \varepsilon)$-normality is the uniform $\varepsilon$-distribution and the range of block sizes $B_j$ (independent of the choice of digits in the block) that will occur in the periodic expansion of $Z^i | m$ to some base $g$ will be those positive integral $j$ values such that $\beta = \omega(1/g') \geq \delta$ or $j = \lfloor \log g/\delta \rfloor$ where $\delta$ such that $0 < \delta < 1/g$ is the lower bound on $\beta = \omega$ for a given $g$ occurring in the definition of the uniform $\varepsilon$-distribution. Based on Theorem 1, we have the following result.

**Theorem 3.** The rational fraction $Z^i | p^j$ in lowest terms for $r \geq 1$ is $(j, \varepsilon)$-normal in the scale $g$ where $g$ is a primitive root mod $p^j$ such that $2 \leq g < p^{j/2}$ for all $j \leq \log g/\varepsilon$ with $\varepsilon = 2\rho(g')$ for $r > 1$, and all $j \leq \log g/\varepsilon$ with $\varepsilon = 1 + (1/g')/(p - 1)$ for $r = 1$.

**Proof.** The proof follows directly from the uniform $\varepsilon$-distribution in Theorem 1 using the comments above Theorem 3. From the argument below (1.8), we can use here $\varepsilon = 1 + (\beta - a)/(p - 1) = 1 + (1/g')/(p - 1)$.

Various consequences of Theorem 3 where $Z^i$ are discussed in ([6], pp. 205-207). Also, in Theorem 3 and 4, we would like to take this opportunity to make a slight correction on the bound for the $f$ values for all primitive roots as stated in ([6], p. 201). For $r > 1$, the upper bound $\log g/\varepsilon$ stated in [6] is adequate for most primitive roots mod $p^j$ such that $2 \leq g < p^j$, i.e., we can say $\log g/\varepsilon = \log g/p$. However, there can be some $g$ with $p$ and $r$ fixed such that $\log g/\varepsilon < \log g/p$. An easily derived criterion for this occurance on the character of $g$ is that $\log g/\varepsilon = \log g/p$ if the fractional parts $\{\log g/p\} - \{\log g/p\} = \log g/p$ and $\log g/\varepsilon = \log g/p$ if $\{\log g/p\} - \{\log g/p\} < \log g/p$.

As an illustration of this point, consider $p = 17$ and $r = 3$ for which the complete set of primitive roots are $3, 5, 6, 7, 10, 11, 12, 14$. One finds that $\log 17/\varepsilon = \log 17/\varepsilon$ for $g = 3, 5, 6, 7, 10, 11, 12, 14$ but $\log 17/\varepsilon$ for $g = 5, 14$. Therefore, $j \leq \log g/\varepsilon$ is satisfactory for all such that $2 \leq g < p^{j/2}$. This conclusion has been entered into Theorems 1 and 3.

3. Residue Progressions, $(j, \varepsilon)$-normality of $Z/m$. In Theorem 4, we prove a fundamental result which states that the complete set of periodic power residues $r_i/m = \{Z^{ik}\rho\}$ for $i = 0, 1, \ldots, \omega(m) - 1$ where $m = 2^m \prod p_i$, $p_i$ are any odd primes, $n_i \geq 1$, and $m \geq 0$ can be partitioned from their irregular or somewhat “random” distribution for consecutive exponents $i$ in $\{Z^{ik}\rho\}$ into sets of residues which are in arithmetic progression. The existence of these “residue progressions” as we shall call them depends on the structure of the odd primes in $m$. The restrictions are related to the powers $n_i$ of the odd primes $p_i$, $s_i$ in $p_i^{n_i}\rho(g^{s_i} - 1)$ which denotes that $p_i^{n_i}\rho(g^{s_i} - 1)$ and $p_i^{n_i+1}\rho(g^{s_i} - 1)$ with $s_i \geq 1$ and $d_i = \omega(p_i)$ as well as the maximum power $s_i$ of the $p_i$-th odd prime contained in any one of the set of least exponents $d_i, d_{i+1}, d_{i+2}, \ldots, d_r$ corresponding to the strictly increasing sequence of odd primes $p_1 < p_2 < \ldots < p_r < \ldots < p_r$, contained in $m$.

The residue progressions are the basis of a summation technique used to prove Theorem 5, i.e., the uniform $\varepsilon$-distribution of the fractional parts $\{Z^{ik}\rho\}$ for $i = 0, 1, \ldots, \omega(m) - 1$. Having established the uniform $\varepsilon$-distribution of the fractional parts $\{Z^{ik}\rho\}$, we obtain, at once, Theorem 6 from Theorem 2. Theorem 6 states that suitable $Z/m$ are $(j, \varepsilon)$-normal.

**Theorem 4.** Let $Z/m = Z/2^n \prod p_i^{n_i}$ where $n > 0$, $r > 1$, each $n_i > 1$ and the $p_i$ are distinct odd primes ($p_1 < p_2 < \ldots < p_r$). Let $d_i = \omega(p_i)$ and suppose that $p_i^{n_i}\rho(g^{s_i} - 1)$, so that $s_i \geq 1$. Let $p_i^{n_i}\rho$ be the largest power of $p_i$, dividing any one of $d_{i+1}, d_{i+2}, \ldots, d_r$. Finally assume $s_i > s_i < s_i$ for at least one $p_i$.

For each $i$, put $t_i = \min(s_i, s_i, s_i)$ and write $D = 2^n \prod p_i^{n_i}$, then the complete set of $\omega(m)$ power residues $R_i = Z^i \rho m \mod m$ can be partitioned into $D$ disjoint arithmetic progressions $P_i$ each containing $\omega(m)\omega(D) = n_i D$ terms, the elements of such progressions $P_i$ being of the form $e_i + KD$ where $Z^{e_i} = r_i \rho m \mod D$, $Z = Z^i \rho m \mod D$ for $e_i = 0, 1, \ldots, \omega(D) - 1$ and $K = 0, 1, \ldots, \omega(m)\omega(D) - 1$.

**Proof.** Consider the complete set of $\omega(m)$ power residues $R_i = Z^i \rho m \mod m$ for $j = 0, 1, \ldots, \omega(m) - 1$. For the composite moduli $m$, we have

\[(3.0) \quad \omega(m) = \langle \omega(2^n), \ldots, \omega(p_i^{n_i}) \rangle = \langle \omega(2^n), \ldots, p_i^{s_i - s_i}d_i \rangle \]

which becomes

\[(3.1) \quad \omega(D) = \langle \omega(2^n), \ldots, p_i^{s_i - s_i}d_i \rangle = \langle \omega(2^n), \ldots, d_i \rangle \]

for $t_i > s_i$ or $t_i < s_i$, resp., since $p_i^{s_i - s_i}d_i$ for some $d_{i+1}, d_{i+2}, \ldots, d_r$ according to the definitions of $s_i$ and $t_i$. Thus the number of residues mod $m$ which lie in these $\omega(D)$ residue classes mod $D$ is

\[(3.2) \quad \omega(D) = \langle \omega(2^n), \ldots, d_i \rangle \]
The proof will be complete if we show that

$$(3.3) \prod_{i=1}^{r} p_i^{e_i-1} \langle \omega(2^{n_i}) \rangle, d_i, \ldots \rangle = \langle \omega(2^{n_i}) \rangle, d_i, \ldots \rangle$$

when $n_i > n_i$ or $n_i < n_i$, respectively. It suffices to consider the powers of $p_i$ that divide each side of (3.3). If $n_i < n_i$, then $e_i = n_i$. Hence $p_i$ appears on each side only as $p_i^{n_i}$ dividing some $d_{i+1}, d_{i+2}, \ldots, d_i$. If $e_i < e_i < e_i$, then $p_i^{e_i-n_i} p_i^r$ which divides some $d_{i+1}, d_{i+2}, \ldots, d_i$ where $p_i^{e_i-n_i} = 1$. Hence again $p_i$ divides both sides of (3.3) to the power $p_i^{n_i}$. If $n_i > n_i + e_i$, then $p_i^{e_i-n_i}$ is the power of $p_i$ dividing the left side of (3.3) and on the right side, we have $p_i^{e_i-n_i} = p_i^{e_i-n_i+1} = p_i^{e_i-n_i}$ as required. Therefore, since $n_i = n_i$ when $n_i = 0, 1, 2, \ldots, z_i, z_i+1, \ldots, z_i + e_i$ and $e_i = z_i + e_i$ when $n_i > z_i + e_i$, we obtain, succinctly stated, $e_i = \min(n_i, z_i + e_i)$.

From (3.3), (3.2), and (3.0), it follows that we have the positive integer

$$\omega(m)/\omega(D) = m/D = \prod_{i=1}^{r} p_i^{n_i-e_i}.$$ 

Since the number of residues in a given progression $p_i = \omega(m)/\omega(D)$ where we have a total of $\omega(D)$ residue progressions, we see that if $n_i < z_i + e_i$ for all $i$ in $m$, then $e_i = \min(n_i, z_i + e_i) = n_i$ and $\omega(m)/\omega(D) = 1$, i.e. we have no residue progressions in this case. However, if at least one $p_i$ has a power $n_i$ such that $n_i > z_i + e_i$, then residue progressions will exist since the number of terms in each progression $\omega(m)/\omega(D) > 1$.

Therefore, the structure of the residue progressions $p_i$ corresponding to a complete set of residues mod $m$ is as follows. Given any $Z/m = Z/2^{n_i} \prod_{i=1}^{r} p_i^{n_i}$ such that $n_i > z_i + e_i$ for at least one $p_i$, then the sequence of power residues $R_i = r_i + KD$ for some fixed $r_i$ with $D = 2^{n_i} \prod_{i=1}^{r} p_i^{n_i}$

and $K = 0, 1, \ldots, \omega(m)/\omega(D)-1$ are such that $Z_0 \equiv R_i \equiv (r_i + KD) \mod m = Z_0 \equiv r_i \mod D$ where $s = j \mod \omega(D)$, $Z_0 \equiv Z \mod D$ and $s = 0, 1, \ldots, \omega(D)-1$, i.e. every $R_i$ for $j = 0, 1, \ldots, \omega(m)/\omega(D)-1$ is contained in one and only one of the disjoint residue classes (residue progressions $P_i$) corresponding to some $Z_i = 0, 1, \ldots, \omega(m)/\omega(D)-1$ whose initial term is some $r_i$.

The proof of Theorem 4 is now complete. (See residue progression example at end of paper.)

Let us keep in mind that the exponents of the odd primes $p_i$ in (3.4) are such that $n_i = t_i = 0$ for those $p_i$ such that $n_i < z_i + e_i$, and $n_i - t_i = n_i - (z_i + e_i)$ for those $p_i$ such that $n_i > z_i + e_i$ which is usually the case.

But more important for some results in a sequel to this paper, is that the $n_i$ increase in a given $Z/m$ for a fixed set of odd primes $p_i$ in $m$, i.e. in $Z_i,m_i, Z_i/m_i^2, \ldots$ the powers of the given set of primes in $m$ increase so that the number of residue progressions $\omega(D)$ remains the same ($D$ does not change under these conditions), but the number of residues $\omega(m)/\omega(D) = \prod_{i=1}^{r} p_i^{n_i-e_i}$ in each $P_i$ may increase indefinitely.

On the maximum power $s_i$ of the $p_i$th odd prime contained in the least exponents $d_{i+1}, d_{i+2}, \ldots, d_i$ of the strictly increasing sequence of odd primes greater than $p_i$, consider some $p_i > p_i$ in the sequence. Clearly, the least exponent $d_i < p_i-1$ and assuming that $d_i = a_i p_i^r$ we have $p_i > 1 + a_i p_i^r$ as a crude estimate on $p_i$ whose least exponent contains $p_i$. For example, if at most $d_i = 2p_i$, then $p_i > 2p_i-1$. Data from a table of least exponents to the base 10, illustrates this estimate. If $p_i = 11$, then $p_i = 23 = 11 + 12 where \omega_{11} = 2(11); if p_i = 17, then p_i = 103 > 2(17) + 1\ where \omega_{17} = 2(17); and for p_i = 57, p_i = 149 > 2(57) + 1\ with \omega_{57} = 2(57). In each case, we have given the prime $p_i$ greater than $p_i$ (and closest to $p_i$ in the strictly increasing sequence of primes) that contains $p_i$ in the exponent $d_i$ to which 10 belongs mod $p_i$. Perhaps, a more precise estimate could be found on the prime $p_i > p_i$ that contains $p_i$ in its least exponent.

In order to study the occurrence of the $(j, e)$-normal property in the rational fractions, we find it convenient to separate the class of all rational fractions into classes for which residue progressions may or may not exist. In the definitions below, we assume $Z/m < 1$ is in lowest terms and the previously stated definitions of $d_i, s_i, \omega(m), \omega(D)$.

**Definition.** Type A. A rational fraction $Z/m = Z/2^{n_i} \prod_{i=1}^{r} p_i^{n_i}$ is of Type A if $n_i > z_i + e_i$ for at least one odd prime.

**Type B.** A rational fraction $Z/m = Z/2^{n_i} \prod_{i=1}^{r} p_i^{n_i}$ is of Type B if $n_i < z_i + e_i$ for all odd primes $p_i$.

**Type C.** A rational fraction $Z/m = Z/2^{n_i} \prod_{i=1}^{r} p_i^{n_i}$ is of Type C when $a_i > 1$.

For Type A, we establish in Theorem 6 of this paper the $(j, e)$-normality using Theorem 2, i.e. we prove in Theorem 5 the uniform $s$-distribution of the fractional parts $[Z_0/m]$ on $(0, 1)$ for Type A.

In the case of Type B for which residue progressions do not exist, numerical studies show that the $(j, e)$-normal phenomenon may or may not exist. The simplest case of this in the Type B fractions is $Z/p$ in Theorem 3 for $r = 1$. Since $g$ is a primitive root mod $p^2$, we have for Theorem 3, $g^{(p-1)(p+1)-1} \equiv 0 \mod p^2$ and $g = p-1, \ x = 1, \ n = 0 \Rightarrow Z/p$ is of Type B, i.e. $n = x + s$ since $s = 1 + 0$. Hence, there are no residue progressions since $t = \min(1, 1+0) = 1 \Rightarrow D = p^2$ so

$$\omega(m)/\omega(D) = m/D = p/p = 1 = \omega(p)/\omega(D) = (p-1)/(p-1) = 1,$$

yet we have shown in [6] that $Z/p$ is $(j, e)$-normal when represented in a primitive root base ([6], Cor. 1, p. 205).
Of course, the essential issue is the uniform \(\varepsilon\)-distribution of the \(\{Zg^j/m\}\) on \([0, 1]\) and a more complex case for which numerical work shows that we have \((j, \varepsilon)\)-normality is, for example, a study of \(1/30 + 1/97 + 1/100\). Here \(10\) is a primitive root of prime index such that \(10^{97} \cdot 10^{108} = 25056\); and computer data shows that \(\min N(B, 10) = 2469\) and \(\max N(B, 10) = 2576\) (note the agreement with the \((j, \varepsilon)\)-normal phenomenon, i.e. if \(g = 10\), the counts of any of the single digits \(B_j = 0, 1, \ldots, 9\) are approximately \(1/10\) of the total number of digits in one period \(25056\)), where the whole data shows \((j, \varepsilon)\)-normality for \(j \leq \log_2 25056 = 4\). A reasonable conjecture for \(j\) is \(j \leq \log_2 \omega(p_1 p_2 \ldots p_r)\), but to date the resolution of the power residue distribution for such a case applies difficult without residue progressions.

We have some results on the case when \(g\) is not a primitive root, i.e. consider \(Z/p\), where \(p = (g^\alpha - 16)\) with \(\alpha = \log_2 p\) which leads to questions concerning the approximately equal distribution of with power residues (i.e. the uniform \(\varepsilon\)-distribution). For example, when \(n = 2\), we have quadratic residues and therefore, we can make use of the Vinogradov--Polya–Burgess inequalities (2), pp. 182–204). The best estimate to date is due to D. Burgess based on character sums (2), p. 198. We will present studies based on such estimates that leads to \((j, \varepsilon)\)-normal for such a case of Type B, i.e. \(Z/p\) where \(g\) is not a primitive root in some future papers. The \((j, \varepsilon)\)-normality for the type illustrated in the example above which is a case of Type B where \(Z/m = Z/p_{1} p_{2} \ldots p_{r}\) appears quite difficult with our present knowledge of their residue distributions. The computer study indicates the above bound on \(j\) as a reasonable conjecture for \(Z^{59} 97 109\). One thing is clear, the \((j, \varepsilon)\)-normality is related to the deeper questions concerning the associated residue distributions on \([0, 1]\) of \(\{Zg^j/m\}\) for \(j = 0, 1, \ldots, \omega(m) - 1\).

An interesting case for Type B, shows that we may construct rational fractions of Type B which may or may not be \((j, \varepsilon)\)-normal. Let \(g = 10\), and write

\[
Z/m = Z/10^4 + Z/10^4 + \cdots + Z/(10^q - 1)
\]

where \(Z\) can be arbitrarily chosen, i.e. the sequence of \(\lambda\) digits may or may not be \((j, \varepsilon)\)-normal such that \((Z, 10^q - 1)\). Since \(10^q - 1\) has a unique prime factorization \(\prod_{i=1} r_i p_i^{\alpha_i}\) where only those \(p_i\) appear such that \(\omega(p_i^{\alpha_i})\) for \(n_i \geq 1\), it follows that \(Z/(10^q - 1)\) is a rational of Type B, i.e. \(n_i = r_i + s_i\) for every \(p_i\) contained in \(10^q - 1\) since the least exponents are divisors of \(\lambda\). For example,

\[
\omega(10^{12} - 1) = \omega(2^3 \cdot 11^2 \cdot 23 \cdot 4063 \cdot 8779 \cdot 21149 \cdot 513239) = \langle 1, 3, 11, 23, 22, 22, 11, 11 \rangle
\]

and the maximum power of \(11\) contained in each \(d_i\) is 1. In our notation, we have \(d_1 = 3, n_1 = 2, s_1 = 2, s_1 = 0\); \(d_2 = 11, n_2 = 2, s_2 = 1, s_2 = 1\); \(d_3 = 23, n_3 = 1, s_3 = 1, s_3 = 0\); \(d_4 = 4063, n_4 = 1, s_4 = 1, s_4 = 0\); etc. In each case, we have \(n_i = s_i + s_i\). To summarize, we may state that if \(Z/m\) is represented in a scale \(g\) such that \(m = g^j - 1\), then \(Z/m < 1\) in lowest terms is a rational of Type B which may or may not be \((j, \varepsilon)\)-normal depending on how the \(\lambda\) digits are selected.

Finally, for Type C with \(n\) subject to certain restrictions, residue progressions exist and \((j, \varepsilon)\)-normality can be easily demonstrated. Due to the need for brevity here, we defer this result to a later paper.

**Theorem 5.** The rational fraction \(Z/m\) of Type B has a uniform \(\varepsilon\)-distribution of fractional parts \(\{Zg^j/m\}\) for \(i = 0, 1, \ldots, \omega(m) - 1\) for all bases \(g\) such that \((g, m) = 1\) and \(2 \leq g < 1/\delta\) where \(\delta = \delta = \omega(D)/\omega(m) = D/m = \sqrt{D} + \sqrt{D^2 - 1} = \sqrt{m} + \sqrt{m - 1} + 1\) and \(k = \min(n_i, s_i + s_i)\).

**Proof.** As in the previous theorem, consider the power residues \(r_i/m = \{Zg^j/m\}\) for \(i = 0, 1, \ldots, \omega(m) - 1\) given by \(Zg^j/r_i \mod m\) and determine for suitable choices of \(\alpha\) and \(\beta\) where \(0 < \alpha < \beta \leq 1\), the number \(N(I)\) of residues \(r_i/m\) contained in \(a < r_i/m < b\), i.e.

\[
ma < r_i < mb.
\]

Using Theorem 4, if we choose the \(r_i\) as least positive residues \(\mod m\), then each residue progression \(P_i\) is a strictly increasing sequence of least positive residues \(\mod m\) in arithmetic progression whose terms differ by \(D\). We may, therefore, determine the number of residues \(r_i\) contained in \(a < r_i/m < b\), i.e.

\[
ma < r_i < mb.
\]

Using Theorem 4, if we choose the \(r_i\) as least positive residues \(\mod D\), then each residue progression \(P_i\) is a strictly increasing sequence of least positive residues \(\mod m\) in arithmetic progression whose terms differ by \(D\). We may, therefore, determine the number of residues \(r_i\) contained in \(a < r_i/m < b\), i.e.

\[
ma < r_i < mb.
\]

Within a given residue progression \(P_i\), the \(N\)th residue is given by \(r_i + (N - 1)D\) where \(N = 1, 2, \ldots, \omega(m)/\omega(D) = m/D\) and therefore, if we replace \(r_i\) by \(r_i + (N - 1)D\) and sum over each residue progression \(P_i\), we obtain the total number of \(r_i\) contained in \(a < r_i/m < b\). We find

\[
N(I) = \sum_{i=0}^{\omega(m)/\omega(D)} \left[ \left( \frac{m}{D} \left( D + (D - r_i) \right) \right) - \left( \frac{ma \cdot D + (D - r_i)}{D} \right) \right]
\]

where \(\sum\) designates that we replace the \(r_i\) by the sequence of \(\omega(D)\) residues \(r_i\) that satisfy \(Zg^j = r_i \mod D\) where \(Z = Z' \mod D\) and then sum. We may write (3.7) as

\[
N(I) = \sum_{i=0}^{\omega(m)/\omega(D)} \left( \frac{ma \cdot (D + \theta_i) - \theta_i \cdot \theta_i}{D} \right)
\]

where \(0 < \theta_i < \theta_i < 1\) are the corresponding fractional parts assuming that \(\alpha\) and \(\beta\) take on continuous real values on \([0, 1]\) such that 0 < a
< \beta \leq 1. Using (3.4) which states that \( m/D = \omega(m)/\omega(D) \) for rationals of Type A, we have from (3.8)
\[
(3.9) \quad N(I) = \omega(D)\omega(m)(\beta - \alpha)/\omega(D) + \beta - \delta
\]
where we have summed over the \( \omega(D) \) residues \( r_\alpha \). Considering the ranges of the \( \delta_i \) and rearranging, we obtain
\[
(3.10) \quad |N(I)/\omega(m) - (\beta - \alpha)| < \omega(D)/\omega(m) = \varepsilon
\]
where we have the equivalent forms for the bound \( \varepsilon \) given by
\[
(3.11) \quad \varepsilon = \omega(D)/\omega(m) = D/m = 1/\prod_{i=1}^{r} p_i^{n_i - t_i}.
\]
In order to ensure that \( N(I) \) has a non-zero value, let us note (3.3) and require that \( \beta - \alpha \) be such that \( m(\beta - \alpha)/D > 1 \). Therefore, for a convenient description of the uniform \( \varepsilon \)-distribution of the fractional parts \( \{\alpha g/k\} \) for rationals of Type A, we shall require that the least \( \beta - \alpha \) on \([0, 1]\) be such that
\[
(3.12) \quad \beta - \alpha > D/m
\]
or \( \delta = \varepsilon = D/m = 1/\prod_{i=1}^{r} p_i^{n_i - t_i} \) as stated in the theorem. The restriction in (3.12) may not be the most stringent to keep \( N(I) > 0 \) for all choices of \( [\alpha, \beta] \) taken anywhere in \([0, 1]\) but the condition does ensure that each residue progression \( P_i \) will contribute some \( r_i/m \) in every sub-interval \( [\alpha, \beta] \) \([0, 1]\) and consequently, in the count \( N(I) \) of the number of points \( r_i/m \) contained in \([\alpha, \beta] \).

The condition (3.12) may also be argued independent of (3.8). Consider the complete sets of reduced residues \( r_i \mod D \) which initiate the residue progressions \( P_i \) which will contain all \( r_i \mod m \) ordered as arithmetic progressions as described in Theorem 1. Since the \( r_i = r_i + KD \) for \( K = 0, 1, \ldots, \omega(m)/\omega(D) - 1 \), the maximum differences between residues \( r_i \) will be the maximum difference between the residues \( r_i \). An upper bound on the maximum difference of residues \( r_i \) is \( D \), therefore, if we require that \( \beta - \alpha > D/m \), the bound \( D/m \) will exceed any maximum difference of residues \( r_i/m \) on \([0, 1]\). This implies that any interval \( \beta - \alpha > D/m \) will surely contain residue points \( r_i/m \) taken anywhere in \([0, 1]\). Also note that we have shown that \( \max(r_i/m - r_j/m) \) which is the maximum distance between adjacent residues out of the complete set is such that \( \max(r_i/m - r_j/m) < \delta = D/m < \frac{1}{2} \) since \( D/m = 1/\prod_{i=1}^{r} p_i^{n_i - t_i} \) for Type A is such that \( r_i > s_i + s_i \) for at least one odd prime, thus surely \( \delta = D/m < \frac{1}{2} \). We have satisfied the requirements for a uniform \( \varepsilon \)-distribution, hence Theorem 5 is complete. The \( (j, \varepsilon) \)-normality for rational fractions of Type A now follows at once using Theorems 2 and 3.

**Theorem 6.** A rational fraction \( z/m < 1 \) in lowest terms of Type A is \((j, \varepsilon)\)-normal to all bases \( g \) such that
\[
(g, m) = 1 \quad \text{and} \quad 2 \leq g < m \quad \text{if} \quad \int D/m = \omega(D)/\omega(m) = 1/\prod_{i=1}^{r} p_i^{n_i - t_i},
\]
where \( t_i \) is \( \min(n_i, z_i + s_i) \), and \( \delta = 2^n \prod_{i=1}^{r} p_i^{n_i} \).

**Proof.** From Theorem 5, \( \delta = D/m \), and we have according to Theorem 2, \((j, \varepsilon)\)-normality for \( j \geq 1 \) such that \( \beta - \alpha = 1/g^r > \delta = D/m \). Also \( \beta = \varepsilon \) which then implies the various equivalent forms
\[
\int \leq \log g + \log \varepsilon \quad \text{where} \quad e = D/m = \omega(D)/\omega(m) = 1/\prod_{i=1}^{r} p_i^{n_i - t_i},
\]
where \( t_i \) is \( \min(n_i, z_i + s_i) \), and \( \delta = 2^n \prod_{i=1}^{r} p_i^{n_i} \).

In order to have \((j, \varepsilon)\)-normality for at least \( j = 1 \), we restrict the bases to those \( g \) such that \((g, m) = 1 \) and \( 2 \leq g < m \). The proof of Theorem 6 is now complete.

One interesting consequence of Theorem 6 is that we may show that the \((j, \varepsilon)\)-normality in \( Z[p^r] \), where \( p \) is an odd prime still exists for \( r \) sufficiently large even though the base \( g \) has no longer a primitive root in contrast to the results in [6].

We have from Theorem 6 for \( m = p^r \).

**Theorem 7.** The rational fraction \( Z[p^r] < 1 \) in lowest terms for \( r > z \geq 1 \) where \( p^r [[p^{r-1} - 1] \) and \( d = \ord_0 g \) is \((j, \varepsilon)\)-normal to all \( j \) such that \( j \leq \log_0 p^{r-1} \) and \( \varepsilon = 1/p^{r-1} \) when represented in bases \( g \) such that \( (g, p) = 1 \) and \( 2 \leq g < p^{r-1} \).

**Proof.** Since \( \omega(p^r) = dp^{r-1} \) for \( r > z \geq 1 \) ([6], p. 52, Theorems 4–6) where \( p^r [[p^{r-1} - 1] \) with \( d = \ord_0 g \) and \( D = p^r \) for \( t = \min(r, z) \) if \( r > z \) (clearly, \( s = 0 \)), we have, using Theorem 6, \( \varepsilon > D/m = 1/p^{r-1} = \omega(D)/\omega(m) = d/p^{r-1} = 1/p^{r-1} \) for \( j \leq [\log_0 p^{r-1}] \). Q.E.D.

Furthermore, consider a crucial observation in relation to uniform distributions on \([0, 1]\) and the behavior of the uniform \( \varepsilon \)-distributions as defined for the fractional parts \( \{\alpha g/k\} \) on \([0, 1]\) for the rational fractions \( Z/m \) which are \((j, \varepsilon)\)-normal. For example, consider the results in Theorem 7 for \( Z[p^r]/[p^r] \) with \( i = 0, 1, \ldots, \omega(p^r) - 1 \) where \( \omega(p^r) = dp^{r-1} \), \( r > z \geq 1 \) for some fixed odd prime \( p \) and base \( g \).

Clearly from the uniform \( \varepsilon \)-distribution of parts for this case, the discrepancy is such that \( D_n < \varepsilon = 1/p^{r-1} \) where \( n = \omega(p^r) = dp^{r-1} \) with \( d \) and \( \varepsilon \) fixed for any suitable \( p \) and \( g \). Therefore, for \( r \) sufficiently large,
we can have \( \lim_{n \to \infty} D_n = \lim_{n \to \infty} D_n = 0 \), i.e. the discrepancy is zero which is a requirement for a uniform distribution ([3], p. 23). However, here we do not have a uniform distribution in a strict sense, even though, \( \lim_{n \to \infty} D_n = 0 \). The reason is that for the uniform distribution, we require that \( \lim \mathcal{N}(I)/n = I \) uniformly in \( n \) for any choice of \( n \), no matter how large. In the \((j, \varepsilon)\)-normal case for the rational fractions, we have the \( n \) increasing; we might say, in discontinuous “jumps” for increasing \( \varepsilon \) since \( n = \alpha(p^n) \) for some consecutive increasing sequence of positive integers \( r \). Therefore, the fractional parts \( \{\log p^n\} \) for \( i = 0, 1, \ldots, \alpha(p^n) - 1 \) distribute themselves on \([0, 1)\) for increasing \( r \), in an increasingly uniform way, i.e.,

\[
D_n = \sup_{\alpha < \beta < \varepsilon} |\mathcal{N}(\alpha, \beta)/\alpha(p^n) - I| < \varepsilon = 1/p^{r-1} \quad \text{where} \quad n = \alpha(p^n),
\]

but do not satisfy all the requirements for a uniform distribution.

Also, consider another aspect of the \((j, \varepsilon)\)-normal property in the rational fractions with reference to block sizes \( j \) for some bounded consecutive sequence of \( j \) values.

For example, the condition \( j \leq [\log_2 1/s] = [\log_2 \sum_{n=1}^{r} n^{2^{n-1}}] \) implies that in the period \( E \) of \( Z/m \) and at most \( j - 1 \) digits into the next period that all blocks \( B_{j} \) whose lengths are restricted to the bounded set of \( j \) values will certainly appear with frequency ratios that satisfy \((3.10)\) independent of the choice of digits they contain. However, blocks \( B_{j} \) may not appear in \( E \) terminating in at most \( j - 1 \) places in the next repetition of \( E \), if \( j > \left[\log_2 1/s\right] \). Therefore, it is convenient for the \((j, \varepsilon)\)-normal characterization to use the notion of “independent” blocks \( B_{j} \) as those whose lengths satisfy \( j \leq \left[\log_2 1/s\right] \), i.e., blocks consisting of any combination of \( j \) digits that will appear with certainty somewhere in \( B_{j} \) and “dependent” blocks as those blocks whose length exceed \( \left[\log_2 1/s\right] \) (or some other prescribed bound) and may or may not depending on the choice of the \( j \) digits in \( B_{j} \) appear in \( E \).

Finally, by means of the results here on the \((j, \varepsilon)\)-normality of Type A, we can give an answer to what we might call a “Brouwer” type question. In 1925 and in his later lectures, L.E.J. Brouwer ([1], p. 3) and others in the intuitionistic school of mathematical logic often stated the following as a possible “undecidable” proposition. Can we prove that the prescribed block \( 0123456789 \) appears in, say, the infinite sequence of digits of \( \pi/4 \) when represented in the base \( 10 \)?

Let us paraphrase and ask for a proof of the question for a given rational fraction: Does the block \( 0123456789 \) appear, for example, somewhere in the decimal expansion of the rational fraction \( 1/17^{1000} \) expanded in the base \( 10 \)? Since \( 1/17^{1000} \) is \((j, \varepsilon)\)-normal by Theorem 3 with a period length of \( \pi(17^{1000}) = 16.17^{999} \) (10 is a primitive root of 17) where \( j \approx [\log_2 17^{1000}/2] = [1000\log_2 17 - \log_2 10] \approx 1236 \), we can say that somewhere in the approximately \( 2.653 \times 10^{1236} \) digits of the period of \( 1/17^{1000} \) that the independent block \( 0123456789 \) will appear (with certainty!) with relative frequency of about \( 1/10^{9} \), i.e.

\[
|\mathcal{N}(I)/16.17^{999} - 1/10^{9}| < 2/16.17^{999}.
\]

An important point here is that we cannot say exactly where the block \( 0123456789 \) will first make its appearance but we do know that it will with the above frequency. This says that now such questions are decidable for given particular rational fractions in the real numbers when their \((j, \varepsilon)\)-normality has been demonstrated.

Of course, the Brouwer question is answered in the affirmative if we could prove that \( \pi/4 \) is a normal number. But, this unresolved question appears quite resistant to solution by our contemporary mathematicians.

We can, however, make a slight advance on the Brouwer question itself with the \((j, \varepsilon)\)-normal properties of the rational fractions of Type A. Consider the \( n \)-th partial product

\[
P_n(\pi/4) = \prod_{i=1}^{n} (1-1/(24i+1)^{1/4}) = p_n/q_n
\]

based on the Wallis infinite product for \( \pi/4 \). In a future paper, we will show that \( p_n/q_n \) is a rational of Type A for \( n \) sufficiently large. In which case, the \((j, \varepsilon)\)-normality shows that the block \( 0123456789 \) will appear somewhere in the arbitrarily long set of digits of one period of \( p_n/q_n \) and, in which, only a comparatively small portion of the long period represents \( \pi/4 \) exactly. We would have an answer to the Brouwer question from another point of view, if we could show in which portion the block \( 0123456789 \) occurs; the exact portion or the set of digits which will change as we consider larger values of \( n \) in

\[
\lim_{n \to \infty} \prod_{i=1}^{n} (1-1/(24i+1)^{1/4}) = \pi/4.
\]

However, we cannot answer this question yet for \((j, \varepsilon)\)-normal rational fractions, i.e. we cannot predict the location of a prescribed block or where this block will make its first appearance within the period.

On the other hand, we can now study the \((j, \varepsilon)\)-normal properties of particular sequences of rational fractions which approximate a given irrational like \( \pi, \varepsilon, \sqrt{2} \), etc.

The author would like to express his gratitude for a valuable correspondence with D. A. Burgess.
An example of residue progressions

Let \( n = 3^4 \cdot 11^2, \ Z = 2; \ p_1 = 5, \ z_1 = 1, \ e_1 = 1, \ d_1 = 4, \ \eta_1 = 3; \ p_2 = 11, \ z_2 = 2, \ e_2 = 0, \ d_2 = 5, \ \eta_2 = 2; \) since \( 11^3 \mid (3^4 - 1) \) and \( 5 \mid (3^4 - 1), \) we have for \( n = 0, \ D = 2^p \prod p_i^n \) where \( t_i = \min(n_0, z_i + e_i) \) implies \( t_1 = \min(3, 2) = 2, \ t_2 = \min(2, 2) = 2, \) hence \( D = 2 \cdot 3^2 \cdot 11^2 = 3205. \) Thus for the \( R_2, \) we have \( 3^2 \equiv R_2 \mod 5^2 \cdot 11^2 \) and the \( r_2 \) in \( R_2 = r_2 \mod 3 \cdot 5 \cdot 11 \) are given by \( 3^2 \equiv r_2 \mod 5^2 \cdot 11^2 \) where \( \omega(m) = \omega(3^2, 11^2) = (4 \cdot 5^1 - 1, \ 5 \cdot 11^2 - 1) = 100 \) with \( K = 0, 1, \ldots, \ \omega(m) / \omega(D) = 100 / 100 = 1 \) given that \( \omega(D) = (4 \cdot 5, 11^2 - 1) = 20. \)

The residues in sequence

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\( D = 3205 \) Residues arranged in their residue progressions \( P_0 \)

\[
\begin{align*}
D = 3205 & \quad \text{Residues arranged in their residue progressions } P_0 \\
\begin{array}{cccccccc}
\eta_0 & 3 & 27 & 81 & 243 & 729 & 2187 & 511 & 1333 \\
3028 & 3028 & 3028 & 3028 & 3028 & 3028 & 3028 & 3028 & 3028 \\
6051 & 6051 & 6051 & 6051 & 6051 & 6051 & 6051 & 6051 & 6051 \\
9075 & 9075 & 9075 & 9075 & 9075 & 9075 & 9075 & 9075 & 9075 \\
12108 & 12108 & 12108 & 12108 & 12108 & 12108 & 12108 & 12108 & 12108 \\
1874 & 1874 & 1874 & 1874 & 1874 & 1874 & 1874 & 1874 & 1874 \\
4599 & 4599 & 4599 & 4599 & 4599 & 4599 & 4599 & 4599 & 4599 \\
7024 & 7024 & 7024 & 7024 & 7024 & 7024 & 7024 & 7024 & 7024 \\
10549 & 10549 & 10549 & 10549 & 10549 & 10549 & 10549 & 10549 & 10549 \\
13674 & 13674 & 13674 & 13674 & 13674 & 13674 & 13674 & 13674 & 13674 \\
\end{array}
\end{align*}
\]

References


THE CITY COLLEGE OF THE CITY UNIVERSITY OF NEW YORK

Reçu par la Réduction le 10.10.1928
A general arithmetic construction of transcendental non-Liouville normal numbers from rational fractions

by

R. G. STONEHAM (New York, N. Y.)

1. Introduction. In this paper, we derive a general arithmetic construction of an extensive class of transcendental non-Liouville normal numbers based on any given rational fraction \( \frac{Z}{m} < 1 \) in lowest terms. The construction is founded on the results in [14] wherein we proved that certain broad classes of rational fractions are \((j, \varepsilon)\)-normal.

In [14], (1.1), we extended the original definition of \((j, \varepsilon)\)-normality due to Besicovitch ([15], p. 261) so as to apply to appropriate rational fractions \( \frac{Z}{m} < 1 \) in lowest terms. Essentially, we showed that the definition of \((j, \varepsilon)\)-normality which Besicovitch defined for finite sets of digits could be applied to the infinite periodic sequences which represent certain broad classes of rational fractions. Therefore, we can consider whether some given rational fraction \( \frac{Z}{m} \) when represented in appropriate bases \( g \) is \((j, \varepsilon)\)-normal or not in this sense.

Consider the real number \( x = .x_1 x_2 \ldots \) represented in the scale \( g \) and let \( N(B_j, X_k) \) denote the number of occurrences of the block \( B_j \) consisting of any combination of \( j \) digits chosen from \( 0, 1, \ldots, g-1 \) in the first \( k \) digits \( x_1 x_2 \ldots x_k \) of \( x \). We have the following definition ([17], p. 95, 104) equivalent to that given by Borel in 1908. Unless otherwise indicated, lower case letters will represent positive integers.

**Definition. Normal number.** The number \( x \) is normal in the scale \( g \) if

\[
\lim_{k \to \infty} \frac{N(B_j, X_k)}{k} = \frac{1}{g^j}
\]

(1.0)

for all \( j = 1, 2, 3, \ldots \).

If \( x \) is any real number, \( x \) is said to be normal to the base \( g \) if \( \{x\} \) \( \{x\} = x - [x] \) is normal to the base \( g \) where \( \{x\} \) is the fractional part of \( x \) and \( [x] \) is the greatest integer not exceeding \( x \). Furthermore, if some \( x \) is to satisfy (1.0), i.e. be a normal number, then it is, necessarily, an irrational.