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On (j, ε)-normality in the rational fractions

by

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1. Introduction. In 1964, we proved [6] that the distribution of the digits in the recurring period of the reciprocal of an integral power of an odd prime when represented in a scale g which is a primitive root $\mathrm{mod}\, p^2$ is (j, ε)-normal in the sense of Besicovitch ([6], p. 201). The computer studies we have carried out show that the (j, ε)-normal phenomenon is quite extensive in the rational fractions.

In this paper, we generalize the results in [6] in several directions and show that broad classes of rational fractions $Z/m < 1$ in lowest terms, when represented in a base g such that $2 \leqslant g < C(m)$ and $(g, m) = 1$ where $C(m)$ is a constant that depends on m, are (j, ε)-normal.

In a sequel to the present paper, we will give a fairly elementary arithmetic construction of normal numbers which can be written in closed arithmetic forms based on any given rational fraction $Z/m < 1$. They are of such generality that we have been able to prove that they are transcendentals of the non-Liouville type. These are, apparently, the first known general class of normal numbers whose irrational character has been demonstrated.

We have found it convenient to extend the Besicovitch definition ([6], p. 201) so as to apply to an infinite periodic representation of Z/m which may or may not have a non-periodic part. Let $N(B_j, g)$ denote the number of occurrences of the block of j digits B_j chosen from $0, 1, \ldots, g-1$ commencing in any period of the representation of Z/m in the scale g and terminating in at most $j-1$ digits of the next period. Let $x = .x_1x_2\ldots$ be the representation of x in the scale g and let X_λ denote the block of the first λ digits in x where $N(B_j, X_\lambda)$ denotes the number of occurrences of the block B_j in X_λ.

For convenience in notation, let us define the base dependent number-theoretic function $\omega(m) = \mathrm{ord}_m g$ which will denote the number of digits in one period of Z/m when represented in the scale g. Without much difficulty, one can prove the following result. Unless otherwise indicated, lower case letters are positive integers.

LEMMA. *Let $Z/m < 1$ in lowest terms be represented in some scale g such that $2 \leqslant g < m$, then if the representation is periodic, we have*

$$\lim_{\lambda\to\infty} N(B_j, X_\lambda)/\lambda = N(B_j, g)/\omega(m). \tag{1.0}$$

Essentially, this lemma states that the limiting relative frequency over the periodic infinite set is the same as the relative frequency over one period.

If g contains all the prime factors of m, then Z/m is terminating, in which case, we can define (j, ε)-normality for the finite set of digits by ([6], p. 201). However, if g contains some but not all prime factors of m, then the expansion of Z/m in the scale g is periodic and may or may not have a periodic part. If this is so, we will use the following definition of (j, ε)-normality.

DEFINITION. *(j, ε)-normal rational fractions.* Let $Z/m < 1$ in lowest terms have a periodic representation that may or may not have a non-periodic part in a scale g such that $2 \leqslant g < m$. If for a given j and $\varepsilon > 0$, every j digit sequence B_j which occurs in the expansion is such that

$$\lim_{\lambda\to\infty} |N(B_j, X_\lambda)/\lambda - 1/g^j| = |N(B_j, g)/\omega(m) - 1/g^j| < \varepsilon \tag{1.1}$$

then Z/m is (j, ε)-*normal in the scale* g.

Let $[x]$ denote the greatest integer not exceeding x and $\{x\}$, the fractional part. If $(g, m) = 1$, then Z/m has no non-periodic part, in which case, we have the periodic set of digits E given by

$$Z/m = .\dot{b}_1 b_2 \ldots b_i b_{i+1} \ldots b_{i+j-1} \ldots \dot{b}_{\omega(m)} = E \tag{1.2}$$

where $B_j = b_i b_{i+1} \ldots b_{i+j-1}$ is any block of j digits chosen from $0, 1, \ldots \ldots, g-1$ whose first digit commences anywhere in E and may extend at most $j-1$ digits into the next period. The digits b_i are given by $b_i = [gr_i/m]$ and $B_j = [g^j r_i/m]$ with the initial digit b_i where the power residues are generated by $Zg^i \equiv r_i \bmod m$ for $i = 0, 1, \ldots, \omega(m)-1$. Since there is a 1-1 correspondance for some bounded consecutive sequence of j values between every B_j for a given $j \geqslant 1$ and the power residues $r_i/m = \{Zg^i/m\}$ for $i = 0, 1, 2, \ldots, \omega(m)-1$, the fundamental issue in order to prove the (j, ε)-normality of Z/m is to show that the fractional parts $\{Zg^i/m\}$ are approximately "uniformly" distributed in some sense on $[0, 1]$.

We find most appropriate for the description of this finite discrete approximately uniform distribution the notion of what we shall call a "uniform ε-distribution" which can be defined in terms of the discrepancy of the finite set $\{Zg^i/m\}$ on $[0, 1]$ as utilized by Weyl [7] and others.

In 1965, LeVeque ([3], p. 23) obtained a precise upper bound on the discrepancy D_n ([3], p. 22) for finite sets in terms of the associated exponential sums by means of the characteristic function technique of probability theory

DEFINITION. *Uniform ε-distribution.* A sequence of real numbers $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1$ has a *uniform ε-distribution* on $[0, 1]$ if for a given n sufficiently large there is an $\varepsilon > 0$ depending on n and a δ such that $\max(x_{i+1} - x_i) \leqslant \delta < \frac{1}{2}$ for $i = 0, 1, \ldots, n$ such that

$$D_n = \sup_{0\leqslant\alpha<\beta\leqslant 1} |N(I)/n - (\beta-\alpha)| < \varepsilon \tag{1.3}$$

for all choices of $\beta - \alpha > \delta$ on $[0, 1]$ where $N(I)$ denotes the number of x_k for $k \leqslant n$ contained in $[\alpha, \beta]$.

For the bound on δ, we have made the minimal requirement on the distribution such that $\max(x_{i+1} - x_i) < \frac{1}{2}$ for $i = 0, 1, \ldots, n$ on $[0, 1]$. The need for this condition will appear near the end of Theorem 2.

As an example to illustrate a uniform ε-distribution, we may prove the following theorem based on the results in ([6], p. 201).

THEOREM 1. *The rational fraction $Z/p^r < 1$ in lowest terms for $r \geqslant 1$ has a uniform ε-distribution of fractional parts $\{Zg^i/p^r\}$ for $i = 0, 1, \ldots \ldots, \varphi(p^r)-1$ where p is an odd prime, g is a primitive root* mod p^2 *such that $2 \leqslant g < p^r/2$, $\varphi(x)$ is the Euler φ-function, $\varepsilon = 2/\varphi(p^r)$, $\delta = 2/p^r$ for $r > 1$; and $\varepsilon = 2/(p-1)$, $\delta = 1/p$ for $r = 1$.*

Proof. Following the procedure in ([6], p. 203), we determine number $N(I)$ of residues r_i/p^r contained in an arbitrary interval $[\alpha, \beta] \in [0, 1]$, i.e. we have $\{Zg^i/p^r\} = r_i/p^r$ with

$$\alpha < r_i/p^r \leqslant \beta \tag{1.4}$$

where $0 \leqslant \alpha < \beta \leqslant 1$. Removing the number of residues in (1.4) not relatively prime to p, we have the number $N(I)$ of $\{Zg^i/p^r\} \in [\alpha, \beta]$

$$N(I) = [p^r\beta] - [p^r\alpha] - ([p^{r-1}\beta] - [p^{r-1}\alpha]) \tag{1.5}$$

similar to ([6], (3.12)). It is clear that (1.5) implies the following for $n = \varphi(p^r)$

$$|N(I)/\varphi(p^r) - (\beta-\alpha)| < 2/\varphi(p^r) \tag{1.6}$$

where $\varphi(p^r) = (p-1)p^{r-1}$. Since we delete the residues not relatively prime to p, then at most 2 adjacent residues in the complete set differ by $2/p^r$. Therefore, for any choice of $[\alpha, \beta] \in [0, 1]$, $\beta - \alpha > \delta = 2/p^r$ will insure that $[\alpha, \beta]$ contains at least one r_i/p^r. For $r = 1$, we find that $\delta = 1/p$ and since

$$N(I) = [p\beta] - [p\alpha] \Rightarrow |N(I)/(p-1) - (\beta-\alpha)| < (1+\beta-\alpha)/(p-1),$$

we have

$$\sup_{0\leqslant\alpha<\beta\leqslant 1} (1+\beta-\alpha)/(p-1) = \varepsilon = 2/(p-1).$$

We see that the above ε and δ satisfy the requirements for a uniform ε-distribution. Q.E.D.

In Theorem 1, note that the set of primitive roots is now confined to $2 \leqslant g < p^r/2$ whereas in ([6], p. 201), we have $2 \leqslant g < p^r$. The reason for this change will be discussed after Theorem 3.

In 1949, D. D. Wall ([5], p. 110) proved that a number x is normal to the base g if and only if $\{xg^i\}$ for $i = 0, 1, 2, \ldots$ are uniformly distributed on $[0, 1]$. There is a considerable literature to date which studies the relations between irrationals, normal numbers, Diophantine approximations, uniform distributions, etc. since the definitive paper of H. Weyl [7] in 1916.

The results which we present here show that there exists analogous properties between certain broad classes of rational fractions, (j, ε)-normality, and uniform ε-distributions. The next theorem is analogous to Wall's theorem in that it shows that the uniform ε-distribution is a necessary and sufficient condition for (j, ε)-normality.

2. Uniform ε-distributions and (j, ε)-normality.

THEOREM 2. *The rational fraction $Z/m < 1$ in lowest terms is (j, ε)-normal in a scale g such that $(g, m) = 1$ and $2 < g < 1/\delta$ for some $\varepsilon > 0$ and $j = 1, 2, \ldots, [\log_g 1/\delta]$ where $0 < \delta < 1/g \leqslant 1/2$ if $\{Zg^i/m\}$ for $i = 0, 1, \ldots, \omega(m)-1$ has a uniform ε-distribution, and conversely, if Z/m is (j, ε)-normal, then there exists an ε_1 such that $\{Zg^i/m\}$ has a uniform ε_1-distribution.*

Proof. Consider (1.2) and the associated description below (1.2). If the $\omega(m)$ residues $r_i/m = \{Zg^i/m\}$ for $i = 0, 1, \ldots, \omega(m)-1$ have a uniform ε-distribution then according to the definition (1.3), for each $n = \omega(m)$, there is an ε and $\delta < \frac{1}{2}$ such that

$$D(\omega(m)) = \sup_{0 \leqslant \alpha < \beta \leqslant 1} |N(I)/\omega(m) - (\beta - \alpha)| < \varepsilon \tag{2.0}$$

for all $\beta - \alpha > \delta$. Let us assume equal sub-intervals on $[0, 1]$ of width $1/g^j$ and choose $\beta = (B_j+1)/g^j$, $\alpha = B_j/g^j$ and $\beta - \alpha = 1/g^j$ for some j value. Thus it follows from (2.0) that

$$|N(B_j, g)/\omega(m) - 1/g^j| < \varepsilon \tag{2.1}$$

and all j such that $1/g^j > \delta$ for some j where $N(I) = N(B_j, g)$, the number of $\{Zg^i/m\}$ contained in the sub-interval $[B_j/g^j, (B_j+1)/g^j]$ for some choice of j digits B_j which is, therefore, the count $N(B_j, g)$ of those B_j whose initial digit b_i is some digit in the recurring period. The block B_j may extend into $j-1$ digits of the next period. Furthermore, (2.1) holds for all $j \geqslant 1$ such that $j \leqslant [\log_g 1/\delta]$ consequently, we require that $0 < \delta < 1/g \leqslant \frac{1}{2}$ in the given uniform ε-distribution so that (2.1) holds for at least $j = 1$ for some given $g \geqslant 2$. Therefore, Z/m is (j, ε)-normal according to (1.1).

Conversely, assume that $[0, 1]$ has been divided into g^j equal sub-intervals $[0, 1/g^j], [1/g^j, 2/g^j], \ldots, [1-1/g^j, 1]$ for some choice of $Z/m < 1$ in lowest terms which is (j, ε)-normal for some given j and $\varepsilon > 0$. Let an arbitrary interval $I = \beta - \alpha$ where $0 \leqslant \alpha < \beta \leqslant 1$ contain some integral number t of sub-intervals $1/g^j$ such that $\beta - \alpha > t/g^j$ or more precisely

$$\beta - \alpha = (t + \theta)/g^j \tag{2.2}$$

where $0 \leqslant \theta < 2$. Since $N(B_j, g)$ denotes the number of $r_i/m = \{Zg^i/m\}$ contained in the interval $[B_j/g^j, (B_j+1)/g^j]$ of width $1/g^j$ for some subset of i values, and assuming j so that $1/g^j > \max(r_i/m - r_j/m)$ for adjacent residues and $N(B_j, g) > 0$, we have

$$tN(B_j, g) < N(I) < (t+2)N(B_j, g) \tag{2.3}$$

as bounds on the number of $\{Zg^i/m\}$ contained in the arbitrary sub-interval $I = \beta - \alpha$. Now (j, ε)-normality implies bounds on $N(B_j, g)$ such that

$$1/g^j - \varepsilon < N(B_j, g)/\omega(m) < \varepsilon + 1/g^j \tag{2.4}$$

for some given j and $\varepsilon > 0$. We write (2.3) as

$$tN(B_j, g)/\omega(m) < N(I)/\omega(m) < (t+2)N(B_j, g)/\omega(m) \tag{2.5}$$

and combining (2.4), (2.5) and (2.2), we obtain

$$-\theta/g^j - \varepsilon t < N(I)/\omega(m) - (\beta - \alpha) < (2 - \theta)/g^j + \varepsilon(t+2). \tag{2.6}$$

Since θ is an absolute constant, and t is fixed for some choice of $[\alpha, \beta]$ of appropriate width on $[0, 1]$, we assume the (j, ε)-normality so that there exists an ε_1 and a δ for ε sufficiently small and j sufficiently large such that

$$D(\omega(m)) = \sup_{0 \leqslant \alpha < \beta \leqslant 1} |N(I)/\omega(m) - (\beta - \alpha)| < \varepsilon_1 \tag{2.7}$$

for all choices of $\beta - \alpha > t/g^j > \delta$ with $t \geqslant 1$ for some fixed g and $\delta < 1/g$. A suitable choice of $\varepsilon_1 > 0$ which can be arbitrarily small for a given $t \geqslant 1$, $0 \leqslant \theta < 2$, and $g \geqslant 2$ for appropriate j and ε is

$$\varepsilon_1 = (t+2)\varepsilon + (2 + \theta)/g^j \tag{2.8}$$

where $-\varepsilon_1 < -t\varepsilon - \theta/g^j$ and $\varepsilon_1 > (t+2)\varepsilon + (2 - \theta)/g^j$ yields (2.7) from (2.6). Therefore, with some $t \geqslant 1$ fixed for a particular choice of $\beta - \alpha$ on $[0, 1]$ not too small, i.e. $\max(r_i/m - r_j/m) \leqslant \delta < t/g^j < \beta - \alpha < \frac{1}{2}$, we have satisfied the requirements for a uniform ε-distribution on $[0, 1]$ by appropriate (j, ε)-normality. This completes the proof of Theorem 2.

Therefore, if we can establish the uniform ε-distribution of the fractional parts $\{Zg^i/m\}$ for $i = 0, 1, \ldots, \omega(m)-1$ on $[0, 1]$, Theorem 2 shows that $Z/m < 1$ in lowest terms is (j, ε)-normal in the sense of definition (1.1). The ε for the (j, ε)-normality is the ε of the uniform ε-distribution and the range of block sizes B_j (independent of the choice of digits in the block) that will occur in the periodic expansion of Z/m to some base g will be those positive integral j values such that $\beta-\alpha = 1/g^j > \delta$ or $j \leqslant [\log_g 1/\delta]$ where δ such that $0 < \delta < 1/g$ is the lower bound on $\beta-\alpha$ for a given g occurring in the definition of the uniform ε-distribution. Based on Theorem 1, we have the following result.

THEOREM 3. *The rational fraction $Z/p^r < 1$ in lowest terms for $r \geqslant 1$ is (j, ε)-normal in the scale g where g is a primitive root* $\bmod p^2$ *such that $2 \leqslant g < p^r/2$ for all $j \leqslant [\log_g p^r/2]$ with $\varepsilon = 2/\varphi(p^r)$ for $r > 1$, and all $j \leqslant [\log_g p]$ with $\varepsilon = (1+1/g^j)/(p-1)$ for $r = 1$.*

Proof. The proof follows directly from the uniform ε-distribution in Theorem 1 using the comments above Theorem 3. From the argument below (1.6), we can use here $\varepsilon = (1+\beta-\alpha)/(p-1) = (1+1/g^j)/(p-1)$.

Various consequences of Theorem 3 where $Z = 1$ are discussed in ([6], pp. 205–207). Also in Theorems 1 and 3, we would like to take this opportunity to make a slight correction on the upper bound for the j values for all primitive roots as stated in ([6], p. 201). For $r > 1$, the upper bound $[\log_g p^r]$ stated in [6] is adequate for most primitive roots $\bmod p^2$ such that $2 \leqslant g < p^r$, i.e. we can say $[\log_g p^r/2] = [\log_g p^r]$. However, there can be some g with p and r fixed such that $[\log_g p^r/2] < [\log_g p^r]$. An easily derived criterion for this occurrence on the character of g is that $[\log_g p^r/2] = [\log_g p^r]$ if the fractional parts $\{\log_g p^r\}-\{\log_g p^r/2\} = \log_g 2$ and $[\log_g p^r/2] < [\log_g p^r]$ if $\{\log_g p^r\}-\{\log_g p^r/2\} < \log_g 2$.

As an illustration of this point, consider $p = 17$ and $r = 3$ for which the complete set of primitive roots are $g = 3, 5, 6, 7, 10, 11, 12, 14$. One finds that $[\log_g 17^3/2] = [\log_g 17^3]$ for $g = 3, 6, 7, 10, 11, 12$ but $[\log_g 17^3/2] < [\log_g 17^3]$ for $g = 5, 14$. Therefore, $j \leqslant [\log_g p^r/2]$ is satisfactory for all g such that $2 \leqslant g < p^r/2$. This conclusion has been entered into Theorems 1 and 3.

3. Residue progressions, (j, ε)-normality of Z/m. In Theorem 4, we prove a fundamental result which states that the complete set of periodic power residues $r_i/m = \{Zg^i/m\}$ for $i = 0, 1, \ldots, \omega(m)-1$ where $m = 2^n \prod_{i=1}^{r} p_i^{n_i}$, p_i are any odd primes, $n_i \geqslant 1$, and $n \geqslant 0$ can be partitioned from their irregular or somewhat "random" distribution for consecutive exponents i in $\{Zg^i/m\}$ into sets of residues which are in arithmetic progression. The existence of these "residue progressions" as we shall call them depends on the structure of the odd primes in m. The restrictions are related to the powers n_i of the odd primes p_i, z_i in $p_i^{z_i} \| (g^{d_i}-1)$ which denotes that $p_i^{z_i} | (g^{d_i}-1)$ and $p_i^{z_i+1} \nmid (g^{d_i}-1)$ with $z_i \geqslant 1$ and $d_i = \operatorname{ord}_{p_i} g$, as well as the maximum power s_i of the p_i-th odd prime contained in any one of the set of least exponents $d_{i+1}, d_{i+2}, \ldots, d_r$ corresponding to the strictly increasing sequence of odd primes $p_1 < p_2 < \ldots \ldots < p_i < \ldots < p_r$ contained in m.

The residue progressions are the basis of a summation technique used to prove Theorem 5, i.e. the uniform ε-distribution of the fractional parts $\{Zg^i/m\}$ for $i = 0, 1, \ldots, \omega(m)-1$. Having established the uniform ε-distribution of the fractional parts $\{Zg^i/m\}$, we obtain, at once, Theorem 6 from Theorem 2. Theorem 6 states that suitable Z/m are (j, ε)-normal.

THEOREM 4. *Let $Z/m = Z/2^n \prod_{i=1}^{r} p_i^{n_i}$ where $n \geqslant 0$, $r \geqslant 1$, each $n_i \geqslant 1$ and the p_i are distinct odd primes $(p_1 < p_2 < \ldots < p_r)$. Let $d_i = \operatorname{ord}_{p_i} g = \omega(p_i)$ and suppose that $p_i^{z_i} \| (g^{d_i}-1)$, so that $z_i \geqslant 1$. Let $p_i^{s_i}$ be the largest power of p_i dividing any one of $d_{i+1}, d_{i+2}, \ldots, d_r$. Finally assume $n_i > z_i+s_i$ for at least one p_i.*

For each i, put $t_i = \min(n_i, z_i+s_i)$ and write $D = 2^n \prod_{i=1}^{r} p_i^{t_i}$, then the complete set of $\omega(m)$ power residues $R_j = Zg^j$ mod *m can be partitioned into $\omega(D)$ disjoint arithmetic progressions P_e each containing $\omega(m)/\omega(D) = m/D$ terms, the elements of such progressions P_e being of the form r_e+KD where $Z'g^e \equiv r_e \bmod D$, $Z \equiv Z' \bmod D$ for $e = 0, 1, \ldots, \omega(D)-1$ and $K = 0, 1, \ldots, \omega(m)/\omega(D)-1$.*

Proof. Consider the complete set of $\omega(m)$ power residues $R_j \equiv Zg^j \bmod m$ for $j = 0, 1, \ldots, \omega(m)-1$. For the composite modulus m, we have

$$\omega(m) = \langle \omega(2^n), \ldots, \omega(p_i^{n_i}), \ldots \rangle = \langle \omega(2^n), \ldots, p_i^{n_i-z_i} d_i \text{ or } d_i \rangle \tag{3.0}$$

power residues if $n_i > z_i$ or $n_i \leqslant z_i$, respectively, according to ([4], p. 52, Theorems 4–6). Modulo $D = 2^n \prod_{i=1}^{r} p_i^{t_i}$, we have

$$\omega(D) = \langle \omega(2^n), \ldots, \omega(p_i^{t_i}) \rangle$$

which becomes

$$\omega(D) = \langle \omega(2^n), \ldots, p_i^{t_i-z_i} d_i \text{ or } d_i \rangle = \langle \omega(2^n), \ldots, d_i, \ldots \rangle \tag{3.1}$$

for $t_i > z_i$ or $t_i \leqslant z_i$, resp., since $p_i^{t_i-z_i} | p_i^{s_i}$ for some $d_{i+1}, d_{i+2}, \ldots, d_r$ according to the definitions of s_i and t_i. Thus the number of residues mod m which lie in these $\omega(D)$ residue classes $\bmod D$ is

$$m\omega(D)/D = \prod_{i=1}^{r} p_i^{n_i-t_i} \langle \omega(2^n), \ldots, d_i, \ldots \rangle. \tag{3.2}$$

The proof will be complete if we show that

$$(3.3)\qquad \prod_{i=1}^{r} p_i^{n_i-t_i}\langle\omega(2^n),\ldots,d_i,\ldots\rangle = \langle\omega(2^n),\ldots,p_i^{n_i-z_i}d_i \text{ or } d_i,\ldots\rangle$$

when $n_i > z_i$ or $n_i \leqslant z_i$, respectively. It suffices to consider the powers of p_i that divide each side of (3.3). If $n_i \leqslant z_i$, then $t_i = n_i$. Hence p_i appears on each side only as $p_i^{s_i}$ dividing some $d_{i+1}, d_{i+2}, \ldots, d_r$. If $z_i < n_i \leqslant z_i+s_i$, then $p_i^{n_i-z_i} | p_i^{s_i}$ which divides some $d_{i+1}, d_{i+2}, \ldots, d_r$ where $p_i^{n_i-t_i} = 1$. Hence again p_i divides both sides of (3.3) to the power $p_i^{s_i}$. If $n_i > z_i+s_i$, then $p_i^{n_i-z_i}$ is the power of p_i dividing the left side of (3.3) and on the right side, we have $p_i^{n_i-t_i}p_i^{s_i} = p_i^{n_i-z_i-s_i+s_t} = p_i^{n_i-z_i}$ as required. Therefore, since $t_i = n_i$ when $n_i = 1, 2, \ldots, z_i, z_i+1, \ldots, z_i+s_i$ and $t_i = z_i+s_i$ when $n_i > z_i+s_i$, we obtain, succinctly stated; $t_i = \min(n_i, z_i+s_t)$. From (3.3), (3.2), and (3.0), it follows that we have the positive integer

$$(3.4)\qquad \omega(m)/\omega(D) = m/D = \prod_{i=1}^{r} p_i^{n_i-t_i}.$$

Since the number of residues in a given progression P_e is $\omega(m)/\omega(D)$ where we have a total of $\omega(D)$ residue progressions, we see that if $n_i \leqslant z_i+s_i$ for all i in m, then $t_i = \min(n_i, z_i+s_i) = n_i$ and $\omega(m)/\omega(D) = 1$, i.e. we have no residue progressions in this case. However, if at least one p_i has a power n_i such that $n_i > z_i+s_i$, then residue progressions will exist since the number of terms in each progression $\omega(m)/\omega(D) > 1$.

Therefore, the structure of the residue progressions P_e corresponding to a complete set of residues $\bmod m$ is as follows. Given any $Z/m = Z/2^n \prod_{i=1}^{r} p_i^{n_i}$ such that $n_i > z_i+s_i$ for at least one p_i, then the sequence of power residues $R_j = r_e+KD$ for some fixed r_e with $D = 2^n \prod_{i=1}^{r} p_i^{t_i}$ and $K = 0, 1, \ldots, \omega(m)/\omega(D)-1$ are such that $Zg^j \equiv R_j \equiv (r_e+KD) \bmod m \equiv Z'g^e \equiv r_e \bmod D$ where $e \equiv j \bmod \omega(D)$, $Z' \equiv Z \bmod D$ and $e = 0, 1, \ldots \ldots, \omega(D)-1$, i.e. every R_j for $j = 0, 1, \ldots, \omega(m)-1$ is contained in one and only one of the disjoint $\omega(D)$ sets (residue progressions P_e) corresponding to some $K = 0, 1, \ldots, \omega(m)/\omega(D)-1$ whose initial term is some r_e. The proof of Theorem 4 is now complete. (See residue progression example at end of paper.)

Let us keep in mind that the exponents of the odd primes p_i in (3.4) are such that $n_i-t_i = 0$ for those p_i such that $n_i \leqslant z_i+s_i$, and $n_i-t_i = n_i-(z_i+s_i)$ for those p_i such that $n_i > z_i+s_i$ which is usually the case.

But more important for some results in a sequel to this paper, is that as the n_i increase in a given Z/m for a fixed set of odd primes p_i in m, i.e. in $Z_2/m^2, Z_3/m^3, \ldots$ the powers of the given set of primes in m increase so that the number of residue progressions $\omega(D)$ remains the same (D does not change under these conditions), but the number of residues $\omega(m)/\omega(D) = \prod_{i=1}^{r} p_i^{n_i-t_i}$ in each P_e may increase indefinitely.

On the maximum power s_i of the p_i-th odd prime contained in the least exponents $d_{i+1}, d_{i+2}, \ldots, d_r$ of the strictly increasing sequence of odd primes greater than p_i, consider some $p_j > p_i$ in the sequence. Clearly, the least exponent $d_j \leqslant p_j-1$ and assuming that $d_j = a_j p_i^{s_i}$, we have $p_j \geqslant 1+a_j p_i^{s_i}$ as a crude estimate on p_j whose least exponent contains p_i. For example, if at most $d_j = 2p_i$, then $p_j \geqslant 2p_j+1$. Data from a table of least exponents to the base 10, illustrates this estimate. If $p_i = 11$, then $p_j = 23 = 2(11)+1$ where $\mathrm{ord}_{23}10 = 2(11)$; if $p_i = 17$, then $p_j = 103 > 2(17)+1$ where $\mathrm{ord}_{103}10 = 2(17)$; and for $p_i = 37$, $p_j = 149 > 2(37)+1$ with $\mathrm{ord}_{149}10 = 2^2(37)$. In each case, we have given the prime p_j greater than p_i (and closest to p_i in the strictly increasing sequence of primes) that contains p_i in the exponent d_j to which 10 belongs $\bmod p_j$. Perhaps, a more precise estimate could be found on the prime $p_j > p_i$ that contains p_i in its least exponent.

In order to study the occurrence of the (j, ε)-normal property in the rational fractions, we find it convenient to separate the class of all rational fractions into classes for which residue progressions may or may not exist. In the definitions below, we assume $Z/m < 1$ is in lowest terms and the previously stated definitions of z_i, d_i, and s_i.

DEFINITION. *Type* A. A rational fraction $Z/m = Z/2^n \prod_{i=1}^{r} p_i^{n_i}$ is of *Type* A if $n_i > z_i+s_i$ for at least one odd prime.

Type B. A rational fraction $Z/m = Z/2^n \prod_{i=1}^{r} p_i^{n_i}$ is of *Type* B if $n_i \leqslant z_i+s_i$ for all odd primes p_i.

Type C. A rational fraction $Z/m = Z/2^n$ is of *Type* C when $n \geqslant 1$.

For Type A, we establish in Theorem 6 of this paper the (j, ε)-normality using Theorem 2, i.e. we prove in Theorem 5 the uniform ε-distribution of the fractional parts $\{Zg^i/m\}$ on $[0, 1]$ for Type A.

In the case of Type B for which residue progressions do not exist, numerical studies show that the (j, ε)-normal phenomenon may or may not exist. The simplest case of this in the Type B fractions is Z/p in Theorem 3 for $r = 1$. Since g is a primitive root $\bmod p^2$, we have for Theorem 4, $p^1 \| (g^{p-1}-1)$, $d = p-1$, $z = 1$, $n = 1$, $s = 0 \Rightarrow Z/p$ is of Type B, i.e. $n = z+s$ since $1 = 1+0$. Hence, there are no residue progressions since $t = \min(1, 1+0) = 1 \Rightarrow D = p$, so

$$\omega(m)/\omega(D) = m/D = p/p = 1 = \omega(p)/\omega(D) = (p-1)/(p-1) = 1,$$

yet we have shown in [6] that Z/p is (j, ε)-normal when represented in a primitive root base ([6], Cor. 1, p. 205).

Of course, the essential issue is the uniform ε-distribution of the $\{Zg^i/m\}$ on $[0,1]$ and a more complex case for which numerical work shows that we have (j,ε)-normality is, for example, a study of $1/59+1/97+1/109$. Here 10 is a primitive root of each prime such that $\omega(59\cdot 97\cdot 109) = \langle 58, 96, 108\rangle = 25056$; and computer data shows that $\min N(B_1, 10) = 2469$ and $\max N(B_1, 10) = 2576$ (note the agreement with the (j,ε)-normal phenomenon, i.e. if $g = 10$, the counts of any of the single digits $B_1 = 0, 1, \ldots, 9$ are approximately $1/10$ of the total number of digits in one period 25056), where the whole data shows (j,ε)-normality for $j \leqslant [\log_{10} 25056] = 4$. A reasonable conjecture for j is $j \leqslant [\log_g \omega(p_1 p_2 \ldots p_r)]$, but to date the resolution of the power residue distribution for such a case appears difficult without residue progressions.

We have some results on the case when g is not a primitive root, i.e. consider Z/p, where $p \| (g^d - 1)$ with $d = (p-1)/n$ which leads to questions concerning the approximately equal distribution of nth power residues (i.e. the uniform ε-distribution). For example, when $n = 2$, we have quadratic residues and therefore, we can make use of the Vinogradov–Pólya–Burgess inequalities ([2], pp. 182–204). The best estimate to date is that due to D. Burgess based on character sums ([2], p. 198). We will present studies based on such estimates that leads to (j,ε)-normal theorems for such a case of Type B, i.e. Z/p where g is not a primitive root in some future papers. The (j,ε)-normality for the type illustrated in the example above which is a case of Type B where $Z/m = Z/p_1 p_2 \ldots p_r$ appears quite difficult with our present knowledge of their residue distributions. The computer study indicates the above bound on j as a reasonable conjecture for $Z/59\cdot 97\cdot 109$. One thing is clear, the (j,ε)-normality is related to the deeper questions concerning the associated residue distributions on $[0,1]$ of $\{Zg^j/m\}$ for $j = 0, 1, \ldots, \omega(m)-1$.

An interesting case for Type B, shows that we may construct rational fractions of Type B which may or may not be (j,ε)-normal. Let $g = 10$, and write

$$Z/m = Z/10^{\lambda} + Z/10^{2\lambda} + \ldots = Z/(10^{\lambda}-1) \tag{3.5}$$

where Z can be arbitrarily chosen, i.e. the sequence of λ digits may or may not be (j,ε)-normal such that $(Z, 10^{\lambda}-1)$. Since $10^{\lambda}-1$ has a unique prime factorization $\prod_{i=1}^{r} p_i^{n_i}$ where only those p_i appear such that $\omega(p_i^{n_i}) \mid \lambda$ for $n_i \geqslant 1$, it follows that $Z/(10^{\lambda}-1)$ is a rational of Type B, i.e. $n_i = z_i + s_i$ for every p_i contained in $10^{\lambda}-1$ since the least exponents are divisors of λ. For example,

$$\begin{aligned}\omega(10^{22}-1) &= \omega(3^2\cdot 11^2\cdot 23\cdot 4093\cdot 8779\cdot 21649\cdot 513239)\\ &= \langle 1, 2\cdot 11, 22, 22, 22, 11, 11\rangle\end{aligned}$$

and the maximum power of 11 contained in each d_i is 1. In our notation, we have $p_1 = 3$, $n_1 = 2$, $z_1 = 2$, $s_1 = 0$; $p_2 = 11$, $n_2 = 2$, $z_2 = 1$, $s_2 = 1$; $p_3 = 23$, $n_3 = 1$, $z_3 = 1$, $s_3 = 0$; $p_4 = 4093$, $n_4 = 1$, $z_4 = 1$, $s_4 = 0$; etc. In each case, we have $n_i = z_i + s_i$. To summarize, we may state that if Z/m is represented in a scale g such that $m = g^{\lambda}-1$, then $Z/m < 1$ in lowest terms is a rational of Type B which may or may not be (j,ε)-normal depending on how the λ digits are selected.

Finally, for Type C with n subject to certain restrictions, residue progressions exist and (j,ε)-normality can be easily demonstrated. Due to the need for brevity here, we defer this result to a later paper.

THEOREM 5. *The rational fraction* Z/m *of Type* A *has a uniform* ε*-distribution of fractional parts* $\{Zg^i/m\}$ *for* $i = 0, 1, \ldots, \omega(m)-1$ *for all bases* g *such that* $(g, m) = 1$ *and* $2 \leqslant g < 1/\delta$ *where* $\varepsilon = \delta = \omega(D)/\omega(m) = D/m = 1/\prod_{i=1}^{r} p_i^{n_i - t_i}$, $D = 2^n \prod_{i=1}^{r} p_i^{t_i}$, *and* $t_i = \min(n_i, z_i + s_i)$.

Proof. As in the proof of Theorem 1, we consider the power residues $r_i/m = \{Zg^i/m\}$ for $i = 0, 1, \ldots, \omega(m)-1$ given by $Zg^i \equiv r_i \bmod m$ and determine for suitable choices of a and β where $0 \leqslant a < \beta \leqslant 1$, the number $N(I)$ of residues r_i/m contained in $a < r_i/m < \beta$, i.e.

$$ma < r_i < m\beta. \tag{3.6}$$

Using Theorem 4, if we choose the r'_e as least positive residues $\bmod D$, then each residue progression P_e is a strictly increasing sequence of least positive residues $\bmod m$ in arithmetic progression whose terms differ by D. We may, therefore, determine the number of residues r_i contained in (3.6) for each P_e and thus evaluate $N(I)$.

Within a given residue progression P_e, the Nth residue is given by $r_e + (N-1)D$ where $N = 1, 2, \ldots, \omega(m)/\omega(D) = m/D$ and therefore, if we replace r_i by $r_e + (N-1)D$ and sum over each residue progression P_e, we obtain the total number of r_i contained in (3.6). We find

$$N(I) = \sum_{(e)} \big([m\beta/D + (D-r_e)/D] - [ma/D + (D-r_e)/D]\big) \tag{3.7}$$

where $\sum\limits_{(e)}$ designates that we replace the r_e by the sequence of $\omega(D) = \omega\big(2^n \prod_{i=1}^{r} p_i^{t_i}\big)$ residues r_e that satisfy $Z'g^e \equiv r_e \bmod D$ where $Z \equiv Z' \bmod D$ and then sum. We may write (3.7) as

$$N(I) = \sum_{(e)} \big(m(\beta - a)/D + \theta_2 - \theta_1\big) \tag{3.8}$$

where $0 \leqslant \theta_1, \theta_2 < 1$ are the corresponding fractional parts assuming that a and β take on continuous real values on $[0,1]$ such that $0 \leqslant a$

$< \beta \leqslant 1$. Using (3.4) which states that $m/D = \omega(m)/\omega(D)$ for rationals of Type A, we have from (3.8)

$$N(I) = \omega(D)\big(\omega(m)(\beta-\alpha)/\omega(D) + \theta_2 - \theta_1\big) \tag{3.9}$$

where we have summed over the $\omega(D)$ residues r_e. Considering the ranges of the θ_i and rearranging, we obtain

$$|N(I)/\omega(m) - (\beta-\alpha)| < \omega(D)/\omega(m) = \varepsilon \tag{3.10}$$

where we have the equivalent forms for the bound ε given by

$$\varepsilon = \omega(D)/\omega(m) = D/m = 1/\prod_{i=1}^{r} p_i^{n_i-t_i}. \tag{3.11}$$

In order to insure that $N(I)$ has a non-zero value, let us note (3.8) and require that $\beta-\alpha$ be such that $m(\beta-\alpha)/D > 1$. Therefore, for a convenient description of the uniform ε-distribution of the fractional parts $\{Zg^i/m\}$ for rationals of Type A, we shall require that the least $\beta-\alpha$ on $[0,1]$ be such that

$$\beta - \alpha > D/m \tag{3.12}$$

or $\delta = \varepsilon = D/m = 1/\prod_{i=1}^{r} p_i^{n_i-t_i}$ as stated in the theorem. The restriction in (3.12) may not be the most stringent to keep $N(I) > 0$ for all choices of $[\alpha, \beta]$ taken anywhere in $[0,1]$ but the condition does insure that each residue progression P_e will contribute some r_i/m in every sub-interval $[\alpha, \beta] \epsilon [0,1]$ and consequently, in the count $N(I)$ of the number of points r_i/m contained in $[\alpha, \beta]$.

The condition (3.12) may also be argued independent of (3.8). Consider the complete sets of reduced residues $r_e \bmod D$ which initiate the residue progressions P_e which will contain all $r_i \bmod m$ ordered as arithmetic progressions as described in Theorem 1. Since the $r_i = r_e + KD$ for $K = 0, 1, \ldots, \omega(m)/\omega(D) - 1$, the maximum differences between residues r_i will be the maximum difference between the residues r_e. An upper bound on the maximum difference of residues r_e is D, therefore, if we require that $\beta - \alpha > D/m$, the bound D/m will exceed any maximum difference of residues r_i/m on $[0,1]$. This implies that any interval $\beta-\alpha > D/m$ will surely contain residue points r_i/m taken anywhere in $[0,1]$. Also note that we have shown that $\max(r_i/m - r_j/m)$ which is the maximum distance between adjacent residues out of the complete set is such that $\max(r_i/m - r_j/m) < \delta = D/m < \frac{1}{2}$ since $D/m = 1/\prod_{i=1}^{r} p_i^{n_i-t_i}$ for Type A is such that $n_i > z_i + s_i$ for at least one odd prime, thus surely $\delta = D/m < \frac{1}{2}$. We have satisfied the requirements for a uniform ε-distribution, hence Theorem 5 is complete. The (j, ε)-normality for rational fractions of Type A now follows at once using Theorems 2 and 5.

THEOREM 6. *A rational fraction* $Z/m < 1$ *in lowest terms of Type* A *is* (j, ε)*-normal to all bases* g *such that*

$$(g, m) = 1 \quad \textit{and} \quad 2 \leqslant g < m/D = \prod_{i=1}^{r} p_i^{n_i-t_i} = 1/\varepsilon$$

for all $j \leqslant [\log_g m/D]$ *where* $\varepsilon = D/m = \omega(D)/\omega(m) = 1/\prod_{i=1}^{r} p_i^{n_i-t_i}$, $t_i = \min(n_i, z_i+s_i)$, *and* $D = 2^n \prod_{i=1}^{r} p_i^{t_i}$.

Proof. From Theorem 5, $\delta = D/m$, and we have according to Theorem 2, (j, ε)-normality for $j \geqslant 1$ such that $\beta - \alpha = 1/g^j > \delta = D/m$. Also $\delta = \varepsilon$ which then implies the various equivalent forms

$$j \leqslant [\log_g m/D] = [\log_g 1/\delta] = [\log_g 1/\varepsilon] = \Big[\log_g \prod_{i=1}^{r} p_i^{n_i-t_i}\Big].$$

In order to have (j, ε)-normality for at least $j = 1$, we restrict the bases to those g such that $(g, m) = 1$ and $2 \leqslant g < m/D = 1/\varepsilon$ from the total set of $\varphi(m) - 1$ integers g such that $2 \leqslant g < m$. The proof of Theorem 6 is now complete.

One interesting consequence of Theorem 6 is that we may show that the (j, ε)-normality in Z/p^r where p is an odd prime still exists for r sufficiently large even though the base g is no longer a primitive root in contrast to the results in [6].

We have from Theorem 6 for $m = p^r$

THEOREM 7. *The rational fraction* $Z/p^r < 1$ *in lowest terms for* $r > z \geqslant 1$ *where* $p^z \| (g^d - 1)$ *and* $d = \operatorname{ord}_p g$ *is* (j, ε)*-normal for all* j *such that* $j \leqslant [\log_g p^{r-z}]$ *and* $\varepsilon = 1/p^{r-z}$ *when represented in bases* g *such that* $(g, p) = 1$ *and* $2 \leqslant g < p^{r-z}$.

Proof. Since $\omega(p^r) = dp^{r-z}$ for $r > z \geqslant 1$ ([4], p. 52, Theorems 4–6) where $p^z \| (g^d - 1)$ with $d = \operatorname{ord}_p g$ and $D = p^t = p^z$ for $t = \min(r, z)$ if $r > z$ (clearly, $s = 0$), we have, using Theorem 6, $\varepsilon = D/m = 1/p^{r-z} = \omega(D)/\omega(m) = d/dp^{r-z} = 1/p^{r-z}$ for $j \leqslant [\log_g p^{r-z}]$. Q.E.D.

Furthermore, consider a crucial observation in relation to uniform distributions on $[0,1]$ and the behavior of the uniform ε-distributions as defined for the fractional parts $\{Zg^i/m\}$ on $[0,1]$ for the rational fractions Z/m which are (j, ε)-normal. For example, consider the result in Theorem 7 for $\{Zg^i/p^r\}$ with $i = 0, 1, \ldots, \omega(p^r) - 1$ where $\omega(p^r) = dp^{r-z}$, $r > z \geqslant 1$ for some fixed odd prime p and base g.

Clearly from the uniform ε-distribution of parts for this case, the discrepancy is such that $D_n < \varepsilon = 1/p^{r-z}$ where $n = \omega(p^r) = dp^{r-z}$ with d and z fixed for any suitable p and g. Therefore, for r sufficiently large, we can have $\lim_{r\to\infty} D_n = \lim_{n\to\infty} D_n = 0$, i.e. the discrepancy is zero which is a requirement for a uniform distribution ([3], p. 22). However, here we do not have a uniform distribution in a strict sense, even though, $\lim_{n\to\infty} D_n = 0$. The reason is that for the uniform distribution, we require that $\lim_{n\to\infty} N(I)/n = I$ uniformly in n for any choice of n, no matter how large. In the (j, ε)-normal case for the rational fractions, we have the n increasing; we might say, in discontinuous "jumps" for increasing r since $n = \omega(p^r)$ for some consecutive increasing sequence of positive integers r. Therefore, the fractional parts $\{Zg^i/p^r\}$ for $i = 0, 1, \ldots, \omega(p^r)-1$ distribute themselves on $[0, 1]$ for increasing r, in an increasingly uniform way, i.e.

$$D_n = \sup_{0\leqslant a<\beta\leqslant 1} |N(a, \beta)/\omega(p^r)-I| < \varepsilon = 1/p^{r-z} \quad \text{where} \quad n = \omega(p^r),$$

but do not satisfy all the requirements for a uniform distribution.

Also, consider another aspect of the (j, ε)-normal property in the rational fractions with reference to block sizes j for some bounded consecutive sequence of j values.

For example, the condition $j \leqslant [\log_g 1/\varepsilon] = [\log_g \prod_{i=1}^{r} p_i^{n_i-t_i}]$ implies that in the period E of Z/m and at most $j-1$ digits into the next period that all blocks B_j whose lengths are restricted to the bounded set of j values will certainly appear with frequency ratios that satisfy (3.10) independent of the choice of digits they contain. However, blocks B_j may or may not appear in E terminating in at most $j-1$ places in the next repetition of E, if j exceeds $[\log_g 1/\varepsilon]$. Therefore, it is convenient for the (j, ε)-normal characterization to use the notion of "independent" blocks B_j as those whose lengths satisfy $j \leqslant [\log_g 1/\varepsilon]$, i.e. blocks consisting of any combination of j digits that will appear *with certainty* somewhere in E, and "dependent" blocks as those blocks whose length exceed $[\log_g 1/\varepsilon]$ (or some other prescribed bound!) and may or may not depending on the choice of the j digits in B_j, appear in E.

Finally, by means of the results here on the (j, ε)-normality of Type A, we can give an answer to what we might call a "Brouwer" type question. In 1925 and in his later lectures, L.E.J. Brouwer ([1], p. 3) and others in the intuitionist school of mathematical logic often stated the following as a possible "undecidable" proposition. Can we prove that the prescribed block 0123456789 appears in, say, the infinite sequence of digits of $\pi/4$ when represented in the base 10?

Let us paraphrase and ask for a proof of the question for a given rational fraction: Does the block 0123456789 appear, for example, somewhere in the decimal expansion of the rational fraction $1/17^{1000}$ expanded in the base 10? Since $1/17^{1000}$ is (j, ε)-normal by Theorem 3 with a period length of $\omega(17^{1000}) = 16\cdot 17^{999}$ (10 is a primitive root of 17) where $j \leqslant [\log_{10} 17^{1000}/2] = [1000\log_{10}17-\log_{10}2] = 1230$, we *can* say that somewhere in the approximately 2.652×10^{1230} digits of the period of $1/17^{1000}$ that the independent block 0123456789 will appear (with certainty!) with relative frequency of about $1/10^{10}$, i.e.

$$|N(I)/16\cdot 17^{999}-1/10^{10}| < 2/16\cdot 17^{999}.$$

An important point here is that we cannot say exactly *where* the block 0123456789 will *first* make its appearance but we do know that *it will* with the above frequency. This says that now such questions are decidable for given particular rational fractions in the real numbers when their (j, ε)-normality has been demonstrated.

Of course, the Brouwer question is answered in the affirmative if we could prove that $\pi/4$ is a normal number. But, this unresolved question appears quite resistant to solution by our contemporary mathematics.

We can, however, make a slight advance on the Brouwer question itself with the (j, ε)-normal properties of the rational fractions of Type A. Consider the nth partial product

$$P_n(\pi/4) = \prod_{i=1}^{n} \left(1-1/(2i+1)^2\right) = p_n/q_n$$

based on the Wallis infinite product for $\pi/4$. In a future paper, we will show that p_n/q_n is a rational of Type A for n sufficiently large. In which case, the (j, ε)-normality shows that the block 0123456789 *will appear* somewhere in the arbitrarily long set of digits of one period of p_n/q_n and, in which, only a comparatively small portion of the long period represents $\pi/4$ exactly. We would have an answer to the Brouwer question from another point of view, if we could show in which portion the block 0123456789 occurs; the exact portion or the set of digits which will change as we consider larger values of n in

$$\lim_{n\to\infty} \prod_{i=1}^{n} \left(1-1/(2i+1)^2\right) = \pi/4.$$

However, we cannot answer this question yet for (j, ε)-normal rational fractions, i.e. we cannot predict the location of a prescribed block or where this block will make its first appearance within the period.

On the other hand, we can now study the (j, ε)-normal properties of particular sequences of rational fractions which approximate a given irrational like π, e, $\sqrt{2}$, etc.

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An example of residue progressions

Let $m = 5^3\cdot 11^2$, $Z = 1$; $p_1 = 5$, $z_1 = 1$, $s_1 = 1$, $d_1 = 4$, $n_1 = 3$; $p_2 = 11$, $z_2 = 2$, $s_2 = 0$, $d_2 = 5$, $n_2 = 2$; since $11^2 \| (3^5-1)$ and $5 \| (3^4-1)$, we have for $n = 0$, $D = 2^n \prod_{i=1}^{r} p_i^{t_i}$ where $t_i = \min(n_i, z_i+s_i)$ implies $t_1 = \min(3, 2) = 2$, $t_2 = \min(2, 2) = 2$, hence $D = 2^0\cdot 5^2\cdot 11^2 = 3025$. Thus for the R_j, we have $3^j \equiv R_j \bmod 5^3\cdot 11^2$ and the r_e in $R_j = r_e+KD$ are given by $3^e \equiv r_e \bmod 5^2\cdot 11^2$ where $\omega(m) = \omega(5^3\cdot 11^2) = \langle 4\cdot 5^{3-1}, 5\cdot 11^{2-2}\rangle = 100$ with $K = 0, 1, \ldots, \omega(m)/\omega(D)-1 = 100/20-1 = 4$ given that $\omega(D) = \langle 4\cdot 5, 11^{2-2}\rangle = 20$.

The residues in sequence

e	r_e	j	R_j	j	R_j	j	R_j	j	R_j	j	R_j
0	1	0	1	21	9078	42	9084	63	3052	84	12181
1	3	1	3	22	12109	43	12127	64	9156	85	6293
2	9	2	9	23	6077	44	6131	65	12343	86	3754
3	27	3	27	24	3106	45	3268	66	6779	87	11262
4	81	4	81	25	9318	46	9804	67	5212	88	3536
5	243	5	243	26	12829	47	14287	68	511	89	10608
6	729	6	279	27	8237	48	12611	69	1533	90	1574
7	2187	7	2187	28	8586	49	7583	70	4599	91	4722
8	511	8	6561	29	13633	50	7624	71	13797	92	14166
9	1533	9	4558	30	10649	51	7747	72	11141	93	12248
10	1574	10	13674	31	1697	52	8116	73	3173	94	6494
11	1697	11	10772	32	5091	53	9223	74	9519	95	9357
12	2066	12	2066	33	148	54	12544	75	13432	96	13071
13	148	13	6198	34	444	55	7382	76	10046	97	8963
14	444	14	3469	35	1332	56	6021	77	15013	98	11764
15	1332	15	10407	36	3996	57	5938	78	14789	99	5042
16	971	16	971	37	11988	58	2689	79	14117	100	1
17	2913	17	2913	38	5714	59	8067	80	12101		
18	2689	18	8739	39	2017	60	9076	81	6053		
19	2017	19	11092	40	6051	61	12103	82	3034		
20	1	20	3026	41	3028	62	6059	83	9102		

$D = 3025$ **Residues arranged in their residue progressions** P_e

$r_e \rightarrow$	1	3	9	27	81	243	729	2187	511	1533
	3026	3028	3034	3052	3106	3268	3754	5212	3536	4558
	6051	6053	6059	6077	6131	6293	6779	8237	6561	7583
	9076	9078	9084	9102	9156	9318	9804	11262	9586	10608
	12101	12103	12109	12127	12181	12343	12829	14287	12611	13633
$r_e \rightarrow$	1574	1697	2066	148	444	1332	971	2913	2689	2017
	4599	4722	5091	3173	3469	4357	3996	5938	5714	5042
	7624	7747	8116	6198	6494	7382	7021	8963	8739	8067
	10649	10772	11141	9223	9519	10407	10046	11988	11764	11092
	13674	13797	14166	12248	12544	13432	13071	15013	14789	14117

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A general arithmetic construction of transcendental non-Liouville normal numbers from rational fractions

by

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1. Introduction. In this paper, we derive a general arithmetic construction of an extensive class of transcendental non-Liouville normal numbers based on any given rational fraction $Z/m < 1$ in lowest terms. The construction is founded on the results in [14] wherein we proved that certain broad classes of rational fractions are (j, ε)-normal.

In [14], (1.1), we extended the original definition of (j, ε)-normality due to Besicovitch ([15], p. 201) so as to apply to appropriate rational fractions $Z/m < 1$ in lowest terms. Essentially, we showed that the definition of (j, ε)-normality which Besicovitch defined for finite sets of digits could be applied to the infinite periodic sequences which represent certain broad classes of rational fractions. Therefore, we can consider whether some given rational fraction Z/m when represented in appropriate bases g is (j, ε)-normal or not in this sense.

Consider the real number $x = .x_1x_2\ldots$ represented in the scale g and let $N(B_j, X_\lambda)$ denote the number of occurrences of the block B_j consisting of any combination of j digits chosen from $0, 1, \ldots, g-1$ in the first λ digits $x_1x_2\ldots x_\lambda$ of x. We have the following definition ([7], p. 95, 104) equivalent to that given by Borel in 1908. Unless otherwise indicated, lower case letters will represent positive integers.

Definition. *Normal number.* The number x is *normal* in the scale g if

$$\lim_{\lambda\to\infty} N(B_j, X_\lambda)/\lambda = 1/g^j \tag{1.0}$$

for all $j = 1, 2, 3, \ldots$

If x is any real number, x is said to be normal to the base g if $\{x\}$ $\{x\} = x-[x]$ is normal to the base g where $\{x\}$ is the fractional part of x and $[x]$ is the greatest integer not exceeding x. Furthermore, if some x is to satisfy (1.0), i.e. be a normal number, then it is, necessarily, an irrational.