et que les d_q sont déterminées par le fait que l'on a pour tout $\theta \in \boldsymbol{R}^k$

$$\sum_{q \in \boldsymbol{Z}^k} d_q \exp\Big(i \sum_{j=1}^{k} q_j \theta_j\Big) = \prod_p \Big(1 - \frac{1}{p}\Big)\Big[1 + \sum_{r=1}^{+\infty} \frac{g_0(p^r)}{p^r}\Big].$$

Comme dans le cas d'une seule fonction, cette égalité peut aussi être établie par la méthode utilisée ici.

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A statistical density theorem for L-functions with applications

by

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§ 1. Introduction

1. In the last years many interesting results in the analytical theory of numbers have been obtained by the so-called "large sieve" method, e.g. new statistical density theorems for L-functions [2], [1] and the mean value theorem of Bombieri [2] ((1.6) below) concerning the distribution of primes in arithmetical progressions.

We shall in this paper combine the large sieve with the method of Rodosski [9] and prove two statistical density results (Theorem 1) for L-functions. The estimate (1.4) is most effective for "high" rectangles and seems to be of a new type. As an arithmetical application of this we shall prove an analogue of Bombieri's theorem, concerning the primes in a "short" interval (Theorem 2). Finally we call attention to the consequences of Theorem 2 to some prime number problems.

2. Let $X \geqslant 1$, $T \geqslant 2$, $\alpha \geqslant \frac{1}{2}$, and let χ be a character $(\operatorname{mod} q)$. Denote by $N(\alpha, T, q, \chi)$ the number of zeros of the function $L(s, \chi)$ in the rectangle

$$1-\alpha \leqslant \sigma \leqslant 1, \quad |t| \leqslant T. \tag{1.1}$$

In the statistical theory of L-functions the main problem is to find an estimate for the sum

$$\sum_{q \leqslant X} \sideset{}{^*}\sum_{\chi \bmod q} N(\alpha, T, q, \chi) \tag{1.2}$$

where the asterisk denotes summation over primitive characters only. Bombieri has in [2] proved that the sum (1.2) is [1]

$$\ll (X^2 + XT)^{\frac{4(1-\alpha)}{3-2\alpha}+\varepsilon} T, \tag{1.3}$$

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[1] We use the following notation: $c_1, c_2, \ldots$ denote positive absolute constants; ε and A stand for positive constants, the former arbitrarily small and the latter arbitrarily large, which need not be always the same. Further, as usual, we write $e(a) = e^{2\pi i a}$, $e_q(a) = e^{2\pi i a/q}$.

and Barban [1] has given other estimates of about the same type. But all these are inconvenient for large values of T. We shall eliminate the factor T in the first of the following estimates:

THEOREM 1. *For the sum* (1.2) *we have the estimates*

$$(X^7T^4)^{\frac{1-\alpha}{\alpha}}\log^{c_1}(X+T), \tag{1.4}$$

$$(X^5T^2)^{\frac{1-\alpha}{\alpha}}T\log^{c_2}(X+T). \tag{1.5}$$

3. As an application of his density theorem, slightly stronger than (1.3), Bombieri proved a result concerning prime numbers which (in a little rough form) runs as follows:

$$\sum_{q\leqslant x^{1/2-\varepsilon}}\max_{z\leqslant x}\max_{(a,q)=1}\left|\psi(z,q,a)-\frac{z}{\varphi(q)}\right|\ll x\log^{-A}x. \tag{1.6}$$

To formulate our theorem, let c be a constant such that

$$\zeta(\tfrac{1}{2}+it,w)\ll t^{c+\varepsilon}$$

for $t\to\infty$ and $0<w\leqslant 1$. It can be proved that $c<\frac{1}{6}$, the best result heretofore obtained being $c\leqslant\frac{6}{37}$ (see [6]).

THEOREM 2. *Let $x\geqslant 2$, $y\geqslant 2$, $y=x^{\theta}$ where θ is a fixed number from the interval $0<\theta<1$. Then*

$$\sum_{q\leqslant x^{\beta}}\max_{z\leqslant y}\max_{(a,q)=1}\left|\psi(x+z,q,a)-\psi(x,q,a)-\frac{z}{\varphi(q)}\right|\ll y\log^{-A}x, \tag{1.7}$$

where

$$\beta=\frac{4c\theta+2\theta-1-4c}{6+4c}-\varepsilon. \tag{1.8}$$

Recently Gallagher [5] has proved (1.6) without using zeros of L-functions. But it seems to us that it is more difficult to prove Theorem 2 by some similar method.

4. Estimations of the type (1.6) and (1.7) are important e.g. in the application of Selberg's sieve method to prime number problems such as the twin-prime problem and Goldbach's problem. By the method [7] it can be deduced from (1.6) e.g. that there is an infinity of primes p such that $p+2$ has at most 3 prime factors. One may ask how the primes of this kind are distributed. Now Theorem 2 offers a possibility for obtaining such results. It can be proved e.g.

THEOREM 3. *For every positive integer $r\geqslant 8$ there exists a real number $\theta(r)$ with $0<\theta(r)<1$ such that for x sufficiently large in any interval $(x,x+x^{\theta(r)})$ there exists a pair $(p,p+2)$ such that $p+2$ has at most r prime factors. We have*

$$\theta(r)<\frac{1+4c}{2+4c}+\varepsilon$$

if $r\geqslant r(\varepsilon)$.

The question about $\theta(r)$ remains open for $r\leqslant 7$.

§ 2. Preliminary lemmas

5. We state first a lemma which follows easily from the considerations carried out in [9].

LEMMA 1. *Let $X\geqslant 1$, $y>1$, $\frac{1}{2}\leqslant\alpha\leqslant 1$, $T\geqslant 2$,*

$$z^{\alpha}=c_2yTX^{1/2}\log X, \tag{2.1}$$

$$a_n=\sum_{\substack{d\mid n\\ d>y}}\mu(d). \tag{2.2}$$

Let further $\lambda=[\log z]+1$,

$$I(\nu,M)=\sum_{q\leqslant X}\sideset{}{^*}\sum_{\chi \bmod q}\int_{\alpha}^{1}\int_{-T}^{T}\Big|\sum_{M\leqslant n\leqslant 2M}a_n\log^{\nu}n\,\chi(n)\,n^{-\sigma-it}\Big|^2d\sigma\,dt. \tag{2.3}$$

Then for the sum (1.2) *we have the estimate*

$$\ll\log^{c_3}z\max_{1\leqslant\nu\leqslant\lambda}\max_{y\leqslant M\leqslant z}\log^{-2\nu}M\,I(\nu,M). \tag{2.4}$$

We shall also need some facts about the divisor function $\tau(n)$.

LEMMA 2. *We have $\tau(n)\ll n^{\varepsilon}$; further,*

$$\sum_{n\leqslant x}\frac{\tau(n)}{n}\ll\log^2x, \tag{2.5}$$

$$\sum_{n\leqslant x}\tau^2(n)\ll x\log^3x, \tag{2.6}$$

$$\sum_{x\leqslant n\leqslant x+x^{\varepsilon}}\tau(n)\ll x^{\varepsilon}\log x. \tag{2.7}$$

Three first properties are well-known, and (2.7) can be proved by the method of [10].

6. The large sieve method we shall apply in the form of

LEMMA 3. *Let d_n, $n = H+1, \ldots, H+K$ be arbitrary complex numbers, and let*

$$S(a) = \sum_{n=H+1}^{H+K} d_n e(na), \quad S(\chi) = \sum_{n=H+1}^{H+K} d_n\chi(n), \quad Z = \sum_{n=H+1}^{H+K} |d_n|^2.$$

Then

$$(2.8)\qquad \sum_{q\leqslant X}\sum_{(a,q)=1}\left|S\left(\frac{a}{q}\right)\right|^2 \ll (X^2+K)Z, \qquad \sum_{q\leqslant X}\sideset{}{^*}\sum_{\chi \bmod q}|S(\chi)|^2 \ll (X^2+K)Z.$$

For a simple proof, see [4].

§ 3. Proof of Theorem 1

7. We show first that (1.5) follows immediately from Lemmas 1, 2, and 3. For by (2.2) we have $|a_n| \leqslant \tau(n)$, and using Lemma 3 in (2.3) we find that

$$I(\nu, M) \ll T(X^2+M)M^{-2a}\log^{2\nu}M\sum_{n=M}^{2M}\tau^2(n).$$

Choosing $y = X^2$ and using (2.6), we establish (1.5) by Lemma 1.

8. Next we turn to the proof of (1.4). Choosing in Lemma 1

$$(3.1)\qquad y = X^3T, \quad z = (c_2X^{7/2}T^2\log X)^{1/a},$$

we consider one particular $I(\nu, M)$ with integral M. Let first σ be a fixed number from the interval $a \leqslant \sigma \leqslant 1$. Obviously

$$(3.2)\qquad J_\chi \overset{\text{def}}{=} \int_{-T}^{T}\Big|\sum_{n=M}^{2M} a_n\log^\nu n\,\chi(n)n^{-\sigma-it}\Big|^2 dt$$
$$= \sum_{n_1=M}^{2M}\sum_{n_2=M}^{2M} d_{n_1}d_{n_2}b_{n_1,n_2}\bar\chi(n_1)\chi(n_2),$$

where

$$(3.3)\qquad b_{n_1,n_2} = \int_{-T}^{T}(n_1/n_2)^{it}dt \ll \begin{cases} T & \text{for} \quad n_1 = n_2,\\ \min\left(T, \dfrac{M}{|n_1-n_2|}\right) & \text{for} \quad n_1 \neq n_2,\end{cases}$$

$$(3.4)\qquad d_n = a_n n^{-\sigma}\log^\nu n.$$

Let $\tau(\chi) = \sum_{(a,q)=1}\chi(a)e_q(a)$ be a Gaussian sum. The first of the following identities is well-known, and the second is a consequence of it:

$$\tau(\chi)\bar\chi(n_1) = \sum_{(a_1,q)=1}\chi(a_1)e_q(a_1n_1), \qquad \overline{\tau(\chi)}\chi(n_2) = \sum_{(a_2,q)=1}\bar\chi(a_2)e_q(-a_2n_2).$$

On multiplying these, multiplying the resulting identity by $d_{n_1}d_{n_2}b_{n_1,n_2}$ and finally summing over n_1 and n_2, we obtain by (3.2)

$$|\tau(\chi)|^2J_\chi = \sum_{n_1}\sum_{n_2}d_{n_1}d_{n_2}b_{n_1,n_2}\sum_{a_1}\sum_{a_2}\chi(a_1)\bar\chi(a_2)e_q(a_1n_1-a_2n_2).$$

Further, summing over all characters $(\bmod q)$, and taking into account that $J_\chi \geqslant 0$ and that for a primitive character $|\tau(\chi)|^2 = q$, we get

$$(3.5)\qquad \sideset{}{^*}\sum_{\chi \bmod q} q\varphi^{-1}(q)J_\chi \leqslant \sum_{n_1}\sum_{n_2}d_{n_1}d_{n_2}b_{n_1,n_2}\sum_{(a,q)=1}e_q\big(a(n_1-n_2)\big).$$

9. Now we set out to estimate the sum on the right of (3.5). Let

$$(3.6)\qquad Y = X^2, \quad M_1 = M-Y, \quad M_2 = M+Y, \quad M_3 = M+2Y,$$

and define the intervals

$$(3.7)\qquad H_\nu\colon\ \nu \leqslant n < \nu+Y, \quad \nu = M_1, M_1+1, \ldots, 2M.$$

LEMMA 4. *Let n_1 and n_2 lie in the intervals H_{ν_1}, and H_{ν_2}, respectively. Then*

$$(3.8)\qquad b_{n_1,n_2} = b_{\nu_1,\nu_2} + O\left(\min\left(T, \frac{TY}{|n_1-n_2|}\right)\right).$$

Proof. By (3.3) the lemma is trivial for $|\nu_1-\nu_2| \leqslant 2Y$, say. Let now $|\nu_1-\nu_2| = \Delta > 2Y$. Then we have to prove that

$$(3.9)\qquad b_{n_1,n_2} - b_{\nu_1,\nu_2} \ll TY\Delta^{-1}.$$

To see this, we remark that

$$(n_1/n_2)^{iT}-(n_1/n_2)^{-iT} = (\nu_1/\nu_2)^{iT}-(\nu_1/\nu_2)^{-iT}+O(TYM^{-1}) \ll T\Delta M^{-1},$$
$$\log^{-1}(n_1/n_2) = \log^{-1}(\nu_1/\nu_2)+O(YM\Delta^{-2}) \ll M\Delta^{-1},$$

whence, by (3.3), the estimate (3.9) follows.

10. Next we consider the sums

$$(3.10)\qquad T_q = \sum_{M_1\leqslant \nu_1,\nu_2\leqslant 2M}\ \sum_{n_1\in H_{\nu_1}}\ \sum_{n_2\in H_{\nu_2}} d_{n_1}d_{n_2}b_{n_1,n_2}\sum_{(a,q)=1}e_q\big(a(n_1-n_2)\big),$$

and assert the crucial

LEMMA 5. *We have*

$$\sum_{q\leqslant X} T_q \ll Y^2 M^{2(1-\sigma)}\log^{2\nu+5} M.$$

Proof. Using (3.8) in (3.10), we get first

$$(3.11)\qquad \sum_{\leqslant X} T_q = \sum_{\nu_1}\sum_{\nu_2} b_{\nu_1,\nu_2} \sum_{q\leqslant X}\sum_{(a,q)=1}\Big\{\sum_{n_1\epsilon H_{\nu_1}} d_{n_1} e_q(an_1)\Big\}\Big\{\sum_{n_2\epsilon H_{\nu_2}} d_{n_2} e_q(-an_2)\Big\}+R,$$

where

$$(3.12)\qquad R \ll M^{-2\sigma}\log^{2\nu} M \sum_{M_1\leqslant\nu_1,\nu_2\leqslant 2M}\ \sum_{n_1\epsilon H_{\nu_1}}\ \sum_{n_2\epsilon H_{\nu_2}} \tau(n_1)\tau(n_2)\times \\ \times\min\left(T, \frac{TY}{|n_1-n_2|}\right)\sum_{q\leqslant X}|S_{q,n_1-n_2}|,$$

and

$$S_{q,n} = \sum_{(a,q)=1} e_q(an)$$

is a Ramanujan sum. It is well-known that

$$S_{q,n} = \sum_{d|(q,n)} \mu\left(\frac{q}{d}\right) d \ll \sum_{d|(q,n)} d,$$

whence

$$(3.13)\qquad \sum_{q\leqslant X} |S_{q,n}| \ll \begin{cases} X^2 & \text{for} \quad n=0,\\ X\tau(n) & \text{for} \quad n\neq 0.\end{cases}$$

To estimate the term R, we subdivide first the pairs (n_1, n_2) into groups such that n_1-n_2 is a constant Δ' for each group. Each pair occurs at most Y^2 times, whence by (3.12) and (3.13) obviously

$$R \ll XY^2M^{-2\sigma}\log^{2\nu} M \sum_{\substack{\Delta'=-M_3\\ \Delta'\neq 0}}^{M_3} \min(T, TY|\Delta'|^{-1})\tau(\Delta')\sum_{n=M_1}^{M_2}\tau(n)\tau(n+\Delta')+ \\ +X^2Y^2TM^{-2\sigma}\log^{2\nu} M\sum_{n=M_1}^{M_2}\tau^2(n).$$

Using (2.5) and (2.6), we obtain the further estimate

$$R \ll XY^3TM^{1-2\sigma}\log^{2\nu+5} M + X^2Y^2M^{1-2\sigma}T\log^{2\nu+3} M.$$

But $M \geqslant y = TX^3$, whence by (3.6) $TYX \leqslant M$, and we find indeed that R is not "too large".

To estimate the main term in (3.11), we apply Lemma 3. First, using Schwartz's inequality, we get

$$(3.14)\qquad \sum_{q\leqslant X} T_q \ll \sum_{\nu_1}\sum_{\nu_2}|b_{\nu_1,\nu_2}|\Big\{\sum_{q_1\leqslant X}\sum_{(a_1,q_1)=1}\Big|\sum_{n_1\epsilon H_{\nu_1}} d_{n_1}e_{q_1}(a_1n_1)\Big|^2\Big\}^{1/2}\times \\ \times\Big\{\sum_{q_2\leqslant X}\sum_{(a_2,q_2)=1}\Big|\sum_{n_2\epsilon H_{\nu_2}} d_{n_2}e_{q_2}(a_2n_2)\Big|^2\Big\}^{1/2}+R.$$

Now the expressions in the brackets are by Lemma 3 and (3.4) (note that $Y = X^2$)

$$\ll Y\sum_{n_i\epsilon H_{\nu_i}}|d_{n_i}|^2 \ll YM^{-2\sigma}\log^{2\nu} M\sum_{n_i\epsilon H_{\nu_i}}\tau^2(n_i),\qquad i=1,2.$$

Hence

$$(3.15)\qquad \sum_{q\leqslant X} T_q \ll YM^{-2\sigma}\log^{2\nu} M\sum_{M_1\leqslant\nu_1,\nu_2\leqslant 2M}|b_{\nu_1,\nu_2}|\times \\ \times\Big(\sum_{n_1\epsilon H_{\nu_1}}\tau^2(n_1)\Big)^{1/2}\Big(\sum_{n_2\epsilon H_{\nu_2}}\tau^2(n_2)\Big)^{1/2}+R.$$

Writing $\nu_2 = \nu_1+\Delta$ and using (3.3), we see that (3.15) takes the form

$$\sum_{q\leqslant X} T_q \ll YM^{-2\sigma}\log^{2\nu} M\sum_{\Delta=-M_2}^{M_2}\min(T, M|\Delta|^{-1})\sum_{\nu_1=M_1}^{2M}\Big(\sum_{n_1\epsilon H_{\nu_1}}\tau^2(n_1)\Big)^{1/2}\times \\ \times\Big(\sum_{n_2\epsilon H_{\nu_1+\Delta}}\tau^2(n_2)\Big)^{1/2}+R \\ \ll YM^{-2\sigma}\log^{2\nu} M\sum_{\Delta=-M_2}^{M_2}\min(T, M|\Delta|^{-1})\Big(\sum_{\nu_1=M_1}^{2M}\sum_{n_1\epsilon H_{\nu_1}}\tau^2(n_1)\Big)^{1/2}\times \\ \times\Big(\sum_{\nu_1=M_1}^{2M}\sum_{n_2\epsilon H_{\nu_1+\Delta}}\tau^2(n_2)\Big)^{1/2}+R \\ \ll Y^2M^{2(1-\sigma)}\log^{2\nu+4} M+R,$$

and the proof of Lemma 5 is complete.

11. We can now complete the proof of Theorem 1. We state first

LEMMA 6. *We have*

$$(3.16)\qquad Y^2\sum_{q\leqslant X}\ \sideset{}{^*}\sum_{\chi \bmod q} J_\chi \ll Y^2M^{2(1-\sigma)}\log^{2\nu+5} M.$$

Proof. Comparing (3.5) and (3.10) we find that each pair (n_1, n_2), occurring in (3.5), occurs exactly Y^2 times in (3.10). In (3.10) there are also some further terms, corresponding to pairs (n_1, n_2) with at least one of the numbers n_1, n_2 not lying in the interval $[M, 2M]$. Let, for example, $M-Y \leqslant n_1 < M$. We estimate the contribution to the sum

$\sum_{q\leqslant X} T_q$ of the pairs with n_1 fixed and n_2 running over the interval $[M_1, M_2]$, and get by (3.10) the estimate

$$Y^2|d_{n_1}|\sum_{n_2}|d_{n_2}|\,|b_{n_1,n_2}|\sum_{q\leqslant X}|S_{q,n_1-n_2}|.$$

Writing $n_2 = n_1+\Delta$, we see that the above expression is by previous arguments

$$Y^2\tau(n_1)M^{-2\sigma}\log^{2\nu}M\Big\{X^2T\tau(n_1)+\sum_{\substack{\Delta=-Y\\ \Delta\neq 0}}^{M_3} MX\Delta^{-1}\tau(n_1+\Delta)\tau(\Delta)\Big\}$$
$$\ll Y^2X^2TM^{-2\sigma}\tau^2(n_1)\log^{2\nu}M+Y^2XM^{1-2\sigma}\tau(n_1)\log^{2\nu+4}M.$$

We separate two cases: $T<X$ and $T\geqslant X$. In the first case we get, summing over n_1, using (2.7), and noting that $\tau(n_1)\ll X$, $M\geqslant TX^3$, $Y = X^{\nu}$, the estimate on the right of (3.16). The second case is clear. So Lemma 6 follows by the above remarks from Lemma 5.

Now by (3.16) and (3.1)

$$I(\nu, M) = \sum_{q\leqslant X}\ \sum_{\chi \bmod q}^{*}\int_{a}^{1} J_\chi d\sigma \ll M^{2(1-\alpha)}\log^{2\nu+5}M$$
$$\ll (X^7X^4)^{\frac{1-\alpha}{\alpha}}\log^{2\nu+7}(X+T),$$

and Lemma 1 completes the proof of (1.4).

12. We state a corollary of (1.4) which is useful in many problems.

Lemma 7. *We have for $T\ll X^{A_1}$, $X\geqslant X_0$, and for a suitable constant a with $0<a<1$*

(3.17) $$\sum_{\chi \bmod q} N(1-\log^{-a}X, T, q, \chi) = 0$$

for all modules from any interval $Q\leqslant q\leqslant 2Q$ with $Q\leqslant X$, with exception of $Q\log^{-A_2}X$ modules at most.

The proof proceeds in a well-known manner, using (1.4), Siegel's theorem and Satz 6.2, p. 295, of [8]. For the constant a may be taken e.g. $\frac{3}{4}+\varepsilon$, by [8]. Usually it is essential only that $a<1$.

§ 4. Arithmetical applications

13. We prove a lemma from which Theorem 2 is an immediate consequence.

Lemma 8. *For all modules q from any interval $Q\leqslant q\leqslant 2Q$ with $Q\leqslant x^\beta$, with exception of $Q\log^{-A_1}x$ modules at most, we have*

(4.1) $$\max_{z\leqslant y}\max_{(a,q)=1}\left|\psi(x+z, q, a)-\psi(x, q, a)-\frac{z}{\varphi(q)}\right| \ll \frac{y}{\varphi(q)}\log^{-A_2}x.$$

Proof. For $Q\leqslant \exp(\log^{1/5}x)$, say, the lemma is clear since then (4.1) holds for all modules, with a possible exception of the "exceptional" modules (see [8], p. 321).

Now let us suppose that $Q>\exp(\log^{1/5}x)$. We start from the well-known estimation

(4.2) $$\left|\psi(x+z, q, a)-\psi(x, q, a)-\frac{z}{\varphi(q)}\right|$$
$$\ll y\varphi^{-1}(q)\sum_{\chi \bmod q}\ \sum_{|\gamma|\leqslant T} x^{\beta-1}+T^{-1}x\log^2x,$$

where $\varrho = \beta+\gamma i$ runs over the zeros of $L(s, \chi)$ (see [8], p. 321). We exclude the same modules as in Lemma 7, and for the remaining ones (3.17) holds. For the non-excluded modules we use the density estimate

(4.3) $$N_q(\alpha) \stackrel{\text{def}}{=} \sum_{\chi \bmod q} N(\alpha, T, q, \chi) \ll \{q^4T^{4c}(T+q)^2\}^{1-\alpha}\log^8(qT)$$

(see [8], p. 299). By this

(4.4) $$\sum_{\chi}\ \sum_{|\gamma|\leqslant T} x^{\beta-1} = \int_{0}^{1-\log^{-4/5}x} x^{\alpha-1}dN_q(\alpha) \ll \log^{-A}x,$$

if

(4.5) $$q^4T^{4c}(T+q)^2 \ll x^{1-\varepsilon}.$$

We choose

(4.6) $$T = y^{-1}qx^{1+\varepsilon},$$

so that $T>q$. Then (4.5) is satisfied if $q\leqslant x^\beta$, where β is given in (1.7). By (4.2), (4.4), and (4.6), this completes the proof.

14. Proof of Theorem 3. Let $\{a_n\}$, $n = 1, \ldots, N$, be a sequence of positive integers, and denote by N_m the number of the a_n's which are divisible by m. Let

$$N_m = \frac{Nf(m)}{m}+R_m,$$

where $f(m)$ is a multiplicative function. Let γ be such that

$$\sum_{m\leqslant N^\gamma}|R(m)| \ll N\log^{-A}N.$$

Let further $\max_{1\leqslant n\leqslant N} a_n\leqslant N^{t+\varepsilon}$, and define $\tau = t\gamma^{-1}$. Then, by [7], among the numbers a_n there are at least

$$0.05\frac{N}{\gamma\log N}\prod_{p}\frac{1-f(p)/p}{1-1/p}$$

numbers with at most k prime factors, where k depends on τ, and $k \to \infty$ when $\tau \to \infty$. E.g. $k = 8$ if $\tau \leqslant 7.02$.

We take now $\{a_n\} = \{p+2\}, f(m) = m\varphi^{-1}(m)$, where p runs over the primes in the interval $[x, x+y]$. Then $N < y$, $t = \theta^{-1}$, $\gamma = \beta\theta^{-1}$, and

$$\tau = \frac{1}{\beta} = \frac{6+4c}{4c\theta+2\theta-1-4c} + \varepsilon .$$

When $\theta \to 1$, then $\tau \to 6+4c+\varepsilon < 7.02$. So $\theta(8) < 1$. The second assertion of Theorem 3 follows also immediately.

15. We remark finally that Theorem 2 is applicable to several other problems. E.g. it is possible to estimate the differences between "short" gaps between prime numbers, along the lines of the paper of Bombieri–Davenport [3]. Also it can be proved that every large even number is representable as a sum of two almost equal integers, one of which is a prime and the other has a finite number ($\leqslant 8$) of prime factors.

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Metrische Theorie einer Klasse zahlentheoretischer Transformationen (Corrigendum)

von

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Wie ich erst nach Drucklegung von [2] bemerkte, ist der Beweis von Satz 5 unrichtig. Die Abschätzung von der drittletzten Zeile zur vorletzten Zeile auf p. 6 ist grob fehlerhaft. Es soll hier ein neuer Beweis gegeben werden, der lediglich eine kleine Zusatzvoraussetzung, nämlich $0 < m \leqslant f_0(x) \leqslant M$ auf B verlangt. Da die Eindeutigkeit aus bekannten Sätzen der Ergodentheorie ohnedies folgt, werden wir Teil (b) von p. 7 an, neu beweisen.

Wir zeigen also: Es sei $f_0(x)$ gegeben mit

$$0 < m \leqslant f_0(x) \leqslant M,$$
$$|f_0(x)-f_0(y)| \leqslant N\cdot\|x-y\|.$$

Definiert man rekursiv

$$f_{s+1}(x) = \sum_k f_s(V_k x)\Delta_k(x)$$

so gilt

$$|f_s(x) - a\varrho(x)| < b\sigma(s)$$

wo $a = \int_B f_0(x)\,dx$ und $b = b(f_0)$ eine Konstante ist.

Zunächst folgern wir:

(a) $f_s(x) = \sum f_0(V_{k_1\ldots k_s}x)\Delta_{k_1\ldots k_s}(x)$.

(b) $|f_s(x)-f_s(y)| \leqslant N_1\|x-y\|$ mit einem passenden $N_1 > N$, unabhängig von s. Der Beweis ist in [2] auf p. 7.

(c) Da $C^{-1} \leqslant \varrho(x) \leqslant C$ ist $c_1\varrho(x) \leqslant f_0(x) \leqslant c_2\varrho(x)$ und daher $0 < m_1 \leqslant f_s(x) \leqslant M_1$ gleichmäßig in s mit $m_1 \leqslant m \leqslant M \leqslant M_1$.

(d) $\int_B f_s(x)\,dx = \int_B f_0(x)\,dx = a$.

(e) Aus (c) folgt nun

$$g_0 f_s(x) < f_{s+t}(x) < G_0 f_s(x)$$

mit $0 < g_0 \leqslant G_0$ gleichmäßig in s und t.