et que les $d_k$ sont déterminées par le fait que l'on a pour tout $0 < R < \infty$

\[
\sum_{a \in \mathbb{Z}} d_k \exp\left( i \sum_{j=1}^{k} g_j(a) \right) = \prod_{p} \left( 1 - \frac{1}{p} \right) \left[ 1 + \sum_{r=1}^{R} \frac{g_r(p)}{p^r} \right].
\]

Comme dans le cas d'une seule fonction, cette égalité peut aussi être établie par la méthode utilisée ici.

Rouge par la Réduction le 14. 2. 1969

A statistical density theorem for L-functions
with applications
by
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§ 1. Introduction

1. In the last years many interesting results in the analytical theory of numbers have been obtained by the so-called "large sieve" method, e.g. new statistical density theorems for L-functions [2], [1] and the mean value theorem of Bombieri [2] (1.6 below) concerning the distribution of primes in arithmetical progressions.

We shall in this paper combine the large sieve with the method of Radoeski [9] and prove two statistical density results (Theorem 1) for L-functions. The estimate (1.4) is most effective for "high" rectangles and seems to be of a new type. As an arithmetical application of this we shall prove an analogue of Bombieri's theorem, concerning the primes in a "short" interval (Theorem 2). Finally we call attention to the consequences of Theorem 2 to some prime number problems.

2. Let $X \geq 1$, $T \geq 2$, $a \geq \frac{1}{2}$, and let $\chi$ be a character (mod $q$). Denote by $N(a, T, q, \chi)$ the number of zeros of the function $L(s, \chi)$ in the rectangle

\[
1 - \sigma \leq \sigma \leq 1, \quad |\tau| \leq T.
\]

In the statistical theory of L-functions the main problem is to find an estimate for the sum

\[
\sum_{\sigma \leq \chi} \sum_{\chi \equiv q, \chi} N(a, T, q, \chi)
\]

where the asterisk denotes summation over primitive characters only. Bombieri has in [2] proved that the sum (1.2) is

\[
\ll (X^{1 + \varepsilon} + XT)^{\frac{1}{2} - \varepsilon} T,
\]

(1.3)

We use the following notation: $c_1, c_2, \ldots$ denote positive absolute constants; $\varepsilon$ and $\Delta$ stand for positive constants, the former arbitrarily small and the latter arbitrarily large, which need not be always the same. Further, as usual, we write $c(\sigma) = e^{\sigma \varepsilon}$, $c_0(\sigma) = e^{\sigma \log q}$. 

\[
(1.4)
\]
and Barban [1] has given other estimates of about the same type. But all these are inconvenient for large values of \( T \). We shall eliminate the factor \( T \) in the first of the following estimates:

**Theorem 1.** For the sum (1.2) we have the estimates

\[
(1.4) \quad (X^2 T^{1 - \varepsilon})^{-1} \log a(X + T),
\]

\[
(1.5) \quad (X^2 T^{1 - \varepsilon})^{-1} T \log a(X + T).
\]

3. As an application of his density theorem, slightly stronger than (1.3), Bombieri proved a result concerning prime numbers which (in a little rough form) runs as follows:

\[
(1.6) \quad \sum_{\alpha \in \mathbb{C}} \max_{\sigma, \theta} \left| \psi(x, \alpha, a) - \frac{e^{\sigma x}}{\phi(q)} \right| = \ll x \log^{-A} x.
\]

To formulate our theorem, let \( c \) be a constant such that

\[
\zeta(1 + it, \sigma) = \ll e^{c t}
\]

for \( t \to \infty \) and \( 0 < \sigma < 1 \). It can be proved that \( c < \frac{1}{4} \), the best result heretofore obtained being \( c = \frac{1}{6} \) (see [6]).

**Theorem 2.** Let \( x \geq 2 \), \( y \geq 2 \), \( y = x^2 \) where \( 0 \) is a fixed number from the interval \( 0 < \theta < 1 \). Then

\[
(1.7) \quad \sum_{\alpha \in \mathbb{C}} \max_{\sigma, \theta} \left| \psi(x + \alpha, \alpha, a) - \frac{e^{\sigma x}}{\phi(q)} \right| = \ll y \log^{-A} x,
\]

where

\[
(1.8) \quad \beta = \frac{4c^2 + 2\theta - 1 - 4c}{6 + 4c} - \varepsilon.
\]

Recently Gallagher [5] has proved (1.6) without using zeros of \( L \)-functions. It seems to us that it is more difficult to prove Theorem 2 by some similar method.

4. Estimations of the type (1.6) and (1.7) are important e.g. in the application of Selberg's sieve method to prime number problems such as the twin-prime problem and Goldbach's problem. By the method [7] it can be deduced from (1.6) e.g. that there is an infinity of primes \( k \) such that \( k + 2 \) has at most 3 prime factors. One may ask how the primes of this kind are distributed. Now Theorem 2 offers a possibility for obtaining such results. It can be proved e.g.

**Theorem 3.** For every positive integer \( r \geq 8 \) there exists a real number \( \theta(r) \) with \( 0 < \theta(r) < 1 \) such that for \( x \) sufficiently large in any interval \( [x, x + x^{r\theta(r)}] \) there exists a pair \((p, p + 2)\) such that \( p + 2 \) has at most \( r \) prime factors. We have

\[
\theta(r) < \frac{1 + 4c}{2 + 4c} + \varepsilon
\]

if \( r \gg r(\varepsilon) \).

The question about \( \theta(r) \) remains open for \( r \leq 7 \).

**§ 2. Preliminary lemmas**

5. We state first a lemma which follows easily from the considerations carried out in [9].

**Lemma 1.** Let \( X > 1 \), \( y > 1 \), \( \frac{1}{2} < a < 1 \), \( T \geq 2 \),

\[
(2.1) \quad z^2 = c_2 y T X^{1 - \varepsilon} \log X,
\]

\[
(2.2) \quad a_n = \sum_{d \leq x} \mu(d).
\]

Let further \( \lambda = [\log z] + 1 \),

\[
(2.3) \quad I(r, M) = \sum_{n \leq X} \sum_{\alpha \in \mathbb{C}} \frac{1}{T} \int_{-T}^{T} \sum_{M \leq n < M + 1} a_n \log^r \gamma(n) n^{-s} n^{-s} \left| \frac{\sigma}{d \alpha} \right| d\sigma d\alpha.
\]

Then for the sum (1.2) we have the estimate

\[
(2.4) \quad \ll \log^2 z \max_{1 \leq \alpha \leq x} \max_{M \leq n < M + 1} \log^{-A} M I(r, M).
\]

We shall also need some facts about the divisor function \( \tau(n) \).

**Lemma 2.** We have \( \tau(n) \ll n^\varepsilon \); further,

\[
(2.5) \quad \sum_{n \leq x} \frac{\tau(n)}{n} \ll \log^2 z,
\]

\[
(2.6) \quad \sum_{n \leq x} \tau^2(n) \ll x \log^2 x,
\]

(2.7)

\[
\sum_{n \leq x} \tau(n) \ll x^\varepsilon \log x.
\]

Three first properties are well-known, and (2.7) can be proved by the method of [10].
6. The large sieve method we shall apply is in the form of

Lemma 3. Let \( d_n, n = H+1, \ldots, H+K \) be arbitrary complex numbers,
and let

\[
S(a) = \sum_{n=H+1}^{H+K} d_n \chi(n), \quad S'(a) = \sum_{n=H+1}^{H+K} d_n \chi(n), \quad Z = \sum_{n=H+1}^{H+K} |d_n|^2.
\]

Then

\[
\sum_{a \mod q} \sum_{n=1}^{\phi(q)} \left| S(a) \right|^2 \leq (X^2+K) Z, \quad \sum_{a \mod q} \sum_{n=1}^{\phi(q)} \left| S'(a) \right|^2 \leq (X^2+K) Z.
\]

For a simple proof, see [4].

\section*{3. Proof of Theorem 1}

7. We show first that (1.5) follows immediately from Lemmas 1, 2, and 3. For by (2.2) we have \( |\alpha| \leq \tau(n) \), and using Lemma 3 in (2.3) we find that

\[
I(\nu, M) \leq T(X^2+M) M^{-\nu} \log^M M \sum_{n=M}^{2M} \tau(n).
\]

Choosing \( y = X^2 \) and using (2.6), we establish (1.5) by Lemma 1.

8. Next we turn to the proof of (1.4). Choosing \( M \) in Lemma 1

\[
y = X^2 T, \quad z = (c_1 X^{1/2} T^2 \log X)^{1/2},
\]

we consider one particular \( I(\nu, M) \) with integral \( M \). Let first \( \sigma \) be a fixed number from the interval \( \alpha \leq \sigma \leq 1 \). Obviously

\[
J_2 \equiv \int_{-T}^{T} \left| \sum_{n=M}^{2M} a_n \log^2 \nu \chi(n) n^{-\nu-\nu^2} dt \right|^2 dt
= \sum_{n_1=M}^{2M} \sum_{n_2=M}^{2M} d_{n_1} d_{n_2} b_{n_1 n_2} \overline{\chi(n_1) \chi(n_2)},
\]

where

\[
b_{n_1 n_2} = \int_{-T}^{T} (n_1 / n_2)^\nu dt \equiv \left\{ \begin{array}{ll} T & \text{for } n_1 = n_2, \\ \min \left( T, \frac{M}{|n_1-n_2|} \right) & \text{for } n_1 \neq n_2, \end{array} \right.
\]

and assert the crucial

Let \( \tau(\nu) = \sum_{|a|=1} \chi(a) \nu \alpha(a) \) be a Gaussian sum. The first of the following identities is well-known, and the second is a consequence of it:

\[
\tau(\nu) \overline{\nu} (n_1) = \sum_{|a|=1} \chi(a_1) \nu \alpha(a_1 n_1), \quad \overline{\tau(\nu) \overline{\nu} (n_2)} = \sum_{|a|=1} \chi(a_2) \nu \alpha(-a_2 n_2).
\]

On multiplying these, multiplying the resulting identity by \( d_{n_1} d_{n_2} b_{n_1 n_2} \) and finally summing over \( n_1 \) and \( n_2 \), we obtain (3.2)

\[
|\tau(\nu)|^2 J_2 = \sum_{n_1} \sum_{n_2} d_{n_1} d_{n_2} b_{n_1 n_2} \sum_{n_1} \sum_{n_2} \chi(a_1) \chi(a_2) \nu \alpha(a_1 n_1 - a_2 n_2).
\]

Further, summing over all characters \( \nu \), and taking into account that \( J_2 \geq 0 \) and that for a primitive character \( |\tau(\nu)|^2 = q \), we get

\[
\sum_{n \mod q} \gamma^q (\nu) J_2 \leq \sum_{n_1} \sum_{n_2} d_{n_1} d_{n_2} b_{n_1 n_2} \sum_{n_1} \sum_{n_2} \alpha(a_1 n_1 - a_2 n_2).
\]

9. Now we set out to estimate the sum on the right of (3.5). Let

\[
Y = X^2, \quad M_1 = M - Y, \quad M_2 = M + Y, \quad M_3 = M + 2Y,
\]

and define the intervals

\[
H_v: v \leq n < v + Y, \quad v = M_1, M_1 + 1, \ldots, 2M.
\]

Lemma 4. Let \( n_1 \) and \( n_2 \) lie in the intervals \( H_n \), and \( H_n \), respectively. Then

\[
|b_{n_1 n_2}| \leq T Y A^{-1}.
\]

Proof. By (3.3) the lemma is trivial for \( |n_1 - n_2| \leq 2Y \), say. Let now

\[
|n_1 - n_2| \geq A > 2Y.\]

Then we have to prove that

\[
|b_{n_1 n_2}| \leq T Y A^{-1}.
\]

To see this, we remark that

\[
|b_{n_1 n_2}| \leq T Y A^{-1} = v_1 / v_2 - v_1 / v_2 - v_1 / v_2 = T Y M^{-1}.
\]

whence, by (3.5), the estimate (3.9) follows.

10. Next we consider the sums

\[
T_q = \sum_{M-q \leq n \leq M} \sum_{|n|=1} d_{n_1} d_{n_2} b_{n_1 n_2} \sum_{n_1 \neq n_2} \alpha(a(n_1 - n_2)),
\]

and assert the crucial
Lemma 5. We have
\[ \sum_{q \leq X} T_q \ll Y^2 M^{2(1-\varepsilon)} \log^{2+\varepsilon} M. \]

Proof. Using (3.8) in (3.10), we get first
\[ (3.11) \quad \sum_{q \leq X} T_q = \sum_{q \leq X} b_{n_1, n_2} \sum_{q \leq X} \sum_{(a, q) = 1} \sum_{n_1 \equiv a \pmod{q}} d_n q(a n_1) \sum_{n_2 \equiv -a \pmod{q}} d_n q(-a n_2) + R, \]
where
\[ (3.12) \quad R \ll M^{-2\varepsilon} \log^{2\varepsilon} M \sum_{M_1 < n_1, n_2 < M} \sum_{n_1 \equiv a \pmod{q}} \sum_{n_2 \equiv b \pmod{q}} \tau(n_1) \tau(n_2) \times \min \left( T, \frac{TY}{|n_1 - n_2|} \right) \sum_{q \leq X} |S_{q, n_1 - n_2}|, \]
and
\[ S_{q, n} = \sum_{(a, q) = 1} \epsilon_q(a n) \]
is a Ramanujan sum. It is well-known that
\[ S_{q, n} = \sum_{d | (q, n)} \mu \left( \frac{q}{d} \right) d \ll \sum_{d | (q, n)} d, \]
whence
\[ (3.13) \quad \sum_{q \leq X} |S_{q, n}| \ll \begin{cases} X^2 & \text{for } n = 0, \\ X \tau(n) & \text{for } n \neq 0. \end{cases} \]
To estimate the term \( R \), we subdivide first the pairs \((n_1, n_2)\) into groups such that \(n_1 - n_2\) is a constant \(\Delta\) for each group. Each pair occurs at most \(Y^2\) times, whence by (3.12) and (3.13) obviously
\[ \sum_{q \leq X} |S_{q, n}| \ll \begin{cases} Y^2 M & \text{for } n = 0, \\ Y M \tau(n) & \text{for } n \neq 0. \end{cases} \]
Now the expressions in the brackets are by Lemma 3 and (3.4) (note that \(Y = X^2\))
\[ \ll \sum_{n \equiv \mu(q)} |\tau(n)| \ll YM^{-2\varepsilon} \log^{2\varepsilon} M \sum_{n \equiv \mu(q)} \tau(n), \quad i = 1, 2. \]
Hence
\[ (3.15) \quad \sum_{q \leq X} T_q \ll Y M^{-2\varepsilon} \log^{2\varepsilon} M \sum_{M_1 < n_1, n_2 < M} |b_{n_1, n_2}| \times \left( \sum_{n_1 \equiv \mu(q)} \tau(n_1) \right)^{1/2} \left( \sum_{n_2 \equiv \mu(q)} \tau(n_2) \right)^{1/2} + R. \]
Writing \(n_2 = n_1 + \Delta\) and using (3.3), we see that (3.15) takes the form
\[ \sum_{q \leq X} T_q \ll Y M^{-2\varepsilon} \log^{2\varepsilon} M \sum_{\Delta = -M_2}^{M_2} \min(T, TY |\Delta|^{-1}) \sum_{n_1 \equiv \mu(q)} \tau(n_1) \times \left( \sum_{n_2 \equiv \mu(q)} \tau(n_2) \right)^{1/2} + R. \]
and the proof of Lemma 5 is complete.

11. We can now complete the proof of Theorem 1. We state first

Lemma 6. We have
\[ (3.16) \quad Y^3 \sum_{q \leq X} \sum_{z \equiv 0} \zeta_q \ll Y^3 M^{2(1-\varepsilon)} \log^{2+\varepsilon} M. \]

Proof. Comparing (3.5) and (3.10) we find that each pair \((n_1, n_2)\), occurring in (3.5), occurs exactly \(Y^3\) times in (3.10). In (3.10) there are also some further terms, corresponding to pairs \((n_1, n_2)\) with at least one of the numbers \(n_1, n_2\) not lying in the interval \([M, 2M]\). Let, for example, \(M - Y \ll n_1 < M\). We estimate the contribution to the sum
\[ \sum_{\sigma \in \mathbb{A}_X} T_\sigma \text{ of the pairs with } n_1 \text{ fixed and } n_2 \text{ running over the interval } [M_1^2, M_2^2], \text{ and get by (3.10) the estimate} \]
\[ Y^{2} |a_n| \sum_{\sigma \in \mathbb{A}_X} |b_{n_1, n_2}| \sum_{\sigma \in \mathbb{A}_X} |S_{n_1, n_2}|. \]

Writing \( n_2 = n_1 + \Delta \), we see that the above expression is by previous arguments
\[ Y^{2} \tau(n_1) M^{-2} \log^{2} X \left\{ X^{2} T \tau(n_1) + \sum_{d \neq \Delta} M \Delta^{-1} \tau(n_1 + \Delta) \tau(\Delta) \right\} \]
\[ \ll Y^{2} X^{2} T M^{-2} \tau(n_1) \log^{2} X + Y^{2} X M^{-2} \tau(n_1) \log^{k+4} X. \]

We separate two cases: \( T < X \) and \( T > X \). In the first case we get, summing over \( n_1 \), using (2.7), and noting that \( \tau(n_1) < X, M > TX, Y > X \), the estimate on the right of (3.16). The second case is clear. So Lemma 6 follows by the above remarks from Lemma 5.

Now by (3.16) and (3.1)
\[ I(\tau, M) = \sum_{\sigma \in \mathbb{A}_X} \sum_{\text{mod } \Delta} \left\{ J_{\sigma} d \sigma \ll M^{-2} \log^{k+4} X \right\} \]
\[ \ll (X^{2} X^{2})^{-\frac{1}{2}} \log^{k+4} (X + T), \]
and Lemma 1 completes the proof of (4.4).

12. We state a corollary of (1.4) which is useful in many problems.

**Lemma 7.** We have for \( T \ll X^{1/2}, X > X \), and for a constant \( a \) with \( 0 < a < 1 \)
\[ n \in \mathbb{A}_X \]
\[ \sum_{\text{mod } \Delta} \tau(n) = 0 \]
for all intervals \( a < q \ll 2 \), with all intervals \( q \ll X \), and with exception of \( Q \log^{A/4} X \) modules at most.

The proof proceeds in a well-known manner, using (1.4), Siegel's theorem and Satz 6.2, p. 295, of [8]. For the constant \( a \) may be taken e.g. \( \frac{1}{3} \), by [8]. Usually it is essential only that \( a < 1 \).

§ 4. Arithmetical applications

13. We prove a lemma from which Theorem 2 is an immediate consequence.

**Lemma 8.** For all modules \( q \) from any interval \( Q < q \ll 2Q \) with \( Q < X \), with exception of \( Q \log^{A/4} X \) modules at most, we have
\[ \max_{n \in \mathbb{N}} \left| \psi(q, x, q, a) - \frac{x}{q} \right| \ll \frac{y}{\psi(q)} \log^{-A/2} x. \]

Proof. For \( Q \ll \exp(\log^{1/2} x) \), say, the lemma is clear since then (4.1) holds for all modules, with a possible exception of the "exceptional" modules (see [8], p. 321).

Now let us suppose that \( Q > \exp(\log^{1/2} x) \). We start from the well-known estimation
\[ \left| \psi(q + x, q, a) - \psi(q, q, a) \right| \ll \frac{y}{\psi(q)} \log^{-A} x, \]
where \( y = \beta + y \) runs over the zeros of \( L(s, \chi) \) (see [8], p. 321). We exclude the same modules as in Lemma 7, and for the remaining ones (3.17) holds. For the non-excluded modules we use the density estimate
\[ N(q) = \sum_{\text{mod } \Delta} \tau(q, T, q, \chi) \ll (q^{2} T^{2} + T^{4} \log^{5} (q T)) \]
(see [8], p. 299). By this
\[ \sum_{n \in \mathbb{N}} a^{n-1} = \int \sum_{n \in \mathbb{N}} a^{n-1} dN(q, a) \ll \log^{-d} x, \]
if
\[ q^{3} T^{2} (T + q)^{2} \ll a^{1-s} \]
We choose
\[ T = y^{-1} q^{1+s} \]
so that \( T > q \). Then (4.5) is satisfied if \( q \ll a^{s} \), where \( \beta \) is given in (1.7). By (4.3), (4.4), and (4.6), this completes the proof.

14. Proof of Theorem 3. Let \( \{a_{n}\}, n = 1, \ldots, N \), be a sequence of positive integers, and denote by \( N \) the number of the \( a_{n} \)'s which are divisible by \( m \). Let
\[ N \left( f(m) \right) = \frac{N f(m)}{m} + R_{m}, \]
where \( f(m) \) is a multiplicative function. Let \( q \) be such that
\[ \sum_{m < N} \left| R_{m} \right| \ll N \log^{-4} N. \]
Let further \( \max_{n \leq N} a_{n} \ll N^{1/2} \), and define \( \tau = ty^{-1} \). Then, by [7], among the numbers \( a_{n} \) there are at least
\[ 0.05 \frac{N}{y \log N} \int_{1}^{y} \frac{1 - f(p)/p}{1 - 1/p} \]
numbers with at most \( k \) prime factors, where \( k \) depends on \( \tau \), and \( k \to \infty \) when \( \tau \to \infty \). E.g. \( k = 8 \) if \( \tau \leq 7.02 \).

We take now \( \{s_n\} = \{p+2\}, f(m) = m \eta^{-1}(m) \), where \( p \) runs over the primes in the interval \([x, x+y]\). Then \( N < y, \gamma = \theta^{-1}, \) and

\[
\frac{1}{\beta} = \frac{6+4c}{4c^2+2\beta-1-4c} + \epsilon.
\]

When \( \theta \to 1 \), then \( \tau \to 6+4c+\epsilon < 7.02 \). So \( \theta(8) < 1 \). The second assertion of Theorem 3 follows also immediately.

15. We remark finally that Theorem 2 is applicable to several other problems. E.g. it is possible to estimate the differences between "short" gaps between prime numbers, along the lines of the paper of Bombieri-Davenport [3]. Also it can be proved that every large even number is representable as a sum of two almost equal integers, one of which is a prime and the other has a finite number (\( \leq 8 \)) of prime factors.

References


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Reçu par la Rédaction le 16. 2. 1969

Metrische Theorie einer Klasse zahlentheoretischer Transformationen (Corrigendum)

von

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\( 0 < m \leq f_0(x) \leq M \text{ auf } B \) verlangt. Da die Eindeutigkeit aus bekannten Sätzen der Ergodentheorie ohnedies folgt, werden wir Teil (b) von p. 7 an, neu beweisen.

Wir zeigen also: Es sei \( f_0(x) \) gegeben mit

\[
f_0(x) = 0 < m \leq f_0(x) \leq M,
\]

so dass

\[
|f_0(x) - f_0(y)| \leq N_1|x-y|.
\]

Definiert man rekursiv

\[
f_{k+1}(x) = \sum_k f_k(V_k x) \Delta_k(x)
\]

so gilt

\[
|f_k(x) - a_k(x)| \leq b_k(x)
\]

wo

\[ a = \int f_0(x) \, dx \text{ und } b = b(f_0) \]

eine Konstante ist.

Zunächst folgen wir:

(a) \( f_0(x) = \sum f_k(V_k x) \Delta_k(x) \).

(b) \( |f_k(x) - f_k(y)| \leq N_1|x-y| \) mit einem passenden \( N_1 > N \), unabhängig von \( s \). Der Beweis ist in [2] auf p. 7.

(c) Da \( C^{-1} \leq \rho(x) \leq C \) ist \( 0 \leq f_0(x) < \rho_0(x) \) und daher \( 0 < m_1 \leq f_0(x) \leq M_1 \) gleichmäßig in \( s \) mit \( m_1 \leq m \leq M \leq M_1 \).

(d) \( \int f_k(x) \, dx = \int f_k(x) \, dx = a \).

(e) Aus (c) folgt nun

\[
g_k(x) < f_{k+1}(x) \leq g_k(x)
\]

mit \( 0 < g_0 \leq g_k \) gleichmäßig in \( s \) und \( t \).