Approximate functional equation
for Hecke’s $L$-functions of quadratic field

by

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Introduction

1. The aim of the present paper is to prove an approximate functional
equation for the Hecke’s $L$-functions $\xi(s, \chi)$ of any quadratic field $K$.
That equation being merely an auxiliary result we will confine our-
selves to proving it merely on the line $\sigma = \frac{1}{2}$ in the plane of complex
numbers $s = \sigma + it$. Having such a very limited purpose in proving the
result, we shall not give here a full account of the existing papers about
approximate functional equations in general, since none of them would
do just as well for the applications which we have in view.

In 1961, Linnik ([10], § 40) proved a shortened functional equation
for the Dirichlet $L$-function $L(s, \chi)$ with a primitive character $\chi \bmod D$
on the line $\sigma = \frac{1}{2} + it$ with $t \ll 1$ and $D$ unbounded. Using the incom-
plete $L$-function Larrivée [8] proved the analogous result for all $s$ in the
strip $0 < \sigma < 1$. He gave [9] also the corresponding result for Hecke’s$L$-
functions with Grössencharakter of imaginary quadratic field. But
if the functional equation contains a higher power of $L$-function than
the first one, his method does not give satisfactory results, since then
the corresponding residue sums do not represent familiar functions.

In the present paper we shall prove the following:

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(1) Which will be used in a later paper for the proof of a sieve theorem of Bomb-
lier’s type (see [1], Theorem 4) but for the rest of primes which are representable
by a given quadratic form.

(2) The result of Lavrière [9] (for example) concerns merely the imaginary quad-
artic field and the simplest case (out of three possible cases) in the real quadratic
field (see further §§ 5 and 6).

(3) With the restriction $\sigma = 1/2$, $t \ll 1$ Linnik’s method is applicable to Hecke’s
$L$-functions of any algebraic field. See further § 11.

(4) A short description of the method and results of the present paper has been
given in [4].
Theorem. Let
\[ \zeta(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} \quad (s > 1) \]
(where the sum is over all integer ideals \( a \neq 0 \)) be the Hecke's L-function of the quadratic field \( K \) (of the discriminant \( \Delta \)) with a primitive character \( \chi \mod \ell \). Let further \( s_0 = 1/2 + it_0, s_1 = 1/2 + it_1 \), \( t_1 < 1 \), \( \ell > 1 \), \( \ell > 1 \).

(2) \[ D = \frac{1}{\pi} \sqrt{\Delta |N|}, \quad X = \begin{cases} \ell^\delta (1+|t_0|)^{2\delta} & \text{if } N > 1, \\ \ell^\delta (1+|t_1|)^{2\delta} & \text{if } N = 1. \end{cases} \]

Then we have uniformly in \( D, t_0 \)
\[ \zeta(s_0, \chi) = \eta(z, s_0, \chi) + \sum_{N \leq 1} \frac{\chi(N)}{N^s} \eta \left( \frac{N}{D}, t_0 \right) \]
\[ + \sum_{N \leq 1} \frac{\overline{\chi}(N)}{N^{-s}} \eta \left( \frac{N}{D}, t_1 \right) + \sum_{N \leq 1} \frac{\chi(N)}{N^{s}} \eta \left( \frac{N}{D}, t_1 \right) \]
\[ + \sum_{N \leq 1} \frac{\overline{\chi}(N)}{N^{-s}} \eta \left( \frac{N}{D}, t_1 \right) + O(1) \]
where \( |\eta| = 1, \) and for \( j = 1, 2, 3, 4 \)
\[ \eta_j(x, t) \ll \begin{cases} 1 & \text{in any case,} \\ \frac{1}{x^{\frac{1}{2}}} & \text{if } x > 1 + \|t\|. \end{cases} \]

(3) It seems likely that (3) remains true after removing all the terms with \( a_1 \). However, we cannot prove such a simple equation by the method of the present paper, since giving \( a_1 \) up we would lose the convergence of some series and integrals during the proof. Let us remark that for the intended applications the terms of (3) with \( s = a_1 \) will not cause much inconvenience.

Corollary. The term \( \eta_i(s, \chi) \) in (3) can be eliminated after replacing \( \eta_3 \) and \( \eta_4 \) by other approximate functions \( \tilde{\eta}_3, \tilde{\eta}_4 \) (say) satisfying the estimates
\[ \tilde{\eta}_j(x, t) \ll \begin{cases} 1 & \text{in any case,} \\ \frac{1}{x^{\frac{1}{2}}} & \text{if } x > 1 + \|t\| \log^2 D \end{cases} \]
with \( l > 0 > 1, j = 3, 4. \)

The theorem will be proved by considering separately the following cases (i)-(iv). Writing
\[ A(f) = \frac{1}{\pi} \sqrt{\Delta |N|} \quad (\text{for } A > 0), \quad A(f) = \frac{1}{2\pi} \sqrt{\Delta |N|} \quad (\text{for } A < 0) \]

and
\[ \xi(s, \chi) = A(f)^2 G(s) \zeta(s, \chi), \]
where in the case of a real field
\[ (i) \quad G(s) = I^\delta \left( \frac{s}{2} \right), \]
\[ (ii) \quad G(s) = I^\delta \left( \frac{s+1}{2} \right) \]
or
\[ (iii) \quad G(s) = I^\delta \left( \frac{s}{2} \right) I^\delta \left( \frac{s+1}{2} \right) \]
(in dependence on the kind of the sign character), and in case of an imaginary field
\[ (iv) \quad G(s) = I^\delta(s), \]
we have (cf. [6], Satz LXI, LVII)
\[ \xi(s, \chi) = \xi(s, \chi) \xi(1-s, \chi), \quad \text{where } \xi(s, \chi) = 1. \]

Dealing with the cases (i)-(iii) we have by (2) and (4) \( A(f) = D \) and thus, by (5),
\[ \xi(s, \chi) = D^\delta G(s) \zeta(s, \chi), \quad D = \frac{1}{\pi} \sqrt{\Delta |N|}. \]

The approximate functional equation for the case (i)

2. Let \( s_0 = 1/2 + it_0, s_1 = 1/2 + it_1 (t_1 < 1) \) and let \( c_0 \) be a positive constant \( < 1/8 \). Write
\[ \int \frac{\xi(s, \chi) \left( \frac{1}{s-s_0} - \frac{1}{s-s_1} \right)}{(s-s_0)(s-s_1)} ds \]
\[ (1/2 - c_0 \ll \xi \ll 1/2 - c_0). \]

By \( F(s) \) denoting the integrand, we have by (7)
\[ F(s) = D^\delta \left( \frac{s}{2} \right) I^\delta \left( \frac{1-s}{2} \right) \zeta(s, \chi) \left( \frac{s_0-s_1}{s_0-s_1} \right) \]
Hence, for all large $|t| \geq 2(1 + |t_0| + |t_1|)$ we have by the asymptotic estimate

\begin{equation}
I'(\sigma + it) \approx 2\pi e^{-\eta t/2} \left(1 + |t|^{-1/2}\right) \left(1 + O\left(\frac{1}{1 + |t|}\right)\right) \quad (\sigma \ll 1)
\end{equation}

(cf. [11], Anhang, Satz 6.2) and by [3], Lemma 4,

\[ F'(\sigma + it) \ll D^2 |t|^{-1/2} D^{1-\varepsilon} (\log D |t|) D |t|^{-s} = D |t|^{-3/2} \log^2 D |t| \]

with the constant in the notation depending on $t_0, t_1$. This proves the absolute convergence of $I_x$. By moving the contour of integration from $\sigma = 1 - c_0$ to $\sigma = 1/2 - c_0$ we prove that

\[ I_{1-c_0} = \frac{\xi(s_0, \chi)}{\Gamma\left(\frac{1-s_0}{2}\right) \Gamma\left(\frac{s_0}{2}\right) \Gamma\left(\frac{1-s_1}{2}\right) \Gamma\left(\frac{s_1}{2}\right)} + I_{1/2-c_0}. \]

By the substitution $s = s_0 - z$ and using (8), (6), (7) we get

\begin{equation}
I_{1/2-c_0} = \frac{1}{2\pi i} \int_{1/2-c_0}^{1/2+c_0} \frac{\xi(s_0, \chi)}{\Gamma\left(\frac{1-s_0}{2}\right) \Gamma\left(\frac{s_0}{2}\right) \Gamma\left(\frac{1-s_1}{2}\right) \Gamma\left(\frac{s_1}{2}\right)} \left(1 + \frac{1}{s - s_0 - s_1}\right) ds
\end{equation}

Let us write

\begin{equation}
R_X(t) = \sum_{N \leq X} \chi(n) \frac{N}{n^{1/2-s_0+it}}\overline{\chi}(n)
\end{equation}

\[ \mathcal{S}_X = \frac{1}{2\pi i} \int_{1-c_0+it}^{1} D^{1-s_0+it} \left(\frac{1}{s-1} + \frac{s}{1+c_0-it}\right) \left(\frac{1}{2\pi} \int_{1/2+c_0}^{1/2+c_0} \frac{\xi(\sigma, \chi)}{\Gamma\left(\frac{1-\sigma}{2}\right) \Gamma\left(\frac{-\sigma}{2}\right) \Gamma\left(\frac{1-\sigma_1}{2}\right) \Gamma\left(\frac{-\sigma_1}{2}\right)} \left(1 + \frac{1}{s - s_0 - s_1}\right) ds\right) \end{equation}

\[ = -\epsilon(\chi) \Gamma', \]

If $N^d > 1$, then the primitive character $\chi \mod f$ is not the principal one and we have, by [3], pp. 298-299,

\[ \sum_{N \leq X} \chi(n) \ll D^2 N^{1/2}. \]

Using this estimate we prove by partial summation (cf. [11], Anhang, Satz 1.4), that

\begin{equation}
R_X(t) \ll D^2 (1 + |t|) \int_{\mathbb{R}} \frac{e^{it} \Gamma(1+it)}{e^{it} - e^{it_0+it}} ds \ll D^2 (1 + |t| + t_0) X^{-1}. \end{equation}
Let us split the integral (13) into parts

\[ \mathcal{J}_X = \frac{1}{2\pi} \int_{-\tau}^{\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{-\tau} \mathcal{J}_N + \mathcal{J}_D + \mathcal{J}_X, \]

say, where \( \tau = 2(1 + |t_1| + |t_0|). \) By (16), (13), (15), (9) and (2) we have

\[ \mathcal{J}_N \ll D^{1/2} T X^{-1/2} \int_{-\tau}^{\tau} \frac{(1 + |t_1|^{1+\epsilon})}{1 + |t_1|} dt \ll D^{1/2} T^{1-\delta_0} X^{-1} \log T \ll l^{-1/2}. \]

By (16), (13), (14) and (2)

\[ \mathcal{J}_D \ll X^{-1/4} D^{1/2} \int_{-\tau}^{\tau} \frac{|t_0 - t_1|}{(1 + t)(t + t_0 - t_1)} \frac{dt}{t + t_0} \ll X^{-1/4} D^{1/2} T^{1-\delta_0} \ll l^{-1/2}. \]

and, similarly,

\[ \mathcal{J}_X \ll l^{-1/2}. \]

The estimates for \( \mathcal{J}_D \) and \( \mathcal{J}_X \) hold as well in the case of \( N \delta = 1 \), whereas estimating \( \mathcal{J}_N \) we use (14) instead of (15) which does not hold anymore. Increasing \( X \) (cf. (2)) we get the same result:

\[ \mathcal{J}_D \ll X^{-1/4} D^{1/2} T \int_{-\tau}^{\tau} \frac{(1 + |t_0 + t_1|^{1-\epsilon})}{1 + |t_1|} \frac{dt}{t + t_0} \ll X^{-1/4} D^{1/2} T^{2-\delta_0} \log T \ll l^{-1/2}. \]

Hence by (16) we have in any case

\[ \mathcal{J}_X \ll l^{-1/2}. \]

By the same arguments we can prove the same estimate for the analogous integral \( \mathcal{J}_D \) (say) of the remainder term of \( \zeta(1-\delta_0+s, \chi) \) in \( \mathcal{I} \) (see (11)). Integrating the terms with \( N \alpha \ll X \) of the \( \zeta^1 \) Dirichlet-expansion (1) in \( \mathcal{I} \) and \( I_{1-\delta_0} \) we get finite sums whose terms are respectively

\[ \frac{1}{2\pi i} \int_{1-\delta_0}^{1-\delta_0 + \infty} x^{-s} \frac{\Gamma(1-\delta_0 + s)}{\Gamma(s - \delta_0 + s)} ds \]

(18)

\[ \frac{1}{2\pi i} \int_{1-\delta_0}^{1-\delta_0 + \infty} x^{-s} \frac{\Gamma(1-s + \delta_0)}{\Gamma(s - \delta_0 + s)} ds, \]

where

\[ x = \frac{N\alpha}{D} \gg \frac{1}{D}. \]

3. Now let us introduce the function

\[ H(s, \alpha) = \frac{1}{2\pi i} \int_{1-\delta_0}^{1-\delta_0 + \infty} x^{-s} \frac{\Gamma(1-\delta_0 + s)}{\Gamma(s - \delta_0 + s)} \frac{dx}{x^{1-\delta_0}} \]

(19)

the convergence of the integral will be established in § 7). Making the substitutions \( s - (\delta_0 - \epsilon) = \zeta, s - (\delta_0 - \epsilon) = \zeta \) (respectively) we prove that

\[ \frac{1}{2\pi i} \int_{1-\delta_0}^{1-\delta_0 + \infty} x^{-s} \frac{\Gamma(1-\delta_0 + s)}{\Gamma(s - \delta_0 + s)} \frac{dx}{x^{1-\delta_0}} \]

\[ = \frac{1}{2\pi i} \int_{1-\delta_0}^{1-\delta_0 + \infty} x^{-s} \frac{\Gamma(1-\alpha + s)}{\Gamma(s + s)} \frac{dx}{x^{1-\delta_0}} \]

\[ = \frac{1}{2\pi i} \int_{1-\delta_0}^{1-\delta_0 + \infty} x^{-s} \frac{\Gamma(1-\delta_0 + s)}{\Gamma(s - \delta_0 + s)} \frac{dx}{x} = x^{\alpha-\delta_0} H(1-\delta_0, \alpha), \]

Hence the integrals in (18) are \( H(1-\delta_0, \alpha) - x^{\alpha-\delta_0} H(1-\delta_0, \alpha) \), \( H(s_0, \alpha) - x^{s_0} H(s_0, \alpha) \), respectively, and by (12), (17)

\[ I_{1-\delta_0} = D^{1-\epsilon} \sum_{N \alpha \ll X} \frac{\chi(a)}{N\alpha} H(\frac{s_0}{D}, \frac{N\alpha}{D}) \frac{1}{(s_0 - s)} H(s_1, \frac{N\alpha}{D}) + O(l^{-1} D^{1/2}). \]

\[ = D^{1-\epsilon} \sum_{N \alpha \ll X} \frac{\chi(a)}{N\alpha} H(\frac{s_0}{D}, \frac{N\alpha}{D}) - D^{1-\epsilon} \sum_{N \alpha \ll X} \frac{\chi(a)}{N\alpha} H(s_1, \frac{N\alpha}{D}) + O(l^{-1} D^{1/2}). \]

Similarly, by (11),

\[ I' = D^{1-\epsilon} \sum_{N \alpha \ll X} \frac{\chi(a)}{N\alpha} H(1-s_0, \frac{N\alpha}{D}) - D^{1-\epsilon} \sum_{N \alpha \ll X} \frac{\chi(a)}{N\alpha} H(1-s_1, \frac{N\alpha}{D}) + O(l^{-1} D^{1/2}). \]
Substituting into (10) and (11), using (7) and dividing through by \( D^s \), we get the approximate functional equation

\[
F(s) - D^{s-\frac{1}{2}} F(s_1) \ll 1
\]

where

\[
F(s) = \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \zeta(s, \chi) - \sum_{N \in \mathbb{X}} \frac{\chi(a)}{Na^s} H\left(s, \frac{Na}{D}\right) - \varepsilon(\chi) D^{1-s} \sum_{N \in \mathbb{X}} \frac{\overline{\chi(a)}}{Na^{1-s}} H\left(1-s, \frac{Na}{D}\right)
\]

Note, that for \( s = s_3 = 1/2 + it_0 \) and \( s = s_1 = 1/2 + it_1 \) the factors before the sums and before \( \zeta(s, \chi) \) are in modulus \( = 1 \).

The case (ii)

4. In this case we have in (7) \( G(s) = \Gamma\left(\frac{s+1}{2}\right) \). Now instead of (3) we start with the integral

\[
I_s = \frac{1}{2\pi i} \int_{c - \infty}^{c + \infty} \xi(s, \chi) \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{2-s}{2}\right) \left(\frac{1}{s-s_0} - \frac{1}{s-s_1}\right) ds
\]

and go on as before. Introducing the function

\[
\tilde{H}(s, x) = \frac{\Gamma\left(\frac{1+s+\frac{s}{2}}{2}\right)}{\Gamma\left(\frac{2-s_0-s}{2}\right)} \frac{1}{s}
\]

we get the approximate functional equation (20) where in the present case

\[
F(s) = \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)} \xi(s, \chi) - \sum_{N \in \mathbb{X}} \frac{\chi(a)}{Na^s} \tilde{H}\left(s, \frac{Na}{D}\right) - \varepsilon(\chi) D^{1-s} \sum_{N \in \mathbb{X}} \frac{\overline{\chi(a)}}{Na^{1-s}} \tilde{H}\left(1-s, \frac{Na}{D}\right)
\]

The case (iii)

5. In this case we have in (7) \( G(s) = \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \). We start with the integral

\[
I_s = \frac{1}{2\pi i} \int_{c - \infty}^{c + \infty} \xi(s, \chi) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{2-s}{2}\right) \left(\frac{1}{s-s_0} - \frac{1}{s-s_1}\right) ds
\]

and go on as before. We get again the approximate functional equation (20) but with

\[
F(s) = \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)} \xi(s, \chi) - \sum_{N \in \mathbb{X}} \frac{\chi(a)}{Na^s} \tilde{H}\left(s, \frac{Na}{D}\right) - \varepsilon(\chi) D^{1-s} \sum_{N \in \mathbb{X}} \frac{\overline{\chi(a)}}{Na^{1-s}} \tilde{H}\left(1-s, \frac{Na}{D}\right)
\]

The case (iv)

6. In this case we have \( \Delta < 0 \) and we have by (2), (4), (5)

\[
\xi(s, \chi) = \left(\frac{1}{2} D\right)^s I(s) \xi(s, \chi).
\]

By the 'duplication formula' ([12], p. 57) we have

\[
I(s) = c_1 \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right), \quad c_1 = (2\sqrt{\pi})^{-1}
\]

and thus

\[
\xi(s, \chi) = c_1 D^s I\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \xi(s, \chi).
\]

Comparing with (7) we remark, that this is exactly the previous case (iii) (except for the constant \( c_1 \) which at the end disappears after dividing (20) through by \( c_1 \)). Therefore we have in this case the same approximate functional equation ((20), (24)).
Finally let us mention that one could deal with all the four cases (i)-(iv) simultaneously by introducing the functions
\[
\xi_j(s, \chi) = e^{Dj} \left[ \Gamma \left( \frac{s}{2} \right) \right]^{1-j} \left[ \Gamma \left( \frac{1+s-j}{2} \right) \right]^{j} \zeta(s, \chi)
\]
(where \(j = 0, 1, 2; c = 1\) if \(A > 0\) and \(c = 1/2\) if \(A < 0\)) and starting with the integral
\[
I = \frac{1}{2\pi i} \int_{C} e^{\lambda s} \frac{\xi_j(s, \chi)}{\left( \frac{s}{2} \right)^{1-j} \left( \frac{1+s-j}{2} \right)^{j}} \zeta(s, \chi) ds
\]
(where \([x]\) denotes the largest integer \(\leq x\). But then the reading would become rather unpleasant.

The functions \(H(s_0, z)\) and \(\hat{H}(s_0, z)\)

7. Let us first establish the convergence of the integral (19) for \(s > x_0\) with any positive \(x_0 < 1\). To this end we take some large fixed \(Y > 1 + 3|x_0| + x^3\) and take any \(T > Y\) and consider the part of (19)

\[
\mathcal{I}_{X,T} = \frac{1}{2\pi i} \int_{C} e^{\lambda s} \frac{\Gamma \left( \frac{s_0 + s}{2} \right)}{\left( \frac{1+s-j}{2} \right)^{s_0-s}} ds
\]

By the asymptotic expansion (cf. [5], Satz 159, 160)

\[
\Gamma(s) = b(s) e^{-4\pi|z|^2} \frac{1}{s} + \frac{1}{s} e^{4\pi s|z|^2} \left( 1 + O \left( \frac{1}{|t|} \right) \right) \quad \text{if} \quad |\sigma| < 1, \quad |t| > 1
\]

we have for \(t > 1\)

\[
\frac{\Gamma \left( \frac{s}{2} \right)}{\Gamma \left( \frac{1}{2} - \frac{s}{2} \right)} = b^{1-s} e^{4\pi s|z|^2} \left( 1 + O \left( \frac{1}{|t|} \right) \right).
\]

Since for \(s = 3/2 - a_0 + it\)

\[
\frac{\xi^{a_0-s}}{s-a_0} = \frac{\xi^{1+a_0-(t-\eta)}}{1-a_0 + i(t-\eta)} = \frac{\xi^{a_0+it}}{t} \left( 1 + O \left( \frac{1}{|t|} \right) \right),
\]

we have, by (25), (26),

\[
\mathcal{I}_{X,T} \ll x_0^{a_0-1} \int_{Y+T}^{T+0} \frac{e^{-\eta t}}{t-a_0} e^{4\pi |z|^2(1/t-\eta)} \left( 1 + O \left( \frac{1}{|t|} \right) \right) dt
\]

To estimate the integral

\[
I = \int_{Y+T}^{T+0} e^{-\eta t} e^{4\pi |z|^2(1/t-\eta)} dt,
\]

we use the following

**Lemma A.** If \(F(u)\) and \(G(u)\) are any real functions such that \(G/F\) is monotonic and \(F/G \geq m > 0\) (or \(\leq -m < 0\)) in \([a, b]\), then we have

\[
\left| \int_{a}^{b} G(u) e^{F(u)} du \right| \leq \frac{4}{m}
\]

(see [13], IV, § 1, Lemma 2). We deduce that \(I \ll Y^{-a_0}\) and hence, by (27), we have uniformly in \(T \geq Y\)

\[
\mathcal{I}_{X,T} \ll x_0^{a_0} Y^{-a_0}
\]

(since \(x_0^{-1} < x_0^{-2}\)). Writing

\[
\mathcal{I}_{X,T} = \frac{1}{2\pi i} \int_{C} e^{\lambda s} \frac{\Gamma \left( \frac{s_0 + s}{2} \right)}{\left( \frac{1+s-j}{2} \right)^{s_0-s}} ds
\]

for any fixed \(X > 1 + 3|x_0| + x^3\) and any \(T \geq X\), we can prove similarly that

\[
\mathcal{I}_{X,T} \ll x_0^{a_0} X^{-a_0}
\]
uniformly in $T \geq X$. Increasing $T$ to infinity we get from (19), (25), (28) and (29)

$$
\frac{1}{2\pi i} \int_{1-\varepsilon_0 - i \infty}^{1-\varepsilon_0 + i \infty} \frac{\Gamma \left( \frac{s_0 + s}{2} \right)}{\Gamma \left( \frac{1 - s_0 - s}{2} \right)} \frac{d \zeta}{s} \sim a_0^{-s} O(X^{-\varepsilon_0} + Y^{-\varepsilon_0}),
$$

by which the convergence of the integral (19) is ensured.

By the same method one can prove the convergence of the integral (22).

8. In this paragraph our aim is to prove the expansion

$$
H(s_0, a) = \frac{\Gamma \left( \frac{s_0}{2} \right)}{\Gamma \left( \frac{1 - s_0}{2} \right)} - 2 \sum_{n=0}^\infty \frac{(-1)^n}{n! \Gamma(n+1/2)} \frac{x^{2n+s_0}}{2n+s_0},
$$

for $s_0 = 1/2 + it_0$, $a > a_0 > 0$.

By the substitution $s + s_0 = s$ we get from (19), (31)

$$
H(s, a) = \frac{1}{2\pi i} \int_{1/2 + iE}^{1/2 + iE} \frac{\Gamma \left( \frac{s}{2} \right)}{\Gamma \left( \frac{1 - s}{2} \right)} \frac{d \zeta}{s - s_0} \sim a_0^{-s} O(X^{-\varepsilon_0} + Y^{-\varepsilon_0})
$$

for any $E > 2(1+e^s)(1+|a_0|)(1+e^{-2s_0})$.

Choosing any large natural number $m$ we replace the path of integration in (32) by the straight lines $L_1, L_2, L_3$, joining the points

$L_1), 3/2 - e_0 - iE, -2m - 1 - iE$;

$L_2) -2m - 1 - iE, -2m - 1 + iE$;

$L_3) -2m - 1 + iE, 3/2 - e_0 + iE$.

By the theorem on residues

$$
H(s, a) = \frac{\Gamma \left( \frac{s_0}{2} \right)}{\Gamma \left( \frac{1 - s_0}{2} \right)} - 2 \sum_{n=0}^\infty \frac{(-1)^n}{n! \Gamma(n+1/2)} \frac{x^{2n+s_0}}{2n+s_0} + \int_{L_1}^{L_3} \int_{L_2}^{+}.
$$

From the functional equation $\Gamma(s+1) = \pi(s) \Gamma(s)$ and the asymptotic estimates of $\Gamma(1/2 + it)$ and $\Gamma(1 - it)$ (see (9)) we deduce that

$$
\Gamma(-m - 1/2 + it/2) \ll \frac{1}{|m + 1/2 + it/2| |m - 1/2 + it/2| \ldots |1/2 + it/2| e^{-|m|t/4},
$$

$$
\Gamma(m + 1 - it/2) \ll \frac{1}{|m + it/2| |m - 1 + it/2| \ldots |1 + it/2| (1 + |t|)^{1/2} e^{-\pi|m|/2}.
$$

For a sufficiently large $m > 6(1 + x)$ the product of $x^{2m+1/2}$ with the factors (34) is in modulus less than $x^{ab}2^{-\frac{1}{2}}(1 + |t|)^{-t/2}$, whence the integral tends to zero (uniformly in $E \geq 2$), as $m \to \infty$.

Since we have on $L_1$ and $L_3$

$$
\frac{\Gamma \left( \frac{s}{2} \right)}{\Gamma \left( \frac{1 - s}{2} \right)} \frac{1}{s - s_0} \ll E^{-s_0}, \quad x^{s_0} \ll x^{2m+1/2} + e_0^{-1},
$$

and the path of integration $L_1 + L_3$ is of length $\ll m$, for a sufficiently large $E = E_0(x, m) \gg m$ we have

$$
\int_{L_1}^{L_3} \ll E^{-s} \ll m^{-e_0}.
$$

Now for every $m$ taking $E = E_0(x, m)$ and increasing $m$ to infinity from (33) we get the expansion (31).

By the same method one can get the expansion

$$
\tilde{H}(s, a) = \frac{\Gamma \left( \frac{1 + s_0}{2} \right)}{\Gamma \left( \frac{1 - s_0}{2} \right)} - 2 \sum_{n=0}^\infty \frac{(-1)^n}{n! \Gamma(n+1/2)} \frac{x^{2n+1+s_0}}{2n+1+s_0} + \int_{L_1}^{L_3} \int_{L_2}^{+}.
$$

for the function $\tilde{H}$ defined by (22).

9. In the present paragraph we shall prove the representations

$$
H(s, a) = \frac{1}{2\pi i} \int_{1/2}^{1/2} J_{-1/2}(2u) u^{a-1/2} du,
$$

$$
\tilde{H}(s, a) = \frac{1}{2\pi i} \int_{1/2}^{1/2} J_{1/2}(2u) u^{a-1/2} du,
$$

where $J_{\nu}(x)$ is the Bessel function of the first kind.
where \( x > s_0 > 0 \), \( s_0 = 1/2 + i t_0 \) and \( J_\nu(z) \) denote the Bessel function

\[
J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\nu}}{m! \Gamma(m+1+\nu)}
\]

(cf. [14], § 3.1).

Proof. By (31) and (38)

\[
H(s_0, z) = \frac{\Gamma\left(\frac{s_0}{2}\right)}{\Gamma\left(\frac{1-s_0}{2}\right)} - 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+1/2)} \cdot \frac{z^{2m+s_0}}{2m+s_0}
\]

\[
= \frac{\Gamma\left(\frac{s_0}{2}\right)}{\Gamma\left(\frac{1-s_0}{2}\right)} - 2 \int_0^z J_{-1/2}(2u) u^{s_0-1/2} du.
\]

In order to get (36) we have to prove that

\[
\frac{\Gamma\left(\frac{s_0}{2}\right)}{\Gamma\left(\frac{1-s_0}{2}\right)} = 2 \int_0^z J_{-1/2}(2u) u^{s_0-1/2} du,
\]

which is equivalent to the proof of the equality

\[
\frac{\Gamma\left(\frac{s_0}{2}\right)}{\Gamma\left(\frac{1-s_0}{2}\right)} = 2^{1/2-s_0} \int_0^z J_{1/2}(t) t^{s_0-1/2} dt.
\]

Using the identity (cf. [14], § 13.24)

\[
\int_0^z J_{1/2}(t) dt = \frac{\Gamma(\mu/2)}{\Gamma(\nu/2 + 1)} (\mu < \nu < \mu/2 + 1)
\]

with \( \nu = -1/2, \mu = s_0 \), we get (39), whence (36) follows. In the same manner one can prove (37).

10. In this paragraph we shall prove the following result:

For all \( x > x_0 > 0 \) we have uniformly in \( x, t_0 \)

\[
H(1/2 + it_0, x) \ll 1, \quad \tilde{H}(1/2 + it_0, x) \ll 1.
\]

And if \( x \geq \max(B, \frac{1}{2}|t_0|) \), where \( B \) stands for appropriate constant \( \geq 1 \), we have uniformly in \( t_0 \)

\[
H(1/2 + it_0, x) \ll x^{-1/2}, \quad \tilde{H}(1/2 + it_0, x) \ll x^{-1/2}.
\]

In the proof we shall need the asymptotic expansions for \( x \to \infty \)

\[
J_{-1/2}(z) = c_1 x^{-1/2} \cos x + O(x^{-2}),
\]

\[
J_{1/2}(z) = c_2 x^{-1/2} \sin x + O(x^{-2}), \quad c_1 = \sqrt{2}/\pi
\]

(which one can get from the results of [14], §§ 7.1, 7.3). Also we shall use Lemma A of § 7 and the analogons

**Lemma B.** If \( F(u) \) and \( G(u) \) are any real functions such that \( G/F \) is monotonic, \( F'(u) \geq r > 0 \) (or \( -r < 0 \)) and \( |G(u)| \leq M \) in \([a, b]\), then we have

\[
\left| \int_a^b G(u)e^{iu0} du \right| \leq \frac{4M}{V_r}.
\]

This is Lemma 4 of [13], IV, § 1.

We are now in a position to begin the proof of (40), (41) for the function \( H \). In the range \( x_0 < x < E \) we use the expansion (31) and get

\[
|H(1/2 + it_0, x)| < c(E) \ll 1.
\]

In what follows we may suppose \( x > E \).

By (36) and (42) we have

\[
H(1/2 + it_0, x) = 2^{1/2} \int_0^x u^{-1/2} (\cos 2u) u^{s_0-1/2} du + O(x^{-3/2})
\]

\[
= 2^{-1/2} \int_0^x u^{-1/2} e^{(2u-t_0)v} du + O(x^{-3/2})
\]

\[
= c_1 V_1 + c_2 V_2 + O(x^{-3/2}),
\]

say. We can estimate the integrals \( V_1 \) and \( V_2 \) by means of Lemma A.

In the notation of that lemma we have in the present case

\[
V'(u)/G(u) = u^{1/2}(\pm 2 + tu/u).
\]

If \( x > \frac{1}{2}|t_0| \), then also \( u > \frac{1}{2}|t_0| \) and we can take \( m = \frac{1}{2}x^{1/2} \). Hence by Lemma A we get (41) for the function \( H \).

It remains to prove that for \( E < x < \frac{1}{2}|t_0| \) we have

\[
V_1 \ll 1 \quad \text{and} \quad V_2 \ll 1
\]

(whence (40) would follow for the function \( H \)). This will be proved by considering different cases.
If \( t_0 > 0 \) and \( x > E \), then by Lemma A we have
\[
V_1 \ll \omega^{-1/2}.
\]
And if besides \( x < \frac{1}{2}t_0 \), then we split \( V_3 \) into the following parts:
\[
V_3 = \int_0^{t_0} \int_0^{t_0} \int_0^x \int_0^{t_0} \frac{d\nu}{\nu} \left( 1 + \frac{d\nu}{\nu} \right) d\nu + \int_0^{t_0} \int_0^{t_0} \frac{d\nu}{\nu} \left( 1 + \frac{d\nu}{\nu} \right) d\nu = V_{31} + V_{32} + V_{33},
\]
say. For the estimation of \( V_{31} \) we use Lemma A with \( m = \omega^{1/2} \), whence
\[
V_{31} \ll \omega^{-1/2}.
\]
For \( V_{32} \) we use the same lemma with \( m = \frac{1}{2} (\frac{1}{4} t_0)^{1/2} > \frac{1}{2} (2\omega)^{1/2} \), whence
\[
V_{32} \ll \omega^{-1/2}.
\]
And \( V_{33} \) can be estimated by means of Lemma B where we can take
\[
M = (\frac{1}{4} t_0)^{1/2}, r = \frac{1}{4} t_0^{-1}
\]
and get
\[
V_{33} \ll 1.
\]
Hence, by (46), (47), (45)
\[
V_3 \ll 1,
\]
and in view of (44) we have established (43) for the case \( t_0 > 0, E < x < \frac{1}{2}t_0 \).

If \( t_0 > 0, x > E \) and \( \frac{1}{2} t_0 < x < \frac{1}{4} t_0 \), then the required estimate for \( V_3 \) can be obtained in the same way, except that now the part \( V_{31} \) of \( V_3 \) in (45) disappears.

In the case of \( t_0 \leq 0, x > E \) we have, by Lemma A, \( V_1 \ll \omega^{-1/2} \).

If in particular \( t_0 = 0, x > E \), then we have also (by the same lemma) \( V_1 \ll \omega^{-1/2} \). If, however, \( t_0 < 0 \) and \( x > E \), then we split \( V_1 \) into parts and by the arguments used before we get the estimate \( V_1 \ll 1 \).

This proves the desired results for the function \( H \). And by the same method, using (35) and (37), we can prove (40) and (41) for the function \( \tilde{H} \).

Proof of the theorem

Theorem of § 1 follows evidently from (20), (21), (23), (24), § 6, (40) and (41). And the Corollary would clearly follow from an approximate functional equation of the following type:
\[
\zeta(s, \chi) = \sum_{\chi \leq x} \frac{\chi(a)}{N^a} \zeta_{\chi} + \gamma' \sum_{\chi \leq x} \frac{\chi(a)}{N^a} \zeta' + O(\epsilon^{-1}),
\]
where \( s_1 = 1 + \frac{1}{2} + t_1, t_1 \ll 1, X_1 = lD \log D (l \geq t_0 > 3), |\gamma'| = 1 \) with the factors \( c_\epsilon, c_\zeta \ll 1 \) depending on \( t_1, N^a \) and \( N^f \).

The equation (48) can be proved by Linnik's method ([10], § 49) arguing as follows.

Let \( e(\chi) \) be defined by (6) and let \( e^{-1} = \sqrt{e(\chi)} \). Then, by (6),
\[
e(\chi (1/2 + it; \chi)) = e(\chi (1/2 + it; \chi)),
\]
whence \( e(\chi (1/2 + it; \chi)) \) is a real number.

Let us write
\[
F(s) = \frac{1}{2\pi i} \int_{2 - \infty}^{2 + \infty} e(\chi(s, \chi)) \frac{ds}{s - s_1}.
\]

Moving the contour of integration to the line \( \Re = 1/2 \) (along which line \( \int \frac{ds}{2\pi i (s - s_1)} \) is purely imaginary), the point \( s = s_1 \) being excluded by a semi-circle with radius tending to zero, we get
\[
\Re F(s) = \frac{1}{2\pi i} \int_{2 - \infty}^{2 + \infty} e(\chi(s, \chi)) \frac{ds}{s - s_1}.
\]

By (49), (1) and since, by § 6,
\[
\xi(s, \chi) = \xi s^G(s, \chi), \quad G(s) = \left( F \left( \frac{s}{2} \right) \right)^{2} \left( F \left( \frac{s - 1/2}{2} \right) \right)^{2}
\]
we have
\[
F(s) = \frac{1}{2\pi i} \int_{2 - \infty}^{2 + \infty} e(\chi(s, \chi)) \frac{ds}{s - s_1}.
\]

For any \( y > 0 \) we have
\[
\frac{1}{2\pi i} \int_{2 - \infty}^{2 + \infty} e^{-y^{-2} G(s)} \frac{ds}{s - s_1} = \int_{2 - \infty}^{2 + \infty} e^{-y^{-2} G(s + s_1)} \frac{ds}{s} = y^{-2} \chi(s_1, y),
\]
say. Moving the line of integration to \( \Re = 0 \), we prove the estimate
\[
|\chi(s_1, y)| < e^{-E_{\text{log} y}} \left( \frac{E}{2} + \frac{3}{4} \right) \leq e^{-E} \left( \frac{E}{2} + \frac{3}{4} \right) \left( e^{-E/2} \right)^{2}
\]
for any fixed \( E > 1, e_1 \) being independent of \( E \).

If \( y > E^2 \), then we move the line of integration to \( \Re = E \) and, supposing \( E \) large enough, we get the estimate
\[
|\chi(s_1, y)| < e^{-E_{\text{log} y}} \left( \frac{E}{2} + \frac{3}{4} \right) \leq e^{-E} \left( \frac{E}{2} + \frac{3}{4} \right) \left( e^{-E/2} \right)^{2}
\]
(cf. [12], p. 57).
Kloosterman sums and finite field extensions*

by

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1. Introduction. Let \( q = p^f \), where \( p \) is prime and \( f \geq 1 \). Put \( F = GF(q) \), the finite field of order \( q \). For arbitrary \( \alpha \in F \) define

\[
t(\alpha) = a + a^p + \ldots + a^{p^{f-1}},
\]

so that \( \alpha \in GF(p) \). Put

\[
e(\alpha) = e^{\pi i d(\alpha)/p},
\]

We now define the Kloosterman sum for \( F \):

\[
S(a) = \sum_{\alpha \in F} e(\alpha a + \alpha^p),
\]

where \( \alpha^p = 1 \). It is easily seen that \( S(a) \) is real for all \( \alpha \in F \).

In addition to \( F \) we consider also the finite field \( F_n = GF(q^n) \) of order \( q^n \), where \( n \geq 1 \) and define the Kloosterman sum for \( F_n \). We denote this sum by \( S^{(n)}(a) \), where \( a \) is an arbitrary element of \( F_n \). Clearly \( S^{(n)}(a) = S(a) \).

It follows at once from the definition that

\[
S^{(n)}(0) = -1 \quad (n = 1, 2, 3, \ldots).
\]

We may accordingly assume that \( a \neq 0 \). For arbitrary \( \alpha \in F \) we investigate the relationship of \( S^{(n)}(a) \) to \( S(a) \). We shall show that

\[
\sum_{\alpha \in F_n} \frac{1}{q^n} S^{(n)}(a) = \log (1 + q^{-1} S(a) + q^{-2n}) \quad (s > 1).
\]

By means of (1.2) we can express \( S^{(n)}(a) \) explicitly in terms of \( S(a) \). In particular we show that

\[
S^{(n)}(a) = (-1)^{n-1} a^{1-n} \sum_{\beta \in F_n} (\beta(a))^{n-1} - 4q^{n-1}.
\]

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