ACTA ARITHMETICA
XVI (1969)

Approximate functional equation for Hecke's L-functions of quadratic field

by

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Introduction

1. The aim of the present paper is to prove an approximate functional equation for the Hecke's L-functions $\zeta(s,\chi)$ of any quadratic field K. That equation being merely an auxiliary result(¹) we will confine ourselves to proving it merely on the line $\sigma = \frac{1}{2}$ in the plane of complex numbers $s = \sigma + it$. Having such a very limited purpose in proving the result, we shall not give here a full account of the existing papers about approximate functional equations in general, since none of them would do just as well for the applications which we have in view(²).

In 1961 Linnik ([10], § 40) proved a shortened functional equation for the Dirichlet L-function $L(s,\chi)$ with a primitive character $\chi \bmod D$ on the line $\sigma = \frac{1}{2}+it$ with $t \ll 1$ and D unbounded(³). Using the incomplete Γ-function Lavrik [8] proved the analogous result for all s in the strip $0<\sigma<1$. He gave [9] also the corresponding result for Hecke's L-functions with Grössencharakter of imaginary quadratic field. But if the functional equation contains a higher power of Γ-function than the first one, his method does not give satisfactory results, since then the corresponding residue sums do not represent familiar functions.

In the present paper(⁴) we shall prove the following

(¹) Which will be used in a later paper for the proof of a sieve theorem of Bombieri's type (see [1], Theorem 4) but for the set of primes which are representable by a given quadratic form.

(²) The result of Lavrik [9] (for example) concerns merely the imaginary quadratic field and the simplest case (out of three possible cases) in the real quadratic field (see further §§ 5 and 6).

(³) With the restriction $\sigma = 1/2$, $t \ll 1$ Linnik's method is applicable to Hecke's L-functions of any algebraic field. See further § 11.

(⁴) A short description of the method and results of the present paper has been given in [4].

THEOREM. *Let*

(1) $$\zeta(s,\chi)=\sum_{\mathfrak{a}}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}\quad(\sigma>1)$$

(where the sum is over all integer ideals $\mathfrak{a}\neq 0$*) be the Hecke's L-function of the quadratic field* K *(of the discriminant* Δ*) with a primitive character* $\chi \bmod \mathfrak{f}$*. Let further* $s_0=1/2+it_0$, $s_1=1/2+it_1$ $(t_1\ll 1)$, $l\geqslant 1$,

(2) $$D=\frac{1}{\pi}\sqrt{|\Delta|N\mathfrak{f}},\quad X=\begin{cases}D^4(1+|t_0|)^4l^4 & \text{if}\quad N\mathfrak{f}>1,\\ D^4(1+|t_0|)^8l^4 & \text{if}\quad N\mathfrak{f}=1.\end{cases}$$

Then we have uniformly in D, t_0

(3) $$\zeta(s_0,\chi)=\eta\zeta(s_1,\chi)+\sum_{N\mathfrak{a}\leqslant X}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{s_0}}\eta_1\left(\frac{N\mathfrak{a}}{D},t_0\right)+$$
$$+\sum_{N\mathfrak{a}\leqslant X}\frac{\overline{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s_0}}\eta_2\left(\frac{N\mathfrak{a}}{D},t_0\right)+\sum_{N\mathfrak{a}\leqslant X}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{s_1}}\eta_3\left(\frac{N\mathfrak{a}}{D},t_1\right)+$$
$$+\sum_{N\mathfrak{a}\leqslant X}\frac{\overline{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s_1}}\eta_4\left(\frac{N\mathfrak{a}}{D},t_1\right)+O(l^{-1}),$$

where $|\eta|=1$, *and for* $j=1,2,3,4$

$$\eta_j(x,t)\ll\begin{cases}1 & \text{in any case},\\ x^{-1/2} & \text{if}\quad x>1+\frac{2}{3}|t|.\end{cases}$$

It seems likely that (3) remains true after removing all the terms with s_1. However, we cannot prove such a simple equation by the method of the present paper, since giving s_1 up we would lose the convergence of some series and integrals during the proof. Let us remark that for the intended applications the terms of (3) with $s=s_1$ will not cause much inconvenience.

COROLLARY. *The term* $\eta\zeta(s_1,\chi)$ *in* (3) *can be eliminated after replacing* η_3 *and* η_4 *by other appropriate functions* $\tilde{\eta}_3$, $\tilde{\eta}_4$ *(say) satisfying the estimates*

$$\tilde{\eta}_j(x,t_1)\ll\begin{cases}1 & \text{in any case},\\ x^{-1/2} & \text{if}\quad x\geqslant l(1+\frac{2}{3}|t_1|)\log^2 D\end{cases}$$

with $l\geqslant l_0>3$, $j=3,4$.

The theorem will be proved by considering separately the following cases (i)-(iv). Writing

(4) $$A(\mathfrak{f})=\frac{1}{\pi}\sqrt{\Delta N\mathfrak{f}}\quad(\text{for }\Delta>0),\quad A(\mathfrak{f})=\frac{1}{2\pi}\sqrt{|\Delta|N\mathfrak{f}}\quad(\text{for }\Delta<0)$$

and

(5) $$\xi(s,\chi)=A(\mathfrak{f})^sG(s)\zeta(s,\chi),$$

where in the case of a real field

(i) $$G(s)=\Gamma^2\left(\frac{s}{2}\right),$$

(ii) $$G(s)=\Gamma^2\left(\frac{s+1}{2}\right)$$

or

(iii) $$G(s)=\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$$

(in dependence on the kind of the sign character), and in case of an imaginary field

(iv) $$G(s)=\Gamma(s),$$

we have (cf. [6], Satz LXI, LVII)

(6) $$\xi(s,\chi)=\varepsilon(\chi)\xi(1-s,\overline{\chi}),\quad\text{where}\quad|\varepsilon(\chi)|=1.$$

Dealing with the cases (i)-(iii) we have by (2) and (4) $A(\mathfrak{f})=D$ and thus, by (5),

(7) $$\xi(s,\chi)=D^sG(s)\zeta(s,\chi),\quad D=\frac{1}{\pi}\sqrt{\Delta N\mathfrak{f}}.$$

The approximate functional equation for the case (i)

2. Let $s_0=1/2+it_0$, $s_1=1/2+it_1$ $(t_1\ll 1)$ and let c_0 be a positive constant $<1/8$. Write

(8) $$I_c=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\xi(s,\chi)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}\left(\frac{1}{s-s_0}-\frac{1}{s-s_1}\right)ds$$
$$(1/2-c_0\leqslant c\leqslant 1-c_0).$$

By $F(s)$ denoting the integrand, we have by (7)

$$F(s)=D^s\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s,\chi)\frac{s_0-s_1}{(s-s_0)(s-s_1)}.$$

Hence, for all large $|t| \geqslant 2(1+|t_0|+|t_1|)$ we have by the asymptotic estimate

$$(9) \qquad \Gamma(\sigma+it) = \sqrt{2\pi}\, e^{-|t|\pi/2}(1+|t|)^{\sigma-1/2}\left\{1+O\left(\frac{1}{1+|t|}\right)\right\} \qquad (\sigma \ll 1)$$

(cf. [11], Anhang, Satz 6.2) and by [2], Lemma 4,

$$F(\sigma+it) \ll D^{\sigma}|t|^{\sigma-1/2}D^{1-\sigma}(\log D|t|)^2|t|^{1-\sigma}|t|^{-2} = D|t|^{-3/2}\log^2 D|t|$$

with the constant in the notation depending on t_0, t_1. This proves the absolute convergence of I_c. By moving the contour of integration from $\sigma = 1-c_0$ to $\sigma = 1/2-c_0$ we prove that

$$(10) \qquad I_{1-c_0} = \frac{\xi(s_0,\chi)}{\Gamma\left(\frac{1-s_0}{2}\right)\Gamma\left(\frac{s_0}{2}\right)} - \frac{\xi(s_1,\chi)}{\Gamma\left(\frac{1-s_1}{2}\right)\Gamma\left(\frac{s_1}{2}\right)} + I_{1/2-c_0}.$$

By the substitution $s = s_0 - z$ and using (8), (6), (7) we get

$$(11) \qquad I_{1/2-c_0} = \frac{1}{2\pi i}\int_{1/2-c_0-i\infty}^{1/2-c_0+i\infty} \frac{\xi(s,\chi)}{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)}\left(\frac{1}{s-s_0}-\frac{1}{s-s_1}\right)ds$$

$$= \frac{1}{2\pi i}\int_{c_0+i\infty}^{c_0-i\infty} \frac{\xi(s_0-z,\chi)}{\Gamma\left(\frac{1-s_0+z}{2}\right)\Gamma\left(\frac{s_0-z}{2}\right)}\left(\frac{1}{z}-\frac{1}{z-(s_0-s_1)}\right)dz$$

$$= -\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty}$$

$$= -\frac{\varepsilon(\chi)}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} \frac{\xi(1-s_0+z,\bar{\chi})}{\Gamma\left(\frac{1-s_0+z}{2}\right)\Gamma\left(\frac{s_0-z}{2}\right)}\left(\frac{1}{z}-\frac{1}{z-(s_0-s_1)}\right)dz$$

$$= -\frac{\varepsilon(\chi)}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} D^{1-s_0+z}\frac{\Gamma\left(\frac{1-s_0+z}{2}\right)}{\Gamma\left(\frac{s_0-z}{2}\right)}\zeta(1-s_0+z,\bar{\chi})\left(\frac{1}{z}-\frac{1}{z-(s_0-s_1)}\right)dz$$

$$= -\frac{\varepsilon(\chi)}{2\pi i}\int_{1-c_0-i\infty}^{1-c_0+i\infty} D^{1-s_0+s}\frac{\Gamma\left(\frac{1-s_0+s}{2}\right)}{\Gamma\left(\frac{s_0-s}{2}\right)}\zeta(1-s_0+s,\bar{\chi})\left(\frac{1}{s}-\frac{1}{s-(s_0-s_1)}\right)ds$$

$$= -\varepsilon(\chi)I',$$

say. By (8) and (7)

$$(12) \qquad I_{1-c_0} = \frac{1}{2\pi i}\int_{1-c_0-i\infty}^{1-c_0+i\infty} \frac{\xi(s,\chi)}{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)}\left(\frac{1}{s-s_0}-\frac{1}{s-s_1}\right)ds$$

$$= \frac{1}{2\pi i}\int_{1/2-c_0-i\infty}^{1/2-c_0+i\infty} \frac{\xi(z+s_0,\chi)}{\Gamma\left(\frac{1-s_0-z}{2}\right)\Gamma\left(\frac{s_0+z}{2}\right)}\left(\frac{1}{z}-\frac{1}{z+s_0-s_1}\right)dz$$

$$= \frac{1}{2\pi i}\int_{1-c_0-i\infty}^{1-c_0+i\infty} D^{s+s_0}\frac{\Gamma\left(\frac{s+s_0}{2}\right)}{\Gamma\left(\frac{1-s_0-s}{2}\right)}\times$$

$$\times\left(\frac{1}{s}-\frac{1}{s-(s_1-s_0)}\right)\zeta(s+s_0,\chi)\,ds.$$

Let us write

$$(13) \qquad R_X(t) = \sum_{N\mathfrak{a}>X}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{3/2-c_0+i(t+t_0)}},$$

$$\mathscr{I}_X = \frac{1}{2\pi}\int_{-\infty}^{\infty} D^{1-c_0+s_0+it}\frac{\Gamma\left(\frac{1-c_0+s_0+it}{2}\right)}{\Gamma\left(\frac{1-s_0-1+c_0-it}{2}\right)}\times$$

$$\times\left(\frac{1}{1-c_0+it}-\frac{1}{1-c_0+i(t-t_1+t_0)}\right)R_X(t)\,dt,$$

X being defined by (2). Since in any quadratic field the number of ideals $\mathfrak{a}$ with the same norm $N\mathfrak{a} = n$ does not exceed the number $d(n)$ of positive divisors of n (cf. [7], Satz 882) and since $c_0 < 1/8$, we have

$$(14) \qquad R_X(t) \ll \sum_{n>X}\frac{d(n)}{n^{3/2-c_0}} \ll X^{-1/4}.$$

If $N\mathfrak{f} > 1$, then the primitive character $\chi \bmod \mathfrak{f}$ is not the principal one and we have, by [3], pp. 298-299,

$$\sum_{N\mathfrak{a}\leqslant x}\chi(\mathfrak{a}) \ll D^3 x^{1/3}.$$

Using this estimate we prove by partial summation (cf. [11], Anhang, Satz 1.4), that

$$(15) \qquad R_X(t) \ll D^3(1+|t+t_0|)\int_X^{\infty}\frac{x^{1/3}}{x^{3/2-c_0+1}}\,dx \ll D^3(1+|t+t_0|)X^{-1}.$$

Let us split the integral (13) into parts

$$(16)\qquad \mathcal{J}_X = \frac{1}{2\pi}\int\limits_{-T}^{T} + \frac{1}{2\pi}\int\limits_{T}^{\infty} + \frac{1}{2\pi}\int\limits_{-\infty}^{-T} = \mathcal{J}_X^{(1)}+\mathcal{J}_X^{(2)}+\mathcal{J}_X^{(3)},$$

say, where $T = 2(1+|t_1|+|t_0|)$. By (16), (13), (15), (9) and (2) we have

$$\mathcal{J}_X^{(1)} \ll D^{9/2}TX^{-1}\int\limits_{-T}^{T}\frac{(1+|t+t_0|)^{1-c_0}}{1+|t|}\,dt \ll D^{9/2}T^{2-c_0}X^{-1}\log T \ll l^{-4}D^{1/2}.$$

By (16), (13), (14) and (2)

$$\mathcal{J}_X^{(2)} \ll X^{-1/4}D^{3/2}\int\limits_{T}^{\infty}|t+t_0|^{1-c_0}\frac{|t_0-t_1|}{(1+t)(t+t_0-t_1)}\,dt$$
$$\ll X^{-1/4}D^{3/2}T^{1-c_0} \ll l^{-1}D^{1/2}$$

and, similarly,

$$\mathcal{J}_X^{(3)} \ll l^{-1}D^{1/2}.$$

The estimates for $\mathcal{J}_X^{(2)}$ and $\mathcal{J}_X^{(3)}$ hold as well in the case of $N\mathfrak{f} = 1$, whereas estimating $\mathcal{J}_X^{(1)}$ we use (14) instead of (15) which does not hold anymore. Increasing X (cf. (2)) we get the same result:

$$\mathcal{J}_X^{(1)} \ll X^{-1/4}D^{3/2}T\int\limits_{-T}^{T}\frac{(1+|t+t_0|)^{1-c_0}}{1+|t|}\,dt \ll X^{-1/4}D^{3/2}T^{2-c_0}\log T \ll l^{-1}D^{1/2}.$$

Hence by (16) we have in any case

$$(17)\qquad \mathcal{J}_X \ll l^{-1}D^{1/2}.$$

By the same arguments we can prove the same estimate for the analogous integral $\mathcal{J}_X'$ (say) of the remainder term of $\zeta(1-s_0+s,\bar{\chi})$ in I' (see (11)). Integrating the terms with $N\mathfrak{a} \leqslant X$ of the ζ Dirichlet-expansion (1) in I' and I_{1-c_0} we get finite sums whose terms are respectively

$$(18)\qquad \begin{aligned} &\frac{1}{2\pi i}\int\limits_{1-c_0-i\infty}^{1-c_0+i\infty} x^{-s}\frac{\Gamma\left(\frac{1-s_0+s}{2}\right)}{\Gamma\left(\frac{s_0-s}{2}\right)}\left(\frac{1}{s}-\frac{1}{s-(s_0-s_1)}\right)ds,\\ &\frac{1}{2\pi i}\int\limits_{1-c_0-i\infty}^{1-c_0+i\infty} x^{-s}\frac{\Gamma\left(\frac{s_0+s}{2}\right)}{\Gamma\left(\frac{1-s_0-s}{2}\right)}\left(\frac{1}{s}-\frac{1}{s-(s_1-s_0)}\right)ds, \end{aligned}$$

where

$$x = \frac{N\mathfrak{a}}{D} \geqslant \frac{1}{D}.$$

3. Now let us introduce the function

$$(19)\qquad H(s_0,x) = \frac{1}{2\pi i}\int\limits_{1-c_0-i\infty}^{1-c_0+i\infty} x^{-s}\frac{\Gamma\left(\frac{s_0+s}{2}\right)}{\Gamma\left(\frac{1-s_0-s}{2}\right)}\frac{ds}{s}\qquad (x>0,\ s_0 = 1/2+it_0)$$

(the convergence of the integral will be established in § 7). Making the substitutions $s-(s_0-s_1) = z$, $s-(s_1-s_0) = z$ (respectively) we prove that

$$\frac{1}{2\pi i}\int\limits_{1-c_0-i\infty}^{1-c_0+i\infty} x^{-s}\frac{\Gamma\left(\frac{1-s_0+s}{2}\right)}{\Gamma\left(\frac{s_0-s}{2}\right)}\frac{ds}{s-(s_0-s_1)}$$
$$= \frac{1}{2\pi i}\int\limits_{1-c_0-i\infty}^{1-c_0+i\infty} x^{-z+s_1-s_0}\frac{\Gamma\left(\frac{1-s_1+z}{2}\right)}{\Gamma\left(\frac{s_1-z}{2}\right)}\frac{dz}{z} = x^{s_1-s_0}H(1-s_1,x),$$

$$\frac{1}{2\pi i}\int\limits_{1-c_0-i\infty}^{1-c_0+i\infty} x^{-s}\frac{\Gamma\left(\frac{s_0+s}{2}\right)}{\Gamma\left(\frac{1-s_0-s}{2}\right)}\frac{ds}{s-(s_1-s_0)}$$
$$= \frac{1}{2\pi i}\int\limits_{1-c_0-i\infty}^{1-c_0+i\infty} x^{-z+s_0-s_1}\frac{\Gamma\left(\frac{s_1+z}{2}\right)}{\Gamma\left(\frac{1-s_1-z}{2}\right)}\frac{dz}{z} = x^{s_0-s_1}H(s_1,x).$$

Hence the integrals in (18) are $H(1-s_0,x)-x^{s_1-s_0}H(1-s_1,x)$, $H(s_0,x)-x^{s_0-s_1}H(s_1,x)$, respectively, and by (12), (17)

$$I_{1-c_0} = D^{s_0}\sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}\leqslant X}}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{s_0}}\left\{H\left(s_0,\frac{N\mathfrak{a}}{D}\right)-\left(\frac{N\mathfrak{a}}{D}\right)^{s_0-s_1}H\left(s_1,\frac{N\mathfrak{a}}{D}\right)\right\}+O(l^{-1}D^{1/2})$$
$$= D^{s_0}\sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}\leqslant X}}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{s_0}}H\left(s_0,\frac{N\mathfrak{a}}{D}\right)-D^{s_1}\sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}\leqslant X}}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{s_1}}H\left(s_1,\frac{N\mathfrak{a}}{D}\right)+O(l^{-1}D^{1/2}).$$

Similarly, by (11),

$$I' = D^{1-s_0}\sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}\leqslant X}}\frac{\bar{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s_0}}H\left(1-s_0,\frac{N\mathfrak{a}}{D}\right)-$$
$$-D^{1-s_1}\sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}\leqslant X}}\frac{\bar{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s_1}}-H\left(1-s_1,\frac{N\mathfrak{a}}{D}\right)+O(l^{-1}D^{1/2}).$$

Substituting into (10) and (11), using (7) and dividing through by D^{s_0}, we get the approximate functional equation

$$F(s_0)-D^{s_1-s_0}F(s_1) \ll l^{-1} \tag{20}$$

where

$$F(s) = \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s,\chi)-\sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}\leqslant X}}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}H\left(s,\frac{N\mathfrak{a}}{D}\right)- \tag{21}$$

$$-\varepsilon(\chi)D^{1-2s}\sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}\leqslant X}}\frac{\bar{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s}}H\left(1-s,\frac{N\mathfrak{a}}{D}\right).$$

Note, that for $s = s_0 = 1/2+it_0$ and $s = s_1 = 1/2+it_1$ the factors before the sums and before $\zeta(s,\chi)$ are in modulus $= 1$.

The case (ii)

4. In this case we have in (7) $G(s) = \Gamma^2\left(\frac{s+1}{2}\right)$. Now instead of (8) we start with the integral

$$I_c = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\xi(s,\chi)}{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{2-s}{2}\right)}\left(\frac{1}{s-s_0}-\frac{1}{s-s_1}\right)ds$$

and go on as before. Introducing the function

$$\tilde{H}(s_0,x) = \frac{1}{2\pi i}\int_{1-c_0-i\infty}^{1-c_0+i\infty}x^{-s}\frac{\Gamma\left(\frac{1+s_0+s}{2}\right)}{\Gamma\left(\frac{2-s_0-s}{2}\right)}\frac{ds}{s} \tag{22}$$

we get the approximate functional equation (20) where in the present case

$$F(s) = \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}\zeta(s,\chi)-\sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}\leqslant X}}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}\tilde{H}\left(s,\frac{N\mathfrak{a}}{D}\right)- \tag{23}$$

$$-\varepsilon(\chi)D^{1-2s}\sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}\leqslant X}}\frac{\bar{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s}}\tilde{H}\left(1-s,\frac{N\mathfrak{a}}{D}\right).$$

The case (iii)

5. In this case we have in (7) $G(s) = \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$. We start with the integral

$$I_c = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\xi(s,\chi)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{2-s}{2}\right)}\left(\frac{1}{s-s_0}-\frac{1}{s-s_1}\right)ds$$

and go on as before. We get again the approximate functional equation (20) but with

$$F(s) = \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}\zeta(s,\chi)-\sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}\leqslant X}}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}\tilde{H}\left(s,\frac{N\mathfrak{a}}{D}\right)- \tag{24}$$

$$-\varepsilon(\chi)D^{1-2s}\sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}\leqslant X}}\frac{\bar{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s}}H\left(1-s,\frac{N\mathfrak{a}}{D}\right).$$

The case (iv)

6. In this case we have $\Delta < 0$ and we have by (2), (4), (5)

$$\xi(s,\chi) = (\tfrac{1}{2}D)^s\Gamma(s)\zeta(s,\chi).$$

By the 'duplication formula' ([12], p. 57) we have

$$\Gamma(s) = c_1 2^s\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right),\qquad c_1 = (2\sqrt{\pi})^{-1},$$

and thus

$$\xi(s,\chi) = c_1D^s\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\zeta(s,\chi).$$

Comparing with (7) we remark, that this is exactly the previous case (iii) (except for the constant c_1 which at the end disappears after dividing (20) through by c_1). Therefore we have in this case the same approximate functional equation {(20), (24)}.

Finally let us mention that one could deal with all the four cases (i)-(iv) simultaneously by introducing the functions

$$\xi_j(s,\chi) = cD^s\left\{\Gamma\left(\frac{s}{2}\right)\right\}^{2-j}\left\{\Gamma\left(\frac{s+1}{2}\right)\right\}^{j}\zeta(s,\chi)$$

(where $j = 0, 1, 2$; $c = 1$ if $\Delta > 0$ and $c = 1/2\sqrt{\pi}$ for $\Delta < 0$) and starting with the integral

$$I_c = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\xi_j(s,\chi)}{\Gamma\left(\frac{s+[j/2]}{2}\right)\Gamma\left(\frac{1+j-[j/2]-s}{2}\right)}\left(\frac{1}{s-s_0}-\frac{1}{s-s_1}\right)ds$$

(where $[x]$ denotes the largest integer $\leqslant x$). But then the reading would become rather unpleasant.

The functions $H(s_0, x)$ and $\tilde{H}(s_0, x)$

7. Let us first establish the convergence of the integral (19) for $x \geqslant x_0$ with any positive $x_0 < 1$. To this end we take some large fixed $Y > 1+3|t_0|+x^2$ and take any $T > Y$ and consider the part of (19)

$$\mathscr{I}_{Y,T} = \frac{1}{2\pi i}\int_{1-c_0+iY}^{1-c_0+iT}\frac{\Gamma\left(\frac{s_0+s}{2}\right)}{\Gamma\left(\frac{1-s_0-s}{2}\right)}x^{-s}\frac{ds}{s} \tag{25}$$

$$= \frac{1}{2\pi i}\int_{3/2-c_0+i(Y+t_0)}^{3/2-c_0+i(T+t_0)}\frac{\Gamma\left(\frac{z}{2}\right)}{\Gamma\left(\frac{1-z}{2}\right)}\frac{x^{s_0-z}}{z-s_0}dz.$$

By the asymptotic expansion (cf. [5], Satz 159, 160)

$$\Gamma(s) = b(\sigma)e^{-|t|\pi/2}|t|^{\sigma-1/2}e^{it(\log|t|-1)}\left\{1+O\left(\frac{1}{|t|}\right)\right\}\qquad (\text{if } \sigma \ll 1,\ |t|>1)$$

we have for $t > 1$

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)} = bt^{\sigma-1/2}e^{it(\log t/2-1)}\left\{1+O\left(\frac{1}{t}\right)\right\}. \tag{26}$$

Since for $s = 3/2-c_0+it$

$$\frac{x^{s_0-s}}{s-s_0} = \frac{x^{-1+c_0-i(t-t_0)}}{1-c_0+i(t-t_0)} = \frac{x^{-1+c_0+it_0}}{i}\cdot\frac{x^{-it}}{t}\left\{1+O\left(\frac{1}{t}\right)\right\},$$

we have, by (25), (26),

$$\mathscr{I}_{Y,T} \ll x^{c_0-1}\int_{Y+t_0}^{T+t_0}\frac{x^{-it}}{t}t^{1-c_0}e^{it(\log t/2-1)}\left\{1+O\left(\frac{1}{t}\right)\right\}dt \tag{27}$$

$$\ll x^{c_0-1}\left\{\int_{Y+t_0}^{T+t_0}t^{-c_0}e^{it(\log t/2-1-\log x)}dt+\int_{Y+t_0}^{T+t_0}t^{-1-c_0}dt\right\}$$

$$= x^{c_0-1}\int_{Y+t_0}^{T+t_0}t^{-c_0}e^{it(\log t/2-1-\log x)}dt + x^{c_0-1}O(Y^{-c_0}).$$

To estimate the integral

$$I = \int_{Y+t_0}^{T+t_0}t^{-c_0}e^{it(\log t/2-1-\log x)}dt,$$

we use the following

LEMMA A. *If $F(u)$ and $G(u)$ are any real functions such that G/F' is monotonic and $F'/G \geqslant m > 0$ (or $\leqslant -m < 0$) in $[a, b]$, then we have*

$$\left|\int_a^b G(u)e^{iF(u)}du\right| \leqslant \frac{4}{m}$$

(see [13], IV, § 1, Lemma 2).

We deduce that $I \ll Y^{-c_0}$ and hence, by (27), we have uniformly in $T \geqslant Y$

$$\mathscr{I}_{Y,T} \ll x_0^{-1}Y^{-c_0} \tag{28}$$

(since $x^{c_0-1} < x_0^{-1}$). Writing

$$\mathscr{I}'_{X,T} = \frac{1}{2\pi i}\int_{1-c_0-iT}^{1-c_0-iX}\frac{\Gamma\left(\frac{s_0+s}{2}\right)}{\Gamma\left(\frac{1-s_0-s}{2}\right)}x^{-s}\frac{ds}{s}$$

for any fixed $X > 1+3|t_0|+x^2$ and any $T \geqslant X$, we can prove similarly that

$$\mathscr{I}'_{X,T} \ll x_0^{-1}X^{-c_0} \tag{29}$$

uniformly in $T \geqslant X$. Increasing T to infinity we get from (19), (25), (28) and (29)

$$(30)\quad \frac{1}{2\pi i}\int_{1-c_0-i\infty}^{1-c_0+i\infty} \frac{\Gamma\left(\frac{s_0+s}{2}\right)}{\Gamma\left(\frac{1-s_0-s}{2}\right)} x^{-s}\frac{ds}{s}$$

$$= \frac{1}{2\pi i}\int_{1-c_0-iX}^{1-c_0+iY} \frac{\Gamma\left(\frac{s_0+s}{2}\right)}{\Gamma\left(\frac{1-s_0-s}{2}\right)} x^{-s}\frac{ds}{s} + x_0^{-1}O(X^{-c_0}+Y^{-c_0}),$$

by which the convergence of the integral (19) is ensured.

By the same method one can prove the convergence of the integral (22).

8. In this paragraph our aim is to prove the expansion

$$(31)\qquad H(s_0, x) = \frac{\Gamma\left(\frac{s_0}{2}\right)}{\Gamma\left(\frac{1-s_0}{2}\right)} - 2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+1/2)}\,\frac{x^{2n+s_0}}{2n+s_0}$$

for $s_0 = 1/2+it_0$, $x \geqslant x_0 > 0$.

By the substitution $s+s_0 = z$ we get from (19), (31)

$$(32)\qquad H(s_0, x) = \frac{1}{2\pi i}\int_{3/2-c_0-iE}^{3/2-c_0+iE} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\,\frac{x^{-s+s_0}}{s-s_0}\,ds + O(E^{-c_0/2})$$

for any $E > 2(1+x^2)(1+|t_0|)(1+x_0^{-2/c_0})$.

Choosing any large natural number m we replace the path of integration in (32) by the straight lines L_1, L_2, L_3, joining the points

$$L_1)\ 3/2-c_0-iE,\ -2m-1-iE;$$

$$L_2)\ -2m-1-iE,\ -2m-1+iE;\qquad L_3)\ -2m-1+iE,\ 3/2-c_0+iE.$$

By the theorem on residues

$$(33)\quad H(s_0, x) = \frac{\Gamma\left(\frac{s_0}{2}\right)}{\Gamma\left(\frac{1-s_0}{2}\right)} - 2\sum_{0\leqslant n\leqslant m}\frac{(-1)^n}{n!\,\Gamma(n+1/2)}\,\frac{x^{2n+s_0}}{2n+s_0} + \int_{L_1,L_3} + \int_{L_2}.$$

From the functional equation $\Gamma(z+1) = z\Gamma(z)$ and the asymptotic estimates of $\Gamma(1/2+it)$ and $\Gamma(1-it)$ (see (9)) we deduce that

$$\Gamma(-m-1/2+it/2) \ll \frac{1}{|m+1/2+it/2|\cdot|m-1/2+it/2|\ldots|1/2+it/2|}\,e^{-|t|\pi/4},$$

(34)

$$\frac{1}{\Gamma(m+1-it/2)} \ll \frac{1}{|m+it/2|\cdot|m-1+it/2|\ldots|1+it/2|\cdot(1+|t|)^{1/2}e^{-|t|\pi/4}}.$$

For a sufficiently large $m > 6(1+x)$ the product of $x^{2m+1+1/2}$ with the factors (34) is in modulus less than $x^{3/2}2^{-m}(1+|t|)^{-5/2}$, whence the integral $\int_{L_2}$ tends to zero (uniformly in $E \geqslant 2$), as $m \to \infty$.

Since we have on L_1 and L_3

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\cdot\frac{1}{s-s_0} \ll E^{-c_0},\qquad x^{-s+s_0} \ll x^{2m+3/2}+x_0^{-1}$$

and the path of integration L_1+L_3 is of length $\ll m$, for a sufficiently large $E \geqslant E_0(x, m) \geqslant m$ we have

$$\int_{L_1,L_3} \ll E^{-c_0/2} \leqslant m^{-c_0/2}.$$

Now for every m taking $E = E_0(x, m)$ and increasing m to infinity from (33) we get the expansion (31).

By the same method one can get the expansion

$$(35)\qquad \tilde H(s_0, x) = \frac{\Gamma\left(\frac{1+s_0}{2}\right)}{\Gamma\left(\frac{2-s_0}{2}\right)} - 2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+1+1/2)}\cdot\frac{x^{2n+1+s_0}}{2n+1+s_0}$$

for the function $\tilde H$ defined by (22).

9. In the present paragraph we shall prove the representations

$$(36)\qquad H(s_0, x) = 2\int_x^{\infty} J_{-1/2}(2u)\,u^{s_0-1/2}\,du,$$

$$(37)\qquad \tilde H(s_0, x) = 2\int_x^{\infty} J_{1/2}(2u)\,u^{s_0-1/2}\,du,$$

where $x \geqslant x_0 > 0$, $s_0 = 1/2 + it_0$ and $J_\nu(z)$ denote the Bessel function

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\nu}}{m!\,\Gamma(m+1+\nu)} \tag{38}$$

(cf. [14], § 3.1).

Proof. By (31) and (38)

$$H(s_0, x) = \frac{\Gamma\left(\frac{s_0}{2}\right)}{\Gamma\left(\frac{1-s_0}{2}\right)} - 2\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+1/2)} \cdot \frac{x^{2m+s_0}}{2m+s_0}$$

$$= \frac{\Gamma\left(\frac{s_0}{2}\right)}{\Gamma\left(\frac{1-s_0}{2}\right)} - 2\int_0^x J_{-1/2}(2u)\, u^{s_0-1/2}\, du.$$

In order to get (36) we have to prove that

$$\frac{\Gamma\left(\frac{s_0}{2}\right)}{\Gamma\left(\frac{1-s_0}{2}\right)} = 2\int_0^\infty J_{-1/2}(2u)\, u^{s_0-1/2}\, du,$$

which is equivalent to the proof of the equality

$$\frac{\Gamma\left(\frac{s_0}{2}\right)}{\Gamma\left(\frac{1-s_0}{2}\right)} = 2^{1/2-s_0}\int_0^\infty J_{-1/2}(t)\, t^{s_0-1/2}\, dt. \tag{39}$$

Using the identity (cf. [14], § 13.24)

$$\int_0^\infty \frac{J_\nu(t)\,dt}{t^{\nu-\mu+1}} = \frac{\Gamma(\mu/2)}{2^{\nu-\mu+1}\Gamma(\nu-\mu/2+1)} \qquad (\text{if } \operatorname{re}\mu < \operatorname{re}\nu + 3/2)$$

with $\nu = -1/2$, $\mu = s_0$, we get (39), whence (36) follows. In the same manner one can prove (37).

10. In this paragraph we shall prove the following result:

For all $x \geqslant x_0 > 0$ we have uniformly in x_0, t_0

$$H(1/2+it_0, x) \ll 1, \qquad \tilde{H}(1/2+it_0, x) \ll 1. \tag{40}$$

And if $x \geqslant \max(E, \frac{2}{3}|t_0|)$, where E stands for appropriate constant $\geqslant 1$, we have uniformly in t_0

$$H(1/2+it_0, x) \ll x^{-1/2}, \qquad \tilde{H}(1/2+it_0, x) \ll x^{-1/2}. \tag{41}$$

In the proof we shall need the asymptotic expansions for $x \to \infty$

$$\begin{aligned} J_{-1/2}(x) &= c_1 x^{-1/2}\cos x + O(x^{-7/2}), \\ J_{1/2}(x) &= c_1 x^{-1/2}\sin x + O(x^{-7/2}), \qquad c_1 = \sqrt{2/\pi} \end{aligned} \tag{42}$$

(which one can get from the results of [14], §§ 7.1, 7.3). Also we shall use Lemma A of § 7 and the analogous

Lemma B. *If $F(u)$ and $G(u)$ are any real functions such that G/F' is monotonic, $F''(u) \geqslant r > 0$ (or $\leqslant -r < 0$) and $|G(u)| \leqslant M$ in $[a, b]$, then we have*

$$\left|\int_a^b G(u)\, e^{iF(u)}\, du\right| \leqslant \frac{8M}{\sqrt{r}}.$$

This is Lemma 4 of [13], IV, § 1.

We are now in a position to begin the proof of (40), (41) for the function H. In the range $x_0 \leqslant x \leqslant E$ we use the expansion (31) and get $|H(1/2+it_0, x)| < c(E) \ll 1$. In what follows we may suppose $x > E$.

By (36) and (42) we have

$$\begin{aligned} H(1/2+it_0, x) &= 2^{1/2}c_1\int_x^\infty u^{-1/2}(\cos 2u)\, u^{it_0}\, du + O(x^{-5/2}) \\ &= 2^{-1/2}c_1\int_x^\infty u^{-1/2} e^{i(2u+t_0\log u)}\, du + \\ &\quad + 2^{-1/2}c_1\int_x^\infty u^{-1/2} e^{i(-2u+t_0\log u)}\, du + O(x^{-5/2}) \\ &= c_2 V_1 + c_2 V_2 + O(x^{-5/2}), \end{aligned}$$

say. We can estimate the integrals V_1 and V_2 by means of Lemma A. In the notation of that lemma we have in the present case

$$F'(u)/G(u) = u^{1/2}(\pm 2 + t_0/u).$$

If $x \geqslant \frac{2}{3}|t_0|$, then also $u \geqslant \frac{2}{3}|t_0|$ and we can take $m = \frac{1}{2}x^{1/2}$. Hence by Lemma A we get (41) for the function H.

It remains to prove that for $E < x < \frac{2}{3}|t_0|$ we have

$$V_1 \ll 1 \quad \text{and} \quad V_2 \ll 1 \tag{43}$$

(whence (40) would follow for the function H). This will be proved by considering different cases.

If $t_0 > 0$ and $x > E$, then by Lemma A we have

$$V_1 \ll x^{-1/2}. \tag{44}$$

And if besides $x < \frac{1}{3}t_0$, then we split V_2 into the following parts:

$$V_2 = \int_x^{\frac13 t_0} u^{-1/2} e^{i(-2u+t_0\log u)}\,du + \int_{\frac13 t_0}^{\frac23 t_0} + \int_{\frac23 t_0}^{\infty} = V_{21}+V_{22}+V_{23}, \tag{45}$$

say. For the estimation of V_{21} we use Lemma A with $m = x^{1/2}$, whence

$$V_{21} \ll x^{-1/2}. \tag{46}$$

For V_{23} we use the same lemma with $m = \frac12(\frac23 t_0)^{1/2} > \frac12(2x)^{1/2}$, whence

$$V_{23} \ll x^{-1/2}. \tag{47}$$

And V_{22} can be estimated by means of Lemma B where we can take $M = (\frac13 t_0)^{-1/2}$, $r = \frac94 t_0^{-1}$ and get

$$V_{22} \ll 1.$$

Hence, by (46), (47), (45)

$$V_2 \ll 1,$$

and in view of (44) we have established (43) for the case $t_0 > 0$, $E < x < \frac13 t_0$.

If $t_0 > 0, x > E$ and $\frac13 t_0 \leqslant x < \frac23 t_0$, then the required estimate for V_2 can be obtained in the same way, except that now the part V_{21} of V_2 in (45) disappears.

In the case of $t_0 \leqslant 0, x > E$ we have, by Lemma A, $V_2 \ll x^{-1/2}$. If in particular $t_0 = 0, x > E$, then we have also (by the same lemma) $V_1 \ll x^{-1/2}$. If, however, $t_0 < 0$ and $x > E$, then we split V_1 into parts and by the arguments used before we get the estimate $V_1 \ll 1$.

This proves the desired results for the function H. And by the same method, using (35) and (37), we can prove (40) and (41) for the function $\tilde{H}$.

Proof of the theorem

11. The theorem of § 1 follows evidently from (20), (21), (23), (24), § 6, (40) and (41). And the Corollary would clearly follow from an approximate functional equation of the following type

$$\zeta(s_1,\chi) = \sum_{N\mathfrak{a}\leqslant X_1} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{s_1}} c_{\mathfrak{a}} + \eta' \sum_{N\mathfrak{a}\leqslant X_1} \frac{\bar{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s_1}} c'_{\mathfrak{a}} + O(l^{-1}), \tag{48}$$

where $s_1 = 1/2+it_1$, $t_1 \ll 1$, $X_1 = lD\log^2 D$ $(l \geqslant l_0 > 3)$, $|\eta'| = 1$ with the factors $c_{\mathfrak{a}} \ll 1$, $c'_{\mathfrak{a}} \ll 1$ depending on t_1, $N\mathfrak{a}$ and $N\mathfrak{f}$.

The equation (48) can be proved by Linnik's method ([10], § 40) arguing as follows.

Let $\varepsilon(\chi)$ be defined by (6) and let $\varepsilon_1^{-1} = \sqrt{\varepsilon(\chi)}$. Then, by (6), $\varepsilon_1\xi(1/2+it,\chi) = \overline{\varepsilon_1\xi(1/2+it,\chi)}$, whence $\varepsilon_1\xi(1/2+it,\chi)$ is a real number. Let us write

$$\mathscr{I}(s_1) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \varepsilon_1\xi(s,\chi)\frac{ds}{s-s_1}. \tag{49}$$

Moving the contour of integration to the line $\sigma = 1/2$ (along which line $ds/2\pi i(s-s_1)$ is purely imaginary), the point $s = s_1$ being excluded by a semi-circle with radius tending to zero, we get

$$\operatorname{re}\mathscr{I}(s_1) = \tfrac12\varepsilon_1\xi(s_1,\chi). \tag{50}$$

By (49), (1) and since, by § 6,

$$\xi(s,\chi) = cD^sG(s)\zeta(s,\chi), \quad G(s) = \left\{\Gamma\left(\frac{s}{2}\right)\right\}^{2-j}\left\{\Gamma\left(\frac{s+1}{2}\right)\right\}^{j} \quad (j = 0,1,2), \tag{51}$$

we have

$$\mathscr{I}(s_1) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \varepsilon_1 c\left(\sum_{\mathfrak{a}}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}\right)D^sG(s)\frac{ds}{s-s_1}. \tag{52}$$

For any $y > 0$ we have

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^{-s}G(s)\frac{ds}{s-s_1} = y^{-s_1}\int_{2-i\infty}^{2+i\infty} y^{-z}G(z+s_1)\frac{dz}{z} = y^{-s_1}\psi(s_1,y), \tag{53}$$

say. Moving the line of integration to $\operatorname{re}z = 0$, we prove the estimate

$$|\psi(s_1,y)| < c_1 \quad \text{if} \quad y \leqslant E^2 \tag{54}$$

for any fixed $E > 1$, c_1 being independent of E.

If $y \geqslant E^2$, then we move the line of integration to $\operatorname{re}z = E$ and, supposing E large enough, we get the estimate

$$|\psi(s_1,y)| < e^{-E\log y}\Gamma^2\left(\frac{E}{2}+\frac34\right) \leqslant e^{-E\log y}\left\{\left(\frac{E}{2}+\frac34\right)^{E/2+1/4}e^{-E/2}\right\}^2 \tag{55}$$

$$\leqslant e^{-E\log y+(E/2+1/4)\log(E/2+3/4)^2} < e^{-\frac12 E\log y}$$

(cf. [12], p. 57).

By (52) and (53)

$$\mathscr{I}(s_1) = \varepsilon_1 c \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \left(\frac{N\mathfrak{a}}{D}\right)^{-s_1} \psi\left(s_1, \frac{N\mathfrak{a}}{D}\right).$$

Hence, by (50),

$$(56) \quad \frac{1}{2}\varepsilon_1 \xi(s_1, \chi) = \mathrm{re}\,\mathscr{I}(s_1) = \frac{1}{2}\{\mathscr{I}(s_1) + \overline{\mathscr{I}(s_1)}\}$$
$$= \frac{\varepsilon_1 c}{2} D^{s_1} \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{s_1}} \psi\left(s_1, \frac{N\mathfrak{a}}{D}\right) + \frac{\bar{\varepsilon}_1 \bar{c}}{2} D^{1-s_1} \sum_{\mathfrak{a}} \frac{\bar{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s_1}} \overline{\psi\left(s_1, \frac{N\mathfrak{a}}{D}\right)}.$$

Now we use (51) and divide (56) through by $\frac{1}{2}c\varepsilon_1 D^{s_1} G(s_1)$. Using (55) (with $y = N\mathfrak{a}/D, E = y^{1/2}$) and considering that $1/G(s_1) \ll 1$ (since $t_1 \ll 1$) we can prove that the remainders of the infinite series with $N\mathfrak{a} > X_1$ are in modulus $\ll l^{-1}$. Thus we get (48) with

$$c_{\mathfrak{a}} = \psi\left(s_1, \frac{N\mathfrak{a}}{D}\right) \Big/ G(s_1), \qquad c'_{\mathfrak{a}} = \overline{\psi\left(s_1, \frac{N\mathfrak{a}}{D}\right)} \Big/ G(s_1)$$

which are $\ll 1$, by (54), (55).

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Reçu par la Rédaction le 7. 2. 1969

ACTA ARITHMETICA
XVI (1969)

Kloosterman sums and finite field extensions*

by

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1. Introduction. Let $q = p^f$, where p is prime and $f \geqslant 1$. Put $F = \mathrm{GF}(q)$, the finite field of order q. For arbitrary $a \in F$ define

$$t(a) = a + a^p + \ldots + a^{p^{f-1}},$$

so that $a \in \mathrm{GF}(p)$. Put

$$e(a) = e^{2\pi i t(a)/p}.$$

We now define the Kloosterman sum for F:

$$(1.1) \qquad S(a) = \sum_{\substack{x \in F \\ x \neq 0}} e(ax + x'),$$

where $xx' = 1$. It is easily seen that $S(a)$ is real for all $a \in F$.

In addition to F we consider also the finite field $F_n = \mathrm{GF}(q^n)$ of order q^n, where $n \geqslant 1$ and define the Kloosterman sum for F_n. We denote this sum by $S^{(n)}(a)$, where a is an arbitrary element of F_n. Clearly $S^{(1)}(a) = S(a)$.

It follows at once from the definition that

$$S^{(n)}(0) = -1 \qquad (n = 1, 2, 3, \ldots).$$

We may accordingly assume that $a \neq 0$. For arbitrary $a \in F$ we investigate the relationship of $S^{(n)}(a)$ to $S(a)$. We shall show that

$$(1.2) \qquad \sum_{n=1}^{\infty} \frac{1}{nq^{ns}} S^{(n)}(a) = \log\{1 + q^{-s}S(a) + q^{1-2s}\} \qquad (s > 1).$$

By means of (1.2) we can express $S^{(n)}(a)$ explicitly in terms of $S(a)$. In particular we show that

$$(1.3) \qquad S^{(n)}(a) = (-1)^{n-1} 2^{1-n} \sum_{2r \leqslant n} \binom{n}{2r} \big(S(a)\big)^{n-2r} \big\{\big(S(a)\big)^2 - 4q\big\}^r$$

* Supported in part by NSF grant GP-7855.