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Reducibility of lacunary polynomials I

by

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To the memory of my teachers
Wacław Sierpiński and Harold Davenport

§ 1. The present paper is in close connection with [9], the notation of that paper is used and extended (for a result which requires little notation see Corollary to Theorem 2). Reducibility means reducibility over the rational field Q. Constants are considered neither reducible nor irreducible. If $f(x_1, \ldots, x_k) \neq 0$ is a polynomial, then

$$f(x_1, \ldots, x_k) \overset{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} f_\sigma(x_1, \ldots, x_k)^{e_\sigma}$$

means that polynomials f_σ are irreducible and relatively prime in pairs.

If $\Phi(x_1, \ldots, x_k) = f(x_1, \ldots, x_k) \prod_{i=1}^{k} x_i^{a_i}$ where f is a polynomial, $\big(f(x_1, \ldots, x_k), x_1 \ldots x_k\big) = 1$ and a_i are integers then

$$J\Phi(x_1, \ldots, x_k) = f(x_1, \ldots, x_k)$$

(this definition is equivalent to one given in [9]). Let

$$J\Phi(x_1, \ldots, x_k) \overset{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} f_\sigma(x_1, \ldots, x_k)^{e_\sigma}.$$

We set

$$K\Phi(x_1, \ldots, x_k) = \text{const} \prod\nolimits_1 f_\sigma(x_1, \ldots, x_k)^{e_\sigma},$$
$$L\Phi(x_1, \ldots, x_k) = \text{const} \prod\nolimits_2 f_\sigma(x_1, \ldots, x_k)^{e_\sigma},$$

where Π_1 is extended over these f_σ which do not divide $J(x_1^{\delta_1} \ldots x_k^{\delta_k} - 1)$ for any $[\delta_1, \ldots, \delta_k] \neq 0$, Π_2 is extended over all f_σ such that

$$(*) \qquad Jf_\sigma(x_1^{-1}, \ldots, x_k^{-1}) \neq \pm f_\sigma(x_1, \ldots, x_k).$$

The leading coefficients of $K\Phi$ and $L\Phi$ are assumed equal to that of $J\Phi$. In particular for $k = 1$, $K\Phi(x)$ equals $J\Phi(x)$ deprived of all its cyclotomic factors and $L\Phi(x)$ equals $J\Phi(x)$ deprived of all its monic irreducible reciprocal factors (a polynomial $f(x)$ is reciprocal if $J(x^{-1}) = \pm f(x)$).

$J0 = K0 = L0 = 0$. Note that $(*)$ implies $Jf_\sigma(x_1^{-1}, \ldots, x_k^{-1}) \neq \text{const} \times \times f_\sigma(x_1, \ldots, x_k)$.

The operations J, K, L are distributive with respect to multiplication, besides for $k = 1$, J and K are commutative with the substitution $x \to x^n$ $(n \geqslant 0)$, L does not share this property and is always performed after the substitution. We have $KJ = JK = K$, $LJ = JL = L$, $LK = KL = L$; the first two formulae follow directly from the definitions, the last one requires a proof (see Lemma 11).

The paper has emerged from unsuccessful efforts to prove the conjecture formulated in [9] concerning the factorization of $KF(x^{n_1}, \ldots, x^{n_k})$ for given F. The operation L has turned out more treatable and the analogue of the conjecture for $LF(x^{n_1}, \ldots, x^{n_k})$ appears below as Lemma 12.

For a polynomial $F(x_1, \ldots, x_k)$ $\|F\|$ is the sum of squares of the absolute values of the coefficients of F; if $F \neq 0$, $|F|$ is the maximum of the degrees of F with respect to x_i $(1 \leqslant i \leqslant k)$,

$$|F|^* = \sqrt{\max\{|F|^2, 2\} + 2},$$

$\exp_1 x = \exp x$, $\exp_j x = \exp(\exp_{j-1} x)$.

From this point onwards all the polynomials considered have integral coefficients unless stated to the contrary. The highest common factor of two polynomials is defined only up to a constant; the formulae involving it should be suitably interpreted; we set $(0, 0) = 0$.

THEOREM 1. *For any polynomial $F \neq 0$ and any integer $n \neq 0$ there exist integers ν and u such that*

(i) $$0 \leqslant \nu \leqslant \exp(10\,|F| \log |F|^* \log \|F\|)^2,$$

(ii) $$n = u\nu,$$

(iii) $$KF(x^\nu) \overset{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} F_\sigma(x)^{e_\sigma} \quad \textit{implies} \quad KF(x^n) \overset{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} F_\sigma(x^u)^{e_\sigma}.$$

This is a quantitative formulation of Corollary to Theorem 1 [9] and a generalization of that theorem.

THEOREM 2. *For any polynomial $F(x_1, \ldots, x_k)$ and any integral vector $\boldsymbol{n} = [n_1, \ldots, n_k] \neq \boldsymbol{0}$ such that $F(x^{n_1}, \ldots, x^{n_k}) \neq 0$ there exist an integral matrix $\boldsymbol{N} = [\nu_{ij}]_{\substack{i \leqslant r \\ j \leqslant k}}$ of rank r and an integral vector $\boldsymbol{v} = [v_1, \ldots, v_r]$ such that*

(i) $$\max |\nu_{ij}| \leqslant c_r(F),$$

(ii) $$\boldsymbol{n} = \boldsymbol{v}\boldsymbol{N},$$

(iii) $$LF\Big(\prod_{i=1}^{r} y_i^{\nu_{i1}}, \ldots, \prod_{i=1}^{r} y_i^{\nu_{ik}}\Big) \overset{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} F_\sigma(y_1, \ldots, y_r)^{e_\sigma} \textit{ implies}$$
$$LF(x^{n_1}, \ldots, x^{n_k}) \overset{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} LF_\sigma(x^{v_1}, \ldots, x^{v_r})^{e_\sigma}.$$

Moreover

$$c_r(F) = \begin{cases} \exp 9k2^{\|F\|-5} & \textit{if} \quad r = k, \\ \exp(5 \cdot 2^{\|F\|^2-4} + 2\,\|F\| \log |F|^*) & \textit{if} \quad r + k = 3, \\ \exp_{(k-r)(k+r-3)}(8k\,|F|^{*\|F\|-1} \log \|F\|) & \textit{otherwise}. \end{cases}$$

COROLLARY. *For any polynomial $f(x) \neq 0$ the number of its irreducible non-reciprocal factors except x counted with their multiplicities does not exceed*

$$\exp_{\|f\|^2 - 5\|f\| + 7}(\|f\| + 2)$$

(a bound independent of $|f|$).

Theorem 2 is the main result of the paper. An essential role in the proof is played by a result of Straus [11]. It is an open question equivalent to the conjecture from [9] whether a similar theorem, possibly with greater constants $c_r(F)$, holds for the operation K instead of L.

The case $k = 1$ is settled by Theorem 1, for $k = 2$ a partial result is given by

THEOREM 3. *For any polynomial $F(x_1, x_2)$ such that $KF(x_1, x_2) = LF(x_1, x_2)$ and any integral vector $\boldsymbol{n} = [n_1, n_2] \neq \boldsymbol{0}$ such that $F(x^{n_1}, x^{n_2}) \neq 0$ there exist an integral matrix $\boldsymbol{N} = [\nu_{ij}]_{\substack{i \leqslant r \\ j \leqslant 2}}$ of rank r and an integral vector $\boldsymbol{v} = [v_1, v_r]$ such that*

(i) $$\max_{i,j} |\nu_{ij}| \leqslant \begin{cases} \exp 9 \cdot 2^{\|F\|-4} & \textit{if} \quad r = 2, \\ \exp\{500\,\|F\|^2 (2\,|F|^*)^{2\|F\|+1}\} & \textit{if} \quad r = 1, \end{cases}$$

(ii) $$\boldsymbol{n} = \boldsymbol{v}\boldsymbol{N},$$

(iii) $$KF\Big(\prod_{i=1}^{r} y_i^{\nu_{i1}}, \prod_{i=1}^{r} y_i^{\nu_{i2}}\Big) \overset{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} F_\sigma(y_1, y_r)^{e_\sigma} \textit{ implies}$$
$$KF(x^{n_1}, x^{n_2}) \overset{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} KF(x^{v_1}, x^{v_r})^{e_\sigma}.$$

This theorem is closely related to Theorem 2 of [9] but is both quantitative and more general, since it does not assume the irreducibility of F.

THEOREM 4. *If $k \geqslant 2$, $a_0 \neq 0$, $a_j \neq 0$ and n_j $(1 \leqslant j \leqslant k)$ are integers then either*

$$L\Big(a_0 + \sum_{j=1}^{k} a_j x^{n_j}\Big)$$

is irreducible or there is an integral vector $[\gamma_1, \ldots, \gamma_k]$ *such that*

$$0 < \max_j |\gamma_j| \leqslant \begin{cases} 2^{4\sum_{j=0}^{2} a_j^2+5} \log \sum_{j=0}^{2} a_j^2 & \text{if} \quad k = 2, \\ \exp_{2k-4}\left(k2^{\sum_{j=0}^{k} a_j^2+2} \log \sum_{j=0}^{k} a_j^2\right) & \text{if} \quad k > 2 \end{cases}$$

and

$$\sum_{j=1}^{k} \gamma_j n_j = 0.$$

THEOREM 5. *If* a, b, c, n, m *are integers,* $n > m > 0$, $abc \neq 0$ *then either* $K(ax^n + bx^m + c)$ *is irreducible or*

$$n/(n, m) \leqslant 2^{4(a^2+b^2+c^2)+5} \log(a^2+b^2+c^2)$$

and there exist integers ν *and* μ *such that* $m/\mu = n/\nu$ *is integral,*

$$0 < \mu < \nu \leqslant \exp(a^2+b^2+c^2)^2 2^{4(a^2+b^2+c^2)+11}$$

and

$$K(ax^\nu + bx^\mu + c) \stackrel{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} F_\sigma(x)^{e_\sigma}$$

implies

$$K(ax^n + bx^m + c) \stackrel{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} F_\sigma(x^{n/\nu})^{e_\sigma}.$$

This is a quantitative formulation of Theorem 3 of [9].

The proofs of Theorems 1, 2, 3, 4, 5 are given in §§ 2, 3, 4, 5, 5 respectively. Some of the proofs could be simplified at the cost of increasing the order of $c_r(F)$ and of other similar constants. Since however simplifications would not be great and the constants already are, I did as much as I could not to increase their order. On the other hand I have refrained from making generalizations to algebraic number fields. The method of proof of Theorem 1 works in any algebraic number field, while the method of proof of Theorems 2 and 3 works only in totally real fields and their totally complex quadratic extensions. The fields of these two types share the property that the trace of a square of the absolute value of any non-zero element is positive. In the case of totally complex fields, the definition of $L\Phi(x_1, \ldots, x_k)$ must be modified, namely condition (*) is to be replaced by

$$Jf_\sigma(x_1^{-1}, \ldots, x_k^{-1}) \neq \text{const}\,\overline{f_\sigma(x_1, \ldots, x_k)}.$$

(There is an error in this respect in [9], see Corrigenda at the end of the paper). A generalization to function fields over totally real fields is also possible.

The following notation is used through the paper in addition to that introduced already.

1. $|\boldsymbol{\Omega}|$ is the degree of a field $\boldsymbol{\Omega}$.
2. ζ_q is a primitive root of unity of degree q.
3. If $\boldsymbol{\Omega}$ is a field and $\alpha \in \boldsymbol{\Omega}$, $\alpha \neq 0$ then

$$e(\alpha, \boldsymbol{\Omega}) = \begin{cases} 0 \text{ if } \alpha = \zeta_q \text{ for some } q, \\ \text{maximal } e \text{ such that } \alpha = \zeta_q \beta^e \text{ with some } q \text{ and } \beta \in \boldsymbol{\Omega}, \text{ otherwise}. \end{cases}$$

4. $h(\boldsymbol{M})$ is the maximum of the absolute values of the elements of a matrix $\boldsymbol{M}$ (the height of $\boldsymbol{M}$).

$\boldsymbol{M}^T$ and $\boldsymbol{M}^\Delta$ are matrices transposed and adjoint to $\boldsymbol{M}$, respectively. The same notation applies to vectors treated as matrices with one row. The elements of a vector denoted by a bold face letter are designated by the same ordinary letter with indices. Bold face capital letters represent matrices except $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$ that are fields.

§ 2. LEMMA 1. *Let* $\boldsymbol{\Omega}$ *be an algebraic number field and* $\alpha \neq 0$ *an element of* $\boldsymbol{\Omega}$ *satisfying an equation* $f(\alpha) = 0$, *where* f *is a polynomial. Then*

$$e(\alpha, \boldsymbol{\Omega}) \leqslant \begin{cases} 20\,|\boldsymbol{\Omega}|^2 \log|\boldsymbol{\Omega}|^* \log\|f\| & \textit{always}, \\ \frac{5}{2}|\boldsymbol{\Omega}| \log\|f\| & \textit{if } \alpha \textit{ is not conjugate to } \alpha^{-1}, \\ (2\log 2)^{-1}|\boldsymbol{\Omega}| \log\|f\| & \textit{if } \alpha \textit{ is not an integer}. \end{cases} \tag{1}$$

Besides, for any algebraic number field $\boldsymbol{\Omega}_1 \supset \boldsymbol{\Omega}$

$$e(\alpha, \boldsymbol{\Omega}_1) \leqslant \frac{|\boldsymbol{\Omega}_1|}{|\boldsymbol{\Omega}|} e(\alpha, \boldsymbol{\Omega}). \tag{2}$$

Proof. If α is a root of unity, the lemma follows from the definition of $e(\alpha, \boldsymbol{\Omega})$. Assume that α is not a root of unity and let

$$\alpha = \zeta_q \beta^e, \quad \beta \in \boldsymbol{\Omega}, \ e = e(\alpha, \boldsymbol{\Omega}). \tag{3}$$

If α is an integer, β is also. It follows that

$$\log\overline{|\alpha|} = e\log\overline{|\beta|}, \tag{4}$$

where $\overline{|\alpha|}$ is the maximal absolute value of the conjugates of α. Now by a recent result of Blanksby and Mongomery [1] and by a slight refinement of a theorem of Cassels [3] (see p. 159 of the present paper)

$$\overline{|\beta|} \geqslant 1 + \begin{cases} (40|\Omega|^2\log|\Omega|^* - 1)^{-1}, \\ (5|\Omega|-1)^{-1} & \text{if } \alpha \text{ is not conjugate to } \alpha^{-1}. \end{cases}$$

Hence

(5) $$\frac{1}{\log\overline{|\beta|}} \leqslant \begin{cases}40|\Omega|^2\log|\Omega|^*,\\ 5|\Omega| & \text{if }\alpha\text{ is not conjugate to }\alpha^{-1}.\end{cases}$$

On the other hand $\overline{|\alpha|}$ does not exceed the maximal absolute value of the zeros of f and by the inequality of Carmichael-Masson (see [5], p. 125)

$$\overline{|\alpha|}\leqslant \|f\|^{\frac12},$$

hence

(6) $$\log\overline{|\alpha|}\leqslant\tfrac12\log\|f\|.$$

The first part of the lemma follows now from (4), (5) and (6). Assume that α is not an integer and let a_0 be the leading coefficient of f. Since $f(\alpha)=0$, $a_0\alpha$ is an integer. Therefore there exists a prime ideal $\mathfrak{p}$ of Ω such that

$$-\mathrm{ord}_{\mathfrak p} a_0 \leqslant \mathrm{ord}_{\mathfrak p}\alpha < 0.$$

It follows from (3) that

$$\mathrm{ord}_{\mathfrak p}\alpha = e\,\mathrm{ord}_{\mathfrak p}\beta$$

and

$$e\leqslant -\mathrm{ord}_{\mathfrak p}\alpha\leqslant \mathrm{ord}_{\mathfrak p}a_0.$$

On the other hand, taking norms N from Ω to Q we get

$$N(\mathfrak p)^{\mathrm{ord}_{\mathfrak p}a_0} \mid a_0^{|\Omega|},$$

whence

$$e\leqslant \mathrm{ord}_{\mathfrak p}a_0\leqslant|\Omega|\frac{\log|a_0|}{\log 2}\leqslant|\Omega|\frac{\log\|f\|}{2\log2}<\frac52|\Omega|\log\|f\|,$$

which proves (1).

In order to prove (2), assume that

$$\alpha=\zeta_r\beta_1^{e_1},\quad \beta_1\in\Omega_1,\ e_1=e(\alpha,\Omega_1)$$

and take norms N_1 from Ω_1 to Ω. We get

$$\alpha^d = N_1(\zeta_r)N_1(\beta_1)^{e_1};\quad e_1\leqslant e(\alpha^d,\Omega),$$

where $d=|\Omega_1|/|\Omega|$. Since by Lemma 1 of [9]

$$e(\alpha^d,\Omega)=de(\alpha,\Omega)$$

(2) follows.

LEMMA 2. *If $\Phi(x)$ is any irreducible polynomial not dividing $x^\delta - x$ ($\delta\neq1$), α is any of its zeros, $\Omega=Q(\alpha)$, n is an integer $\neq0$,*

$$\nu=\left(n,\,2^{e(\alpha,\Omega)-1}e(\alpha,\Omega)!\right),$$

then

$$\Phi(x^\nu)\overset{\text{can}}{=}\Phi_1(x)\dots\Phi_r(x)$$

implies

$$J\Phi(x^n)\overset{\text{can}}{=}J\Phi_1(x^{n/\nu})\dots J\Phi_r(x^{n/\nu}).$$

Proof for $n>0$ does not differ from the proof of Theorem 1 of [9]. The case $n<0$ can be reduced to the former in view of the identity $J\Phi(x^n)=\Psi(x^{-n})$, where $\Psi(x)=J\Phi(x^{-1})$.

Proof of Theorem 1. Let

$$KF(x)\overset{\text{can}}{=}\text{const}\prod_{i=1}^{\varrho}\Phi_i(x)^{e_i}.$$

For each Φ_i we denote by α_i,Ω_i,ν_i the relevant parameters from Lemma 2 and set

$$\nu=\left(n,\max_{1\leqslant i\leqslant\varrho}2^{e(\alpha_i,\Omega_i)-1}e(\alpha_i,\Omega_i)!\right),\qquad u=n\nu^{-1}.$$

We may assume that either $\|F\|\geqslant5$ or $|F|\geqslant3$, $\|F\|\geqslant3$ because otherwise $s=0$.

Since $2^{m-1}m!\leqslant m^m$ and $|\Omega_i|\leqslant|F|$ $(i=1,\dots,\varrho)$ we get by Lemma 1

$$\begin{aligned}\nu&\leqslant\exp\left(20|F|^2\log|F|^*\log\|F\|(\log20|F|^2+\log_2|F|^*+\log_2\|F\|)\right)\\&\leqslant\exp(10|F|\log|F|^*\log\|F\|)^2,\end{aligned}$$

which proves (i). (ii) is clear. In order to prove (iii) we notice that $2^{m_1-1}m_1!\mid 2^{m_2-1}m_2!$ for $m_1\leqslant m_2$, thus $\nu_i\mid\nu$ for $i\leqslant\varrho$. By Lemma 2

$$\Phi_i(x^{\nu_i})\overset{\text{can}}{=}\prod_{j=1}^{r_i}\Phi_{ij}(x)$$

implies

$$\Phi_i(x^\nu)\overset{\text{can}}{=}\prod_{j=1}^{r_i}\Phi_{ij}(x^{\nu/\nu_i}),$$

$$J\Phi_i(x^n)\overset{\text{can}}{=}\prod_{j=1}^{r_i}J\Phi_{ij}(x^{n/\nu_i}),$$

whence

$$KF(x^\nu)\overset{\text{can}}{=}\text{const}\prod_{i=1}^{\varrho}\prod_{j=1}^{r_i}\Phi_{ij}(x^{\nu/\nu_i})^{e_i},$$

$$KF(x^n)\overset{\text{can}}{=}\text{const}\prod_{i=1}^{\varrho}\prod_{j=1}^{r_i}J\Phi_{ij}(x^{n/\nu_i})^{e_i}.$$

Denoting the polynomials $\Phi_{ij}(x^{v/v_i})$ $(1\leqslant i\leqslant \varrho, 1\leqslant j\leqslant r_i)$ by $F_1,\ldots,F_s$ we obtain (iii).

§ 3. Lemma 3. *Let $P(x_1,\ldots,x_{k+1})\neq 0$, $Q(x_1,\ldots,x_{k+1})\neq 0$ be polynomials with complex coefficients, $(P,Q)=G$ and $P=GT$, $Q=GU$. The resultant of T, U with respect to x_i divides a certain nonvanishing minor of Sylvester's matrix $\boldsymbol{R}$ of P,Q formed with respect to x_i ($|\boldsymbol{R}|$ being the resultant of P,Q).*

Proof. Consider polynomials $A(x), B(x), C(x)$ of degrees $|A|>0$, $|B|>0$, $|C|$ with indeterminate coefficients $a_0,\ldots,b_0,\ldots,c_0,\ldots$, the resultant D of A, B and any minor S of degree $|A|+|B|+|C|$ of Sylvester's matrix $\boldsymbol{R}$ of AC, BC. Since D is absolutely irreducible and prime to a_0b_0 (see [6], Satz 120), we have either $S=DV$, where V is a polynomial in the coefficients of A, B, C or there exist complex values of the coefficients such that $D=0$ and $a_0b_0c_0S\neq 0$ (cf. [6], Satz 136). $A(x)$ and $B(x)$ with these coefficients have a common factor of positive degree, hence AC and BC have a common factor of degree $>|C|$ and by a well known theorem ([6], Satz 114) the rank of $\boldsymbol{R}$ is less than $|A|+|B|+|C|$. The contradiction obtained with $S\neq 0$ proves that

$$S=DV \tag{7}$$

for any minor S of degree $|A|+|B|+|C|$ of $\boldsymbol{R}$.

Now, if neither T nor U is constant with respect to x_i we set $A(x_i)=T(x_1,\ldots,x_{k+1})$, $B(x_i)=U(x_1,\ldots,x_{k+1})$, $C(x_i)=G(x_1,\ldots,x_{k+1})$.

Since $(AC,BC)=C$, it follows from the quoted theorem that at least one of the minors of degree $|A|+|B|+|C|$ of $\boldsymbol{R}$ does not vanish. By (7) this minor has the property asserted in the lemma.

If T, say, is constant with respect to x_i and the relevant degree of U is u, the diagonal minor S of degree u has the said property (if $u=0$ we take $S=1$).

Lemma 4. *Let $T(x_1,x_2)$, $U(x_1,x_2)$ be polynomials with complex coefficients, $(T,U)=1$. The number of pairs $\langle\eta,\vartheta\rangle$ such that $T(\eta,\vartheta)=U(\eta,\vartheta)=0$ does not exceed the degree of the resultant of T, U with respect to x_i $(i=1,2)$.*

Remark. The lemma must be notorious but it is not readily found in the literature.

Proof. It suffices to consider $i=2$. Let t, u be the degrees of T, U with respect to x_2 and for a given η let t_η, u_η be the degrees of $T(\eta,x_2)$, $U(\eta,x_2)$. Let $\boldsymbol{R}(x_1)$ be Sylvester's matrix of T, U formed with respect to x_2, $R(x_1)$ its determinant and $\boldsymbol{R}_\eta$ Sylvester's matrix of $T(\eta,x_2)$, $U(\eta,x_2)$.

If $t_\eta=t$, $u_\eta=u$ then $\boldsymbol{R}_\eta=\boldsymbol{R}(\eta)$, otherwise $\boldsymbol{R}_\eta$ can be obtained from $\boldsymbol{R}(\eta)$ by crossing out step by step row i, column i $(1\leqslant i\leqslant u-u_\eta)$, row $u+i$, column i $(u-u_\eta<i\leqslant(u-u_\eta)+(t-t_\eta))$. At each step all non-zero elements crossed out are in a row, thus the rank diminishes by at most one. We get

$$\text{rank of } \boldsymbol{R}_\eta\geqslant \text{rank of } \boldsymbol{R}(\eta)-(t-t_\eta)-(u-u_\eta).$$

Now if there are k_η different ϑ such that $T(\eta,\vartheta)=U(\eta,\vartheta)=0$, $T(\eta,x_2)$, $U(\eta,x_2)$ have a common factor of degree at least k_η, thus ([6], Satz 114)

$$\text{rank of } \boldsymbol{R}_\eta\leqslant t_\eta+u_\eta-k_\eta.$$

It follows that the rank of $\boldsymbol{R}(\eta)$ does not exceed $t+u-k_\eta$, whence by differentiation

$$(x_1-\eta)^{k_\eta}\,|\,R(x_1).$$

Giving η all the possible values, we obtain

$$\sum k_\eta\leqslant|R|,\qquad \text{q.e.d.}$$

Lemma 5. *Let $P(x_1,\ldots,x_{k+1})\neq 0$, $Q(x_1,\ldots,x_{k+1})\neq 0$ be polynomials and $S\neq 0$ a minor of their Sylvester's matrix formed with respect to x_i $(1\leqslant i\leqslant k+1)$. The following inequalities hold*

$$|S|\leqslant 2\,|P|\,|Q|,$$

$$\|S\|\leqslant\|P\|^{2|Q|}\|Q\|^{2|P|}.$$

Proof. We assume without loss of generality $i=k+1$ and set

$$P=\sum_{i=0}^{m}P_i(x_1,\ldots,x_k)x_{k+1}^{m-i},\qquad Q=\sum_{j=0}^{n}Q_j(x_1,\ldots,x_k)x_{k+1}^{n-j}.$$

Since $m\leqslant|P|$, $n\leqslant|Q|$ and Sylvester's matrix of P, Q is

$$\begin{bmatrix} P_0 & P_1\ldots P_m & & \\ \cdots & \cdots & \cdots & \\ & & P_0 & P_1\ldots P_m \\ Q_0 & Q_1\ldots Q_n & & \\ \cdots & \cdots & \cdots & \\ & & Q_0 & Q_1\ldots Q_n \end{bmatrix}\begin{matrix}\left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} n \text{ times}\\ \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} m \text{ times}\end{matrix}$$

it follows that

$$|S|\leqslant n\max|P_i|+m\max|Q_j|\leqslant 2\,|P|\,|Q|.$$

In order to estimate $\|S\|$ we note that

$$\|S\|=(2\pi)^{-k}\int_0^{2\pi}\ldots\int_0^{2\pi}|S(e^{i\varphi_1},\ldots,e^{i\varphi_k})|^2\,d\varphi_1\,d\varphi_2\ldots d\varphi_k$$

(cf. [2], Lemma 6 of Chapter VIII), hence

$$\|S\| \leqslant \max_{0\leqslant\varphi\leqslant 2\pi} |S(e^{i\varphi_1}, \ldots, e^{i\varphi_k})|^2. \tag{8}$$

On the other hand, for any polynomial R with integral coefficients

$$\max_{0\leqslant\varphi\leqslant 2\pi} |R(e^{i\varphi_1}, \ldots, e^{i\varphi_k})|^2 \leqslant \|R\|^2. \tag{9}$$

Using (8), Hadamard's inequality and (9) we obtain

$$\|S\| \leqslant \max_{0\leqslant\varphi\leqslant 2\pi} \Big(\sum_{j=0}^{m} |P_j(e^{i\varphi_1}, \ldots, e^{i\varphi_k})|^2\Big)^n \Big(\sum_{j=0}^{n} |Q_j(e^{i\varphi_1}, \ldots, e^{i\varphi_k})|^2\Big)^m$$
$$\leqslant \Big(\sum_{j=0}^{m} \max_{0\leqslant\varphi\leqslant 2\pi} |P_j(e^{i\varphi_1}, \ldots, e^{i\varphi_k})|^2\Big)^n \Big(\sum_{j=0}^{n} \max_{0\leqslant\varphi\leqslant 2\pi} |Q_j(e^{i\varphi_1}, \ldots, e^{i\varphi_k})|^2\Big)^m$$
$$\leqslant \Big(\sum_{j=0}^{m} \|P_j\|^2\Big)^n \Big(\sum_{j=0}^{n} \|Q_j\|^2\Big)^m \leqslant \Big(\sum_{j=0}^{m} \|P_j\|\Big)^{2n} \Big(\sum_{j=0}^{n} \|Q_j\|\Big)^{2m} \leqslant \|P\|^{2|Q|}\|Q\|^{2|P|}.$$

LEMMA 6. *If an m-dimensional sublattice of the n-dimensional integral lattice contains m linearly independent vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ then it has a basis of the form*

$$\sum_{j=1}^{m} c_{1j}\boldsymbol{v}_j, \ \ldots, \ \sum_{j=1}^{m} c_{mj}\boldsymbol{v}_j,$$

where

$$0 \leqslant c_{ij} < c_{jj} \leqslant 1 \ \ (i \neq j), \qquad c_{ij} = 0 \ \ (i < j).$$

Proof is obtained by a standard method (see [2], Appendix A). For a more precise result see [7].

LEMMA 7. *Let k_i $(0 \leqslant i \leqslant l)$ be an increasing sequence of integers. Let $k_{j_p} - k_{i_p}$ $(1 \leqslant p \leqslant p_0)$ be all the numbers which appear only once in the double sequence $k_j - k_i$ $(0 \leqslant i \leqslant j \leqslant l)$. Suppose that for each p*

$$k_{j_p} - k_{i_p} = \sum_{q=1}^{k} c_{pq} n_q,$$

where c_{pq} are integers, $|c_{pq}| \leqslant c$. Then either there exist integral matrices

$$\boldsymbol{K} = [\varkappa_{qi}]_{\substack{q\leqslant k\\ i\leqslant l}} \quad \textit{and} \quad \boldsymbol{\Lambda} = [\lambda_{qt}]_{\substack{q\leqslant k\\ t\leqslant k}}$$

and an integral vector $\boldsymbol{u}$ such that

$$[k_1 - k_0, \ldots, k_l - k_0] = \boldsymbol{u}\boldsymbol{K}, \quad \boldsymbol{n} = [n_1, \ldots, n_k] = \boldsymbol{u}\boldsymbol{\Lambda}, \tag{10}$$
$$h(\boldsymbol{K}) \leqslant k(\max\{c^2, 2\}+2)^{l/2},$$

$$0 \leqslant \lambda_{qt} < \lambda_{tt} \leqslant 2^{l-1} \ \ (q \neq t), \quad \lambda_{qt} = 0 \ \ (q < t) \tag{11}$$

or there exists an integral vector $\boldsymbol{\gamma}$ such that

$$\boldsymbol{\gamma}\boldsymbol{n} = 0 \quad \textit{and} \quad 0 < h(\boldsymbol{\gamma}) \leqslant k^{k-1}(\max\{kc^2, 2\}+2)^{(l+1)(k-1)/2}.$$

Proof. By the assumption for each pair $\langle i, j\rangle$ where $0 \leqslant i \leqslant j \leqslant l$ and $\langle i, j\rangle \neq \langle i_p, j_p\rangle$ $(1 \leqslant p \leqslant p_0)$ there exists a pair $\langle g_{ij}, h_{ij}\rangle \neq \langle i, j\rangle$ such that

$$k_j - k_i = k_{h_{ij}} - k_{g_{ij}}.$$

Let us consider the system of linear homogeneous equations

$$\begin{aligned} &x_0 = 0,\\ &x_j - x_i - x_{h_{ij}} + x_{g_{ij}} = 0, \qquad \langle i, j\rangle \neq \langle i_1, j_1\rangle, \ldots, \langle i_{p_0}, j_{p_0}\rangle,\\ &x_{j_p} - x_{i_p} - \sum_{q=1}^{k} c_{pq} y_q = 0 \qquad (1 \leqslant p \leqslant p_0)\end{aligned} \tag{12}$$

satisfied by $x_i = k_i - k_0$ $(0 \leqslant i \leqslant l)$, $y_q = n_q$ $(1 \leqslant q \leqslant k)$.

Let $\boldsymbol{A}$ be the matrix of the system obtained from (12) by cancelling the first equation and substituting $x_0 = 0$ in the others, $\boldsymbol{B}$ be the matrix of the coefficients of the x's, $-\boldsymbol{\Gamma}$ the matrix of the coefficients of the y's so that $\boldsymbol{A} = \boldsymbol{B}|-\boldsymbol{\Gamma}$ in the sense of juxtaposition (the vertical line is added in order to avoid a confusion with the subtraction).

We assert that (12) has at most k linearly independent solutions. Indeed, if we had $k+1$ such solutions $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{k+1}$ then taking as $\xi_1, \ldots, \xi_{k+1}$ real numbers rationally independent we should find a set of reals $\sum_{m=1}^{k+1} a_{mi}\xi_m$ $(0 \leqslant i \leqslant l)$, where all the differences would span over the rationals a space of dimension $k+1$, while the differences occurring only once

$$\sum_{m=1}^{k+1} (a_{mj_p} - a_{mi_p})\xi_m = \sum_{m=1}^{k+1} \xi_m \sum_{q=1}^{k} c_{pq} a_{m,l+q} = \sum_{q=1}^{k} c_{pq}\Big(\sum_{m=1}^{k+1} a_{m,l+q}\xi_m\Big)$$

would span a space of dimension at most k contrary to the theorem of Straus [11].

It follows that the rank of $\boldsymbol{A}$ is $l+\varrho$, where $0 \leqslant \varrho < k$. If the rank of $\boldsymbol{B}$ is l then since one row of $\boldsymbol{B}$ (corresponding to $\langle i, j\rangle = \langle 0, l\rangle$) is $[0, \ldots, 0, 1]$ there exists a nonsingular submatrix $\boldsymbol{\Delta}$ of $\boldsymbol{B}$ of degree l containing this row. Solving the system by means of Cramer formulae we find a system of k linearly independent integral solutions which can be written (horizontally) in the form $\boldsymbol{K}'|\boldsymbol{\Lambda}'$, where elements of $\boldsymbol{K}'$ are determinants obtained from $\boldsymbol{\Delta}$ by replacing one column by a column of $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda}' = D\boldsymbol{I}_k$, $D = |\boldsymbol{\Delta}|$, $\boldsymbol{I}_k$ is the identity matrix of degree k.

By Hadamard's inequality

$$|D| \leqslant 2^{l-1}, \qquad h(\boldsymbol{K}') \leqslant (\max\{c^2, 2\}+2)^{l/2}.$$

From $\boldsymbol{K}'|\boldsymbol{\Lambda}'$ we obtain by Lemma 6 a fundamental system of integral solutions $\boldsymbol{K}|\boldsymbol{\Lambda}$ satisfying (11). Since the system is fundamental there exists an integral vector $\boldsymbol{u}$ satisfying (10).

If the rank of $\boldsymbol{B}$ is less than l, we find a system of $k-\varrho$ linearly independent integral solutions in the form $\boldsymbol{K}'|\boldsymbol{\Lambda}'$, where elements of $\boldsymbol{\Lambda}'$ are up to a sign minors of $\boldsymbol{\Lambda}$ of degree $l+\varrho$. The rank of $\boldsymbol{\Lambda}'$ is less than k, otherwise the equality $\boldsymbol{B}\boldsymbol{K}'^T = \boldsymbol{\Gamma}\boldsymbol{\Lambda}'^T$ would imply

$$\boldsymbol{\Gamma} = \boldsymbol{B}\boldsymbol{K}'^T(\boldsymbol{\Lambda}'^T)^{-1}, \qquad \boldsymbol{\Lambda} = \boldsymbol{B}|-\boldsymbol{\Gamma} = \boldsymbol{B}\big(\boldsymbol{I}_l|-\boldsymbol{K}'^T(\boldsymbol{\Lambda}'^T)^{-1}\big)$$

and the rank of $\boldsymbol{\Lambda}$ would be less than l, which is impossible. By Hadamard's inequality

$$h(\boldsymbol{\Lambda}') \leqslant (2+\max\{kc^2, 2\})^{(l+\varrho)/2}.$$

By a well known lemma ([2], Lemma 3 of Chapter VI) there exists an integral vector $\gamma \neq \boldsymbol{0}$ such that $\boldsymbol{\Lambda}'\gamma^T = \boldsymbol{0}$ and

$$h(\gamma) \leqslant [h(\boldsymbol{\Lambda}')k]^{\frac{k-\max\{\varrho,1\}}{\max\{\varrho,1\}}} \leqslant k^{k-1}(\max\{kc^2, 2\}+2)^{\frac{(l+1)(k-1)}{2}}.$$

Since $\boldsymbol{n} = \boldsymbol{u}'\boldsymbol{\Lambda}'$ ($\boldsymbol{u}'$ not necessarily integral) we get

$$\gamma\boldsymbol{n} = \boldsymbol{n}\gamma^T = \boldsymbol{u}'\boldsymbol{\Lambda}'\gamma^T = 0.$$

Remark. The proof of Straus can be transformed into a proof that (12) has at most k linearly independent solutions, which does not use any irrationalities and is in this respect nearer to the proof of Lemma 4 in [9].

Suppose that $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{k+1}$ are solutions,

$$\boldsymbol{a}_m = [0, a_{m1}, \ldots, a_{ml}, a_{m,l+1}, \ldots, a_{m,l+k}].$$

There exist integers $b_1, \ldots, b_{k+1}$ not all zero such that

$$\sum_{m=1}^{k+1} b_m a_{m,l+q} = 0 \qquad (1 \leqslant q \leqslant k).$$

Consider the vector $\boldsymbol{a} = \sum_{m=1}^{k+1} b_m\boldsymbol{a}_m = [0, a_1, \ldots, a_l, 0, \ldots, 0]$. It is also a solution of (12). Set

i' = the least i such that $a_i = \min\limits_{0\leqslant j\leqslant l} a_j$ or $\max\limits_{0\leqslant j\leqslant l} a_j$,

j' = the greatest i such that $a_i = \min\limits_{0\leqslant j\leqslant l} a_j + \max\limits_{0\leqslant j\leqslant l} a_j - a_{i'}$.

The equality $a_{j'}-a_{i'} = a_h - a_g$ implies $a_{i'} = a_g$, $a_{j'} = a_h$, $i' \leqslant g$, $j' \geqslant h$ and either $\langle i', j'\rangle = \langle g, h\rangle$ or $k_{j'}-k_{i'} > k_h - k_g$. It follows that $\langle i', j'\rangle$ is identical with some $\langle i_p, j_p\rangle$ $(1 \leqslant p \leqslant p_0)$ and we get

$$a_{j'} - a_{i'} = \sum_{q=1}^{k} c_{pq}a_{l+q} = 0.$$

Hence $a_i = 0$ $(0 \leqslant i \leqslant l+k)$ and

$$\sum_{m=1}^{k+1} b_m\boldsymbol{a}_m = \boldsymbol{0}.$$

LEMMA 8 ($\mathrm{L8}_k$). *Let $P(x_1, \ldots, x_k) \neq 0$, $Q(x_1, \ldots, x_k) \neq 0$ be polynomials and $(P, Q) = G$. For any integral vector $\boldsymbol{n} = [n_1, \ldots, n_k]$ we have either*

$$\big(LP(x^{n_1}, \ldots, x^{n_k}), LQ(x^{n_1}, \ldots, x^{n_k})\big) = LG(x^{n_1}, \ldots, x^{n_k})$$

or $|P||Q| > 0$ and there exists an integral vector β such that

$$\beta\boldsymbol{n} = 0, \tag{13}$$

$$0 < h(\beta) < \begin{cases} 5|P||Q|\log\|P\|^{2|Q|}\|Q\|^{2|P|} & \text{if } k = 2, \\ \exp_{2k-5}(2\|P\|^{2|Q|}\|Q\|^{2|P|}\log 5|P||Q|+\log 7k) & \text{if } k > 2. \end{cases} \tag{14}$$

LEMMA 9 ($\mathrm{L9}_k$). *For any polynomial $F(x_1, \ldots, x_k) \neq 0$, any integral vector $\boldsymbol{n} = [n_1, \ldots, n_k]$ and any irreducible factor $f(x)$ of $LF(x^{n_1}, \ldots, x^{n_k})$ either there exist an integral matrix $\boldsymbol{\Lambda} = [\lambda_{qt}]$ of degree k, an integral vector $\boldsymbol{u} = [u_1, \ldots, u_k]$ and a polynomial $T(z_1, \ldots, z_k)$ such that*

$$0 \leqslant \lambda_{qt} < \lambda_{tt} \leqslant 2^{\|F\|-2} \ (q \neq t), \qquad \lambda_{qt} = 0 \ (q < t), \tag{15}$$

$$\boldsymbol{n} = \boldsymbol{u}\boldsymbol{\Lambda}, \tag{16}$$

$$T(z_1, \ldots, z_k) \,|\, F\Big(\prod_{q=1}^{k} z_q^{\lambda_{q1}}, \ldots, \prod_{q=1}^{k} z_q^{\lambda_{qk}}\Big),$$

$$f(x) = \mathrm{const}\, LT(x^{u_1}, \ldots, x^{u_k})$$

or $\|F\| \geqslant 3$ and there exists an integral vector γ such that

$$\gamma\boldsymbol{n} = 0, \tag{17}$$

$$0 < h(\gamma) < \begin{cases} 120(2|F|^*)^{2\|F\|-1}\log\|F\| & \text{if } k = 2, \\ \exp_{2k-4}(7k|F|^{*\|F\|-1}\log\|F\|) & \text{if } k > 2. \end{cases}$$

We prove these lemmata by induction showing first $\mathrm{L8}_2$ and then the implications $\mathrm{L8}_k \to \mathrm{L9}_k$ $(k \geqslant 1)$, $\mathrm{L9}_k \to \mathrm{L8}_{k+1}$ $(k > 1)$. Since $\mathrm{L8}_1$ is obvious this argumentation is sufficient.

Proof of $\mathrm{L8}_2$. If $P = GT$, $Q = GU$ and

$$\big(LP(x^{n_1}, x^{n_2}), LQ(x^{n_1}, x^{n_2})\big) \neq LG(x^{n_1}, x^{n_2})$$

then for some ξ not conjugate to ξ^{-1}: $T(\xi^{n_1}, \xi^{n_2}) = 0 = U(\xi^{n_1}, \xi^{n_2})$. Let R_i be the resultant of $T(x_1, x_2)$, $U(x_1, x_2)$ with respect to x_i and S_i a non-vanishing minor of Sylvester's matrix of P, Q, divisible by R_i, whose existence is asserted in Lemma 3. Set

$$a_i = \xi^{n_i}, \qquad \boldsymbol{\Omega} = \boldsymbol{Q}(a_1, a_2). \tag{18}$$

$|\Omega|$ does not exceed the number of distinct pairs $\langle\eta, \vartheta\rangle$ satisfying $T(\eta, \vartheta) = U(\eta, \vartheta) = 0$ thus by Lemma 4

$$|\Omega| \leqslant |R_i| \leqslant |S_i| \quad (i = 1, 2).$$

Since $\xi^{(n_1,n_2)} \in \Omega$, it follows

$$|Q(\xi)| \leqslant (n_1, n_2)|\Omega|.$$

Moreover $R_{3-i}(a_i) = 0$, $S_{3-i}(a_i) = 0$ and if a_i is not an integer or $n_i = 0$ we get from (18) and Lemma 1

$$|n_i| \leqslant e\big(a_i, Q(\xi)\big) \leqslant (2\log 2)^{-1}|Q(\xi)|\log\|S_{3-i}\| \leqslant (2\log 2)^{-1}(n_1, n_2)|S_i|\log\|S_{3-i}\|. \tag{19}$$

If a_i is an integer and $n_i \neq 0$, $\xi^{\operatorname{sgn} n_i}$ is also an integer. It is not conjugate to $\xi^{-\operatorname{sgn} n_i}$, thus by the already quoted refinement of Theorem 1 of [3]

$$\overline{|\xi^{\operatorname{sgn} n_i}|} > 1 + \frac{1}{5|Q(\xi)|-1}; \qquad \frac{1}{\log\overline{|\xi^{\operatorname{sgn} n_i}|}} < 5|Q(\xi)|.$$

On the other hand, by the inequality of Carmichael–Masson

$$\overline{|a_i|} \leqslant \|S_{3-i}\|^{\frac{1}{2}}; \qquad \log\overline{|a_i|} \leqslant \tfrac{1}{2}\log\|S_{3-i}\|.$$

It follows from (18) that

$$|n_i| = \frac{\log\overline{|a_i|}}{\log\overline{|\xi^{\operatorname{sgn} n_i}|}} < \frac{5}{2}|Q(\xi)|\log\|S_{3-i}\| \leqslant \frac{5}{2}(n_1, n_2)|S_i|\log\|S_{3-i}\|.$$

In view of Lemma 5 this inequality together with (19) implies $L8_2$ on taking $\beta = \left[\frac{n_2}{(n_1, n_2)}, \frac{-n_1}{(n_1, n_2)}\right]$.

Proof of the implication $L8_k \to L9_k$. Let

$$F(x_1, \ldots, x_k) = \sum_{i=0}^{I} a_i x_1^{a_{i1}} \ldots x_k^{a_{ik}}$$

where a_i are integers $\neq 0$ and the vectors $\boldsymbol{a}_i$ are all different. Let further

$$F(x^{n_1}, \ldots, x^{n_k}) = f(x)g(x),$$

where f and g have integral coefficients (if necessary we may change $f(x)$ by a constant factor without impairing the assertion of the lemma).

We set

$$f(x)g(x) = \sum_{i=0}^{l} c_i x^{k_i} \quad (c_i \text{ integers} \neq 0,\ k_0 < k_1 < \ldots < k_l)$$

and consider two expressions for $F(x^{n_1}, \ldots, x^{n_k})F(x^{-n_1}, \ldots, x^{-n_k})$:

$$F(x^{n_1}, \ldots, x^{n_k})F(x^{-n_1}, \ldots, x^{-n_k}) = \sum_{i=0}^{I} a_i^2 + \sum_{\substack{0 \leqslant i,j \leqslant I \\ i \neq j}} a_i a_j x^{\boldsymbol{n}\boldsymbol{a}_j - \boldsymbol{n}\boldsymbol{a}_i},$$

$$\big(f(x^{-1})g(x)\big)\big(f(x)g(x^{-1})\big) = \sum_{i=0}^{l} c_i^2 + \sum_{\substack{0 \leqslant i,j \leqslant l \\ i \neq j}} c_i c_j x^{k_j - k_i}.$$

If for any pair $\langle i, j\rangle$

$$i \neq j \quad \text{and} \quad \boldsymbol{n}\boldsymbol{a}_j - \boldsymbol{n}\boldsymbol{a}_i = 0 \tag{20}$$

we have (17) with $h(\gamma) \leqslant |F|$.

If no pair $\langle i, j\rangle$ satisfies (20), it follows that $F(x^{n_1}, \ldots, x^{n_k}) \neq 0$

$$\sum_{i=0}^{l} c_i^2 = \sum_{i=0}^{I} a_i^2 = \|F\|, \quad l \leqslant \|F\| - 1, \tag{21}$$

each number $k_j - k_i$ which appears only once in the double sequence $k_j - k_i$ $(0 \leqslant i \leqslant j \leqslant l)$ has a value $\sum_{q=1}^{k} n_q d_q$ with $|d_q| \leqslant |F|$.

Applying Lemma 7 with $c = |F|$ we find either integral matrices $\boldsymbol{K} = [\varkappa_{qi}]$, $\boldsymbol{\Lambda} = [\lambda_{qt}]$ and an integral vector $\boldsymbol{u}$ satisfying (15), (16) and

$$k_i - k_0 = \sum_{q=1}^{k} \varkappa_{qi} u_q, \quad h(\boldsymbol{K}) < k|F|^{*\|F\|-1}$$

or an integral vector γ satisfying (17) with

$$h(\gamma) < k(k|F|^{*2})^{\|F\|(k-1)/2} < \begin{cases} 120(2|F|^*)^{2\|F\|-1}\log\|F\| & \text{if} \quad k = 2, \\ \exp_{2k-4}(7k|F|^{*\|F\|-1}\log\|F\|) & \text{if} \quad k > 2. \end{cases}$$

We notice that $\|F\| \geqslant 3$ since otherwise $LF(x^{n_1}, \ldots, x^{n_k}) = \text{const}$. Set

$$P(z_1, \ldots, z_k) = \sum_{i=0}^{I} a_i \prod_{q=1}^{k} z_q^{\sum_{t=1}^{k} \lambda_{qt} a_{it}},$$

$$Q(z_1, \ldots, z_k) = J \sum_{i=0}^{l} c_i \prod_{q=1}^{k} z_q^{\varkappa_{qi}}.$$

Clearly

$$|P| \leqslant k|F|2^{\|F\|-2}, \quad |Q| \leqslant 2k|F|^{*\|F\|-1},$$

whence

$$|P|+|Q| \leqslant 3k|F|^{*\|F\|-1}, \qquad |P|\,|Q| \leqslant k^2 2^{\|F\|-1}|F|^{*\|F\|}. \tag{22}$$

The vectors $[\varkappa_{1i},\dots,\varkappa_{ki}]$ $(0\leqslant i\leqslant l)$ are all different since such are the numbers k_i-k_0. Similarly, by (16) the vectors $\left[\sum_{t=1}^{n}\lambda_{1t}\alpha_{it},\dots,\sum_{t=1}^{n}\lambda_{kt}\alpha_{it}\right]$ $(0\leqslant i\leqslant l)$ are all different since such are the numbers $\sum_{t=1}^{k}\alpha_{it}n_t$. Therefore, by (21)

$$\|P\| = \|Q\| = \|F\|. \tag{23}$$

We get from $L8_k$ that either

$$\big(LP(x^{u_1},\dots,x^{u_k}),\, LQ(x^{u_1},\dots,x^{u_k})\big) = LG(x^{u_1},\dots,x^{u_k})$$

or $\beta\boldsymbol{u} = 0$ with β satisfying (14).

In the former case

$$\begin{aligned} Lg(x) &= \operatorname{const}\big(LF(x^{n_1},\dots,x^{n_k}),\, Lf(x^{-1})g(x)\big)\\ &= \operatorname{const}\big(LP(x^{u_1},\dots,x^{u_k}),\, LQ(x^{u_1},\dots,x^{u_k})\big)\\ &= \operatorname{const} LG(x^{u_1},\dots,x^{u_k}),\end{aligned}$$

$$f(x) = \frac{LF(x^{n_1},\dots,x^{n_k})}{Lg(x)} = \frac{LP(x^{u_1},\dots,x^{u_k})}{\operatorname{const} LG(x^{u_1},\dots,x^{u_k})} = \operatorname{const} LT(x^{u_1},\dots,x^{u_k}),$$

where $T = PG^{-1}$.

In the latter case we have $k\geqslant 2$,

$$\gamma\boldsymbol{n} = 0 \quad \text{with} \quad \gamma = \beta\boldsymbol{A}^{\Delta},$$

$$h(\gamma) \leqslant kh(\beta)h(\boldsymbol{A}^{\Delta}) \leqslant k(k-1)^{(k-1)/2}h(\boldsymbol{A})^{k-1}h(\beta)$$

and we estimate $h(\gamma)$ separately for $k=2$ and for $k>2$, using (14), (15), (22), (23) and $|F|^*\geqslant 2$, $\|F\|\geqslant 3$.

For $k=2$ we obtain

$$\begin{aligned} h(\gamma) &\leqslant 2h(\boldsymbol{A})\cdot 5|P|\,|Q|\log\|P\|^{2|Q|}\|Q\|^{2|P|}\\ &\leqslant 5\cdot 2^{\|F\|-1}\cdot 2^{\|F\|+1}|F|^{*\|F\|}\cdot 12|F|^{*\|F\|-1}\log\|F\|\\ &\leqslant 120(2|F|^*)^{2\|F\|-1}\log\|F\|.\end{aligned}$$

For $k>2$ we use the inequality

$$k(k-1)^{(k-1)/2}h(\boldsymbol{A})^{k-1} < k^{k-1}2^{(k-1)(\|F\|-2)} < \exp_{2k-4}(6k|F|^{*\|F\|-1}\log\|F\|)$$

and obtain

$$\begin{aligned} h(\gamma) &\leqslant k(k-1)^{(k-1)/2}h(\boldsymbol{A})^{k-1}\times\\ &\quad\times\exp_{2k-4}(6k|F|^{*\|F\|-1}\log\|F\| + \log\log 5k^2 2^{\|F\|-1}|F|^{*\|F\|} + \log 3)\\ &\leqslant \exp^2_{2k-4}(6k|F|^{*\|F\|-1}\log\|F\| + \log\tfrac{5}{2}k^2 + \|F\|\log 2|F|^* + \log 3 - 1)\\ &< \exp_{2k-4}(7k|F|^{*\|F\|-1}\log\|F\|).\end{aligned}$$

Proof of the implication $L9_k\to L8_{k+1}$ $(k>1)$. Let $P=GT$, $Q=GU$, let R_j be the resultant of T, U with respect to x_j and let S_j be a nonvanishing minor of Sylvester's matrix of P, Q divisible by R_j, whose existence is asserted in Lemma 3.

If

$$\big(LP(x^{n_1},\dots,x^{n_{k+1}}),\, LQ(x^{n_1},\dots,x^{n_{k+1}})\big) \neq LG(x^{n_1},\dots,x^{n_{k+1}})$$

then $|P|\,|Q|>0$ and there exists an irreducible polynomial $f(x)$ such that

$$f(x)\,|\,\big(LT(x^{n_1},\dots,x^{n_{k+1}}),\, LU(x^{n_1},\dots,x^{n_{k+1}})\big).$$

Clearly for each $j\leqslant k+1$

$$f(x)\,|\,R_j(x^{n_1},\dots,x^{n_{k+1}})\,|\,S_j(x^{n_1},\dots,x^{n_{k+1}}),$$

where x^{n_j} does not occur among the arguments of R_j and S_j. By $L9_k$ either there exist an integral nonsingular triangular matrix $\boldsymbol{A}_j$ with nonnegative entries, an integral vector $\boldsymbol{u}_j$ and a polynomial T_j such that

$$h(\boldsymbol{A}_j) \leqslant 2^{\|S_j\|-2}, \tag{24}$$

$$[n_1,\dots,n_{j-1},n_{j+1},\dots,n_{k+1}] = \boldsymbol{A}_j\boldsymbol{u}_j, \tag{25}$$

$$T_j\,|\,S_j\Big(\prod_{q=1}^{k}z_q^{\lambda_{q1}},\dots,\prod_{q=1}^{k}z_q^{\lambda_{qk}}\Big), \qquad f(x) = \operatorname{const} T_j(x^{u_{j1}},\dots,x^{u_{jk}}) \tag{26}$$

or

$$\gamma_j[n_1,\dots,n_{j-1},n_{j+1},\dots,n_{k+1}] = 0$$

with

$$0 < h(\gamma_j) < \begin{cases} 120(2|S_j|^*)^{2\|S_j\|-1}\log\|S_j\| & \text{if} \quad k=2,\\ \exp_{2k-4}(7k|S_j|^{*\|S_j\|-1}\log\|S_j\|) & \text{if} \quad k>2.\end{cases}$$

In the latter case we have $\beta\boldsymbol{n}=0$, where

$$0 < h(\beta) \leqslant \max_{1\leqslant j\leqslant k+1} h(\gamma_j).$$

If $k=2$ we obtain from Lemma 5

$$\begin{aligned} h(\beta) &\leqslant 120(2|S_j|^*)^{2\|S_j\|-1}\log\|S_j\|\\ &< \exp\big(\log(120\log\|S_j\|) + (\|S_j\|-\tfrac{1}{2})\log(16|P|^2|Q|^2+8)\big)\\ &< \exp\big(\log\log\|P\|^{2|Q|}\|Q\|^{2|P|} + \|P\|^{2|Q|}\|Q\|^{2|P|}\log(16|P|^2|Q|^2+8) + \log 5\big)\\ &< \exp(2\|P\|^{2|Q|}\|Q\|^{2|P|}\log 5|P|\,|Q| + \log 21).\end{aligned}$$

If $k>2$ we have similarly

$$\begin{aligned} h(\beta) &\leqslant \exp_{2k-4}(7k|S_j|^{*\|S_j\|-1}\log\|S_j\|)\\ &< \exp_{2k-3}\big(\tfrac{1}{2}\|S_j\|\log(4|P|^2|Q|^2+2) + \log\log\|S_j\| + \log 7k\big)\\ &< \exp_{2k-3}(\|P\|^{2|Q|}\|Q\|^{2|P|}\log 5|P|\,|Q| + \log 7k).\end{aligned}$$

In the former case we set $\boldsymbol{u}_{k+1} = \boldsymbol{v} = [v_1, \ldots, v_k]$, find

$$f(x) = \operatorname{const} LT_{k+1}(x^{v_1}, \ldots, x^{v_k}),$$
$$Jf(x^{-1}) = \operatorname{const} LT_{k+1}(x^{-v_1}, \ldots, x^{-v_k})$$

and

$$\frac{Jf(x^{-1})}{f(x)} = \frac{LT_{k+1}(x^{-v_1}, \ldots, x^{-v_k})}{LT_{k+1}(x^{v_1}, \ldots, x^{v_k})} = \frac{JT_{k+1}(x^{-v_1}, \ldots, x^{-v_k})}{JT_{k+1}(x^{v_1}, \ldots, x^{v_k})}. \tag{27}$$

Let

$$T_{k+1}(z_1, \ldots, z_k) = \sum_{i=0}^{I} a_i z_1^{a_{i1}} z_2^{a_{i2}} \ldots z_k^{a_{ik}},$$

where $a_i \neq 0$ $(0 \leqslant i \leqslant I)$ and the vectors $\boldsymbol{a}_i$ are all different. Since $S_{k+1} \neq 0$, $|\boldsymbol{A}_{k+1}| \neq 0$ we get by (26)

$$h(\boldsymbol{a}_i) \leqslant k|S_{k+1}|h(\boldsymbol{A}_{k+1}) \quad (0 \leqslant i \leqslant I). \tag{28}$$

Let $\boldsymbol{a}_i\boldsymbol{u}$ takes its minimum for $i = m$, maximum for $i = M$. We have

$$JT_{k+1}(x^{v_1}, \ldots, x^{v_k}) = x^{-\boldsymbol{a}_m\boldsymbol{v}} \sum_{i=0}^{I} a_i x^{-\boldsymbol{a}_i\boldsymbol{v}},$$
$$JT_{k+1}(x^{-v_1}, \ldots, x^{-v_k}) = x^{\boldsymbol{a}_M\boldsymbol{v}} \sum_{i=0}^{I} a_i x^{-\boldsymbol{a}_i\boldsymbol{v}}. \tag{29}$$

Since $Jf(x^{-1}) \neq \operatorname{const} f(x)$ we get from (27)

$$d(x) = a_m JT_{k+1}(x^{-v_1}, \ldots, x^{-v_k}) - a_M JT_{k+1}(x^{v_1}, \ldots, x^{v_k}) \neq 0.$$

By (29) the lowest term in $d(x)$ is of the form $ax^{\gamma\boldsymbol{v}}$, where $\gamma = \boldsymbol{a}_i - \boldsymbol{a}_m$ or $\boldsymbol{a}_M - \boldsymbol{a}_i$ so that

$$a \neq 0; \quad \gamma\boldsymbol{v} > 0 \tag{30}$$

and by (28)

$$h(\gamma) \leqslant k|S_{k+1}|h(\boldsymbol{A}_{k+1}). \tag{31}$$

It follows that

$$\frac{Jf(x^{-1})}{f(x)} = \frac{JT_{k+1}(x^{-v_1}, \ldots, x^{-v_k})}{JT_{k+1}(x^{v_1}, \ldots, x^{v_k})} = \frac{a_M}{a_m} + \frac{a}{a_m^2} x^{\gamma\boldsymbol{v}} \bmod x^{\gamma\boldsymbol{v}+1}. \tag{32}$$

By (25) $|\boldsymbol{A}_{k+1}|\gamma\boldsymbol{v} = (\gamma\boldsymbol{A}_{k+1}^{\mathcal{A}})[n_1, \ldots, n_k]$ and since

$$\gamma' = \gamma\boldsymbol{A}_{k+1}^{\mathcal{A}} \neq \boldsymbol{0} \tag{33}$$

we have for some $j \leqslant k$, $\gamma_j' \neq 0$. Applying (25) and (26) we find as above

$$\frac{Jf(x^{-1})}{f(x)} = \frac{b_N}{b_n} + \frac{b}{b_n^2} x^{\boldsymbol{\delta}\boldsymbol{v}_j} \bmod x^{\boldsymbol{\delta}\boldsymbol{v}_j+1} \tag{34}$$

with

$$b \neq 0, \quad \boldsymbol{\delta}\boldsymbol{v}_j > 0, \tag{35}$$
$$h(\boldsymbol{\delta}) \leqslant k|S_{j+1}|h(\boldsymbol{A}_{j+1}). \tag{36}$$

It follows from (30), (32), (34) and (35) that

$$\gamma\boldsymbol{v} = \boldsymbol{\delta}\boldsymbol{v}_j,$$

which gives

$$|\boldsymbol{A}_j|\gamma'[n_1, \ldots, n_k] = |\boldsymbol{A}_{k+1}|\boldsymbol{\delta}'[n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_{k+1}]$$

with

$$\boldsymbol{\delta}' = \boldsymbol{\delta}\boldsymbol{A}_j^{\mathcal{A}}. \tag{37}$$

Hence

$$\sum_{i=1}^{j-1} (|\boldsymbol{A}_j|\gamma_i' - |\boldsymbol{A}_{k+1}|\delta_i')n_i + |\boldsymbol{A}_j|\gamma_j' n_j + $$
$$+ \sum_{i=j+1}^{k} (|\boldsymbol{A}_j|\gamma_i' - |\boldsymbol{A}_{k+1}|\delta_{i-1}')n_i + |\boldsymbol{A}_{k+1}|\gamma_k' n_{k+1} = 0,$$

which is the desired equality (13) with

$$0 < h(\boldsymbol{\beta}) \leqslant |\boldsymbol{A}_j|h(\gamma') + |\boldsymbol{A}_{k+1}|h(\boldsymbol{\delta}').$$

It follows from (24), (31), (33), (36), (37) and Lemma 5 that

$$\begin{aligned} h(\boldsymbol{\beta}) &\leqslant h(\boldsymbol{A}_j)^k k(k-1)^{(k-1)/2} h(\boldsymbol{A}_{k+1})^{k-1} h(\gamma) + \\ &\qquad + h(\boldsymbol{A}_{k+1})^k k(k-1)^{(k-1)/2} h(\boldsymbol{A}_j)^{k-1} h(\boldsymbol{\delta}) \\ &\leqslant k^2(k-1)^{(k-1)/2} h(\boldsymbol{A}_j)^k h(\boldsymbol{A}_{k+1})^k (|S_j| + |S_{k+1}|) \\ &< \exp\left(\frac{k+3}{2}\log k + k(\|S_j\| + \|S_{k+1}\|)\log 2 + \log(|S_j| + |S_{k+1}|)\right) \\ &< \exp\left(\frac{k+3}{2}\log k + 2k\|P\|^{2|Q|}\|Q\|^{2|P|}\log 2 + \log 4|P||Q|\right). \end{aligned}$$

For $k = 2$ we get

$$h(\boldsymbol{\beta}) < \exp(2\|P\|^{2|Q|}\|Q\|^{2|P|}\log 5|P||Q| + \log 21),$$

for $k > 2$ we use the inequality

$$kx < \exp_{2k-4} x \quad (x \geqslant 0)$$

and obtain

$$\begin{aligned} h(\boldsymbol{\beta}) &\leqslant \exp(2k\|P\|^{2|Q|}\|Q\|^{2|P|} + k\log 4|P||Q|k) \\ &< \exp_{2k-3}(2\|P\|^{2|Q|}\|Q\|^{2|P|}\log 5|P||Q| + \log 7k). \end{aligned}$$

LEMMA 10. *If $Q \neq 0$ is a polynomial,*

$$JQ(y_1^{-1}, \ldots, y_k^{-1}) \neq \pm JQ(y_1, \ldots, y_k) \quad \text{and} \quad LQ(x^{v_1}, \ldots, x^{v_k}) = \text{const},$$

then

$$\beta \boldsymbol{v} = 0 \quad \text{with} \quad h(\beta) \leqslant 2|Q|. \tag{38}$$

Proof. Let the degree of JQ with respect to y_j be q_j and

$$JQ(y_1, \ldots, y_k) = \sum a_{\boldsymbol{a}} y_1^{a_1} \ldots y_k^{a_k},$$

where the summation is taken over all integral vectors $\boldsymbol{a}$ satisfying $0 \leqslant a_j \leqslant q_j$. Clearly

$$JQ(y_1^{-1}, \ldots, y_k^{-1}) = \sum a_{\boldsymbol{q}-\boldsymbol{a}} y_1^{a_1} \ldots y_k^{a_k}$$

and there exist integral vectors $\boldsymbol{a}_j$ and $\boldsymbol{a}_{-j}$ $(1 \leqslant j \leqslant k)$ such that $a_{jj} = q_j$, $a_{\boldsymbol{a}_j} \neq 0$, $a_{-jj} = 0$, $a_{\boldsymbol{a}_{-j}} \neq 0$.

In view of the condition $JQ(y_1^{-1}, \ldots, y_k^{-1}) \neq \pm JQ(y_1, \ldots, y_k)$ we have for some $\boldsymbol{a}_l$, $\boldsymbol{a}_{-l}$

$$a_{\boldsymbol{a}_l} \neq a_{\boldsymbol{q}-\boldsymbol{a}_l}, \quad a_{\boldsymbol{a}_{-l}} \neq -a_{\boldsymbol{q}-\boldsymbol{a}_{-l}}. \tag{39}$$

Let the product $\boldsymbol{a}\boldsymbol{v}$ taken over all $\boldsymbol{a}$ for which $a_{\boldsymbol{a}} \neq 0$, attains its minimum for $\boldsymbol{a} = \boldsymbol{a}_m$, maximum for $\boldsymbol{a} = \boldsymbol{a}_n$. We have

$$JQ(x^{v_1}, \ldots, x^{v_k}) = x^{-\boldsymbol{a}_m \boldsymbol{v}} \sum a_{\boldsymbol{a}} x^{\boldsymbol{a}\boldsymbol{v}},$$
$$JQ(x^{-v_1}, \ldots, x^{-v_k}) = x^{\boldsymbol{a}_n \boldsymbol{v}} \sum a_{\boldsymbol{a}} x^{-\boldsymbol{a}\boldsymbol{v}}.$$

All the exponents $\boldsymbol{a}\boldsymbol{v}$ are different unless (38) holds (even with $h(\beta) \leqslant |Q|$). In particular, $Q(x^{v_1}, \ldots, x^{v_k}) \neq 0$.

The equality $LQ(x^{v_1}, \ldots, x^{v_k}) = \text{const}$ implies

$$JQ(x^{v_1}, \ldots, x^{v_k}) = \text{const}\, JQ(x^{-v_1}, \ldots, x^{-v_k})$$

and by the comparison of constant terms

$$a_{\boldsymbol{a}_n} JQ(x^{v_1}, \ldots, x^{v_k}) = a_{\boldsymbol{a}_m} JQ(x^{-v_1}, \ldots, x^{-v_k}).$$

Comparing the leading coefficients on both sides we get

$$a_{\boldsymbol{a}_n}^2 = a_{\boldsymbol{a}_m}^2, \quad \text{i.e.} \quad a_{\boldsymbol{a}_n} = \pm a_{\boldsymbol{a}_m},$$

$$\sum a_{\boldsymbol{a}} x^{\boldsymbol{a}\boldsymbol{v}} = \pm x^{(\boldsymbol{a}_m+\boldsymbol{a}_n)\boldsymbol{v}} \sum a_{\boldsymbol{a}} x^{-\boldsymbol{a}\boldsymbol{v}}. \tag{40}$$

In particular, we have for each $j \leqslant k$ and a suitable β_j

$$a_{\boldsymbol{a}_j} x^{\boldsymbol{a}_j \boldsymbol{v}} = \pm a_{\beta_j} x^{(\boldsymbol{a}_m+\boldsymbol{a}_n-\beta_j)\boldsymbol{v}}.$$

If $\boldsymbol{a}_j + \beta_j - \boldsymbol{a}_m - \boldsymbol{a}_n \neq 0$ we get again (38), otherwise

$$a_{mj} + a_{nj} = a_{jj} + \beta_{jj} \geqslant a_{jj} = q_j. \tag{41}$$

Similarly we have for each $j \leqslant k$ and a suitable β_{-j}

$$a_{\boldsymbol{a}_{-j}} x^{\boldsymbol{a}_{-j}\boldsymbol{v}} = \pm a_{\beta_{-j}} x^{(\boldsymbol{a}_m+\boldsymbol{a}_n-\beta_{-j})\boldsymbol{v}};$$

thus either (38) holds or

$$a_{mj} + a_{nj} = a_{-jj} + \beta_{-jj} = \beta_{-jj} \leqslant q_j.$$

The last inequality together with (41) implies

$$\boldsymbol{a}_m + \boldsymbol{a}_n = \boldsymbol{q}$$

and

$$x^{(\boldsymbol{a}_m+\boldsymbol{a}_n)\boldsymbol{v}} \sum a_{\boldsymbol{a}} x^{-\boldsymbol{a}\boldsymbol{v}} = \sum a_{\boldsymbol{q}-\boldsymbol{a}} x^{\boldsymbol{a}\boldsymbol{v}}.$$

It follows now from (39) and (40) that with a suitable sign and a suitable integral $\boldsymbol{a}$

$$\boldsymbol{a}_{\pm l}\boldsymbol{v} = \boldsymbol{a}\boldsymbol{v}, \quad \boldsymbol{a} \neq \boldsymbol{a}_{\pm l}$$

which gives (38) again.

LEMMA 11. *For any polynomial $F(x_1, \ldots, x_k) \neq 0$*

$$LKF(x_1, \ldots, x_k) = KLF(x_1, \ldots, x_k) = LF(x_1, \ldots, x_k).$$

Proof. In view of the definition of the operations K and L it is enough to prove that for any integral vector $[\delta_1, \ldots, \delta_k] \neq \boldsymbol{0}$ and any factor $Q(y_1, \ldots, y_k)$ of $J(y_1^{\delta_1} \ldots y_k^{\delta_k} - 1)$

$$JQ(y_1^{-1}, \ldots, y_k^{-1}) = \pm JQ(y_1, \ldots, y_k).$$

Supposing the contrary we apply Lemma 10 with

$$v_i = \big(4h(\boldsymbol{\delta}) + 1\big)^i \quad (1 \leqslant i \leqslant k).$$

Since the conditions $\beta\boldsymbol{v} = 0$, $h(\beta) \leqslant 2|Q| \leqslant 2h(\boldsymbol{\delta})$ imply $\beta = 0$, it follows from that lemma $LQ(x^{v_1}, \ldots, x^{v_k}) \neq \text{const}$. On the other hand

$$LQ(x^{v_1}, \ldots, x^{v_k}) \mid L(x^{\boldsymbol{v}\boldsymbol{\delta}} - 1)$$

and since all factors of $x^{|\boldsymbol{v}\boldsymbol{\delta}|} - 1$ are reciprocal we get a contradiction.

LEMMA 12. *For any polynomial $F(x_1, \ldots, x_k)$ and any integral vector $\boldsymbol{n} = [n_1, \ldots, n_k]$ such that $F(x^{n_1}, \ldots, x^{n_k}) \neq 0$ there exist an integral matrix $\boldsymbol{M} = [\mu_{ij}]$ of degree k and an integral vector $\boldsymbol{v} = [v_1, \ldots, v_k]$ such that*

$$0 \leqslant \mu_{ij} < \mu_{jj} \leqslant \exp 9k \cdot 2^{\|F\|-5} \ (i \neq j), \quad \mu_{ij} = 0 \ (i < j); \tag{42}$$

$$\boldsymbol{n} = \boldsymbol{v}\boldsymbol{M}, \tag{43}$$

and either

$$(44)\qquad LF\Big(\prod_{i=1}^{k} y_i^{\mu_{i1}}, \prod_{i=1}^{k} y_i^{\mu_{i2}}, \ldots, \prod_{i=1}^{k} y_i^{\mu_{ik}}\Big) \overset{\text{can}}{=} \text{const} \prod_{\sigma=i}^{s} F_\sigma(y_1, \ldots, y_k)^{e_\sigma}$$

implies

$$(45)\qquad LF(x^{n_1}, \ldots, x^{n_k}) \overset{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} LF_\sigma(x^{v_1}, \ldots, x^{v_k})^{e_\sigma}$$

or $\|F\| \geqslant 3$ *and there exists an integral vector* $\vec{\gamma}$ *such that*

$$(46)\qquad \gamma n = 0,$$

where

$$(47)\qquad 0 < h(\gamma) < \begin{cases} \max\{120(2|F|^*)^{2\|F\|-1}\log\|F\|,\ 8|F|\exp 9\cdot 2^{\|F\|-3}\} & \text{if } k = 2,\\ \exp_{2k-4}(7k|F|^{*\|F\|-1}\log\|F\|) & \text{if } k > 2.\end{cases}$$

If $k = 2$ *and some* $LF_\sigma(x^{v_1}, x^{v_2})$ *in* (45) *are allowed to be constants then* (47) *can be replaced by*

$$0 < h(\gamma) < 120(2|F|^*)^{2\|F\|-1}\log\|F\|.$$

Proof. If $\|F\| \leqslant 2$ then by Lemma 11 $s = 0$, $LF(x_1^{n_1}, \ldots, x_k^{n_k}) = \text{const}$ and it suffices to take $M = I_k$ (the identity matrix). Therefore we assume $\|F\| \geqslant 3$.

Let S be the set of all integral matrices $\Lambda = [\lambda_{qt}]$ of degree k satisfying

$$(48)\qquad 0 \leqslant \lambda_{qt} < \lambda_{tt} \leqslant 2^{\|F\|-2}\ (q \neq t), \qquad \lambda_{qt} = 0\ (q < t),$$

$$(49)\qquad n = u\Lambda \quad \text{with integral } u.$$

Integral vectors m such that for all $\Lambda \in S$ and a suitable integral vector v_Λ

$$m = v_\Lambda \Lambda$$

form a module $\mathfrak{M}$, say. By (48) for any $\Lambda \in S$, $|\Lambda|$ divides

$$\exp k\psi(2^{\|F\|-2}) = \mu,$$

where ψ is Čebyšev's function. Clearly vectors $[\mu, 0, \ldots, 0]$, $[0, \mu, \ldots, 0]$, $\ldots$, $[0, \ldots, 0, \mu]$ belong to $\mathfrak{M}$. It follows from Lemma 5 that $\mathfrak{M}$ has a basis $\mu_1, \ldots, \mu_k$ such that

$$0 \leqslant \mu_{ij} < \mu_{jj} \leqslant \mu\ (i \neq j), \qquad \mu_{ij} = 0\ (i \leqslant j).$$

Since by Theorem 12 of [8], $\psi(x) < 1{,}04x < \frac{9}{8}x$ for all x, the matrix M satisfies (42), since $n \in \mathfrak{M}$ it satisfies also (43).

In order to prove the alternative (45) or (46) and (47) we set

$$(50)\qquad P(y_1, \ldots, y_k) = F\Big(\prod_{i=1}^{k} y_i^{\mu_{i1}}, \ldots, \prod_{i=1}^{k} y_i^{\mu_{ik}}\Big) \overset{\text{can}}{=} \text{const} \prod_{i=1}^{k} y_i^{a_i} \prod_{\sigma=1}^{s_1} F_\sigma(y_1, \ldots, y_k)^{e_\sigma},$$

$$H_i(x_1, \ldots, x_k) = \sum_{j=1}^{k} \mu_{ij} x_j \frac{\partial F}{\partial x_j}$$

(note that $P \neq 0$ since $F(x^{n_1}, \ldots, x^{n_k}) \neq 0$). It follows

$$(51)\qquad \frac{\partial P}{\partial y_i} y_i = H_i\Big(\prod_{i=1}^{k} y_i^{\mu_{i1}}, \ldots, \prod_{i=1}^{k} y_i^{\mu_{ik}}\Big) = P\Big(y_i \sum_{\sigma=1}^{s_1} e_\sigma F_\sigma^{-1} \frac{\partial F_\sigma}{\partial y_i} + a_i\Big)$$

and by (43)

$$(52)\qquad P(x^{v_1}, \ldots, x^{v_k}) = F(x^{n_1}, \ldots, x^{n_k}),$$

$$(53)\qquad x^{v_i} \frac{\partial P}{\partial y_i}(x^{v_1}, \ldots, x^{v_k}) = H_i(x^{n_1}, \ldots, x^{n_k}).$$

(44) implies

$$(54)\qquad JF_\sigma(y_1^{-1}, \ldots, y_k^{-1}) = \pm F_\sigma(y_1, \ldots, y_k) \qquad (\sigma > s).$$

Assume now that for some distinct $\varrho, \tau \leqslant s_1$

$$(55)\qquad D(x) = \big(LF_\varrho(x^{v_1}, \ldots, x^{v_k}), LF_\tau(x^{v_1}, \ldots, x^{v_k})\big) \neq 1.$$

We consider two cases:

1. for some j: $\dfrac{\partial F_\varrho}{\partial y_j} \neq 0$ and $\dfrac{\partial F_\tau}{\partial y_j} \neq 0$,

2. for each i: $\dfrac{\partial F_\varrho}{\partial y_i} \cdot \dfrac{\partial F_\tau}{\partial y_i} = 0$.

1. Here $H_j \neq 0$ and we set $G = (F, H_j)$. It follows from (50) and (51), that

$$G\Big(\prod_{i=1}^{k} y_i^{\mu_{i1}}, \ldots, \prod_{i=1}^{k} y_i^{\mu_{ik}}\Big) = \text{const}\Big(P, \frac{\partial P}{\partial y_j} y_j\Big) = \text{const}\, P \prod_{\sigma=1}^{s} F_\sigma^{-1}(y_1, \ldots, y_k),$$

where the product is taken over all σ satisfying $\dfrac{\partial F_\sigma}{\partial y_j} \neq 0$. On substituting $y_i = x^{v_i}$ $(1 \leqslant i \leqslant k)$ we obtain from (50), (51)

$$D(x)\, LG\Big(\prod_{i=1}^{k} x^{\mu_{i1} v_i}, \ldots, \prod_{i=1}^{k} x^{\mu_{ik} v_i}\Big) \Big| \Big(LP(x^{v_1}, \ldots, x^{v_k}),\ Lx^{v_j} \frac{\partial P}{\partial y_j}(x^{v_1}, \ldots, x^{v_k})\Big),$$

which in view of (43), (52) and (53) gives

$$D(x)LG(x^{n_1},\ldots,x^{n_k})\,|\,\big(LF(x^{n_1},\ldots,x^{n_k}),\,LH_j(x^{n_1},\ldots,x^{n_k})\big).$$

By (55) and Lemma 8 we have (46) with

$$0<h(\gamma)<\begin{cases}5|F|\,|H_j|\log\|F\|^{2|H_j|}\|H_j\|^{2|F|} & \text{if} \quad k=2,\\ \exp_{2k-5}(2\|F\|^{2|H_j|}\|H_j\|^{2|F|}\log 5|F|\,|H_j|+\log 7k) & \text{if} \quad k>2.\end{cases}$$

2. Here we have for some h, j

$$\frac{\partial F_\varrho}{\partial y_h}\neq 0,\qquad \frac{\partial F_\tau}{\partial y_h}=0;\qquad \frac{\partial F_\varrho}{\partial y_j}=0,\qquad \frac{\partial F_\tau}{\partial y_j}\neq 0,$$

thus $H_h\neq 0, H_j\neq 0$.

We set $G=(H_h, H_j)$. It follows from (50) and (51) that

$$\text{(56)}\qquad \begin{aligned}\frac{\partial P}{\partial y_h}y_h &= F_\varrho^{e_\varrho-1}F_\tau^{e_\tau}U, & U&\not\equiv 0 \bmod F_\varrho,\\ \frac{\partial P}{\partial y_j}y_j &= F_\varrho^{e_\varrho}F_\tau^{e_\tau-1}V, & V&\not\equiv 0 \bmod F_\tau,\end{aligned}$$

hence

$$G\Big(\prod_{i=1}^{k}y_i^{\mu_{i1}},\ldots,\prod_{i=1}^{k}y_i^{\mu_{ik}}\Big)=F_\varrho^{e_\varrho-1}F_\tau^{e_\tau-1}(U,V)(y_1,\ldots,y_k).$$

On substituting $y_i=x^{v_i}$ we obtain from (56)

$$D(x)LG\Big(\prod_{i=1}^{k}x^{\mu_{i1}v_i},\ldots,\prod_{i=1}^{k}x^{\mu_{ik}v_i}\Big)\Big|\Big(Lx^{v_h}\frac{\partial P}{\partial y_h}(x^{v_1},\ldots,x^{v_k}),\,Lx^{v_j}\frac{\partial P}{\partial y_j}(x^{v_1},\ldots,x^{v_k})\Big),$$

which in view of (43) and (53) gives

$$D(x)LG(x^{n_1},\ldots,x^{n_k})\,|\,\big(LH_h(x^{n_1},\ldots,x^{n_k}),\,LH_j(x^{n_1},\ldots,x^{n_k})\big).$$

By (55) and Lemma 8 we have (46) with

$$0<h(\gamma)<\begin{cases}5|H_h|\,|H_j|\log\|H_h\|^{2|H_j|}\|H_j\|^{2|H_h|} & \text{if} \quad k=2,\\ \exp_{2k-5}(2\|H_h\|^{2|H_j|}\|H_j\|^{2|H_h|}\log 5|H_h|\,|H_j|+\log 7k) & \text{if} \quad k>2.\end{cases}$$

Since for all i: $|H_i|\leqslant|F|$,

$$\|H_i\|\leqslant k\sum_{j=1}^{k}\Big\|\mu_{ij}x_j\frac{\partial F}{\partial x_j}\Big\|\leqslant k^2h(\boldsymbol{M})^2|F|^2\|F\|,$$

it follows in both cases that if $k=2$

$$\begin{aligned}0<h(\gamma)&<20|F|^3\log 4h(\boldsymbol{M})^2|F|^2\|F\|\\ &<20|F|^3\log 4|F|^2\|F\|+20|F|^3\cdot 9\cdot 2^{\|F\|-3}<120(2|F|^*)^{2\|F\|-1}\log\|F\|,\end{aligned}$$

if $k>2$

$$\begin{aligned}0<h(\gamma)&<\exp_{2k-4}\big(4|F|\log k^2h(\boldsymbol{M})^2|F|^2\|F\|+\log\log 5|F|^2+\log 3\big)\\ &<\exp_{2k-4}(5|F|\log k^2|F|^2\|F\|+|F|\cdot 9k\cdot 2^{\|F\|-2})\\ &<\exp_{2k-4}(7k|F|^{*\|F\|-1}\log\|F\|).\end{aligned}$$

Assume, therefore, that for all distinct $\varrho,\tau\leqslant s_1$

$$\text{(57)}\qquad \big(LF_\varrho(x^{v_1},\ldots,x^{v_k}),\,LF_\tau(x^{v_1},\ldots,x^{v_k})\big)=1$$

and let $f(x)$ be any irreducible factor of $LF(x^{n_1},\ldots,x^{n_k})$. By Lemma 9 either (46)-(47) hold or there exist an integral matrix $\boldsymbol{A}=[\lambda_{qt}]$ of degree k, an integral vector $\boldsymbol{u}=[u_1,\ldots,u_k]$ satisfying (48)-(49) and a polynomial T such that

$$\text{(58)}\qquad T(z_1,\ldots,z_k)\,|\,F\Big(\prod_{q=1}^{k}z_q^{\lambda_{q1}},\ldots,\prod_{q=1}^{k}z_q^{\lambda_{qk}}\Big),$$

$$\text{(59)}\qquad f(x)=\operatorname{const}LT(x^{u_1},\ldots,x^{u_k}).$$

Since $\boldsymbol{A}\in\mathcal{S}$ and by the choice of $\boldsymbol{M}$: $\mu_1,\ldots,\mu_n\in\mathfrak{M}$ we have for some integral vectors $\boldsymbol{\vartheta}_1,\ldots,\boldsymbol{\vartheta}_n$: $\mu_i=\boldsymbol{\vartheta}_i\boldsymbol{A}$, thus

$$\begin{aligned}&\text{(60)}\qquad \boldsymbol{M}=\boldsymbol{\Theta}\boldsymbol{A}\\ &\text{(61)}\qquad \boldsymbol{u}=\boldsymbol{v}\boldsymbol{\Theta}\end{aligned}\Bigg\},\qquad \boldsymbol{\Theta}=\begin{bmatrix}\boldsymbol{\vartheta}_1\\ \vdots\\ \boldsymbol{\vartheta}_n\end{bmatrix}.$$

Set

$$W(y_1,\ldots,y_k)=JT\Big(\prod_{i=1}^{k}y_i^{\vartheta_{i1}},\ldots,\prod_{i=1}^{k}y_i^{\vartheta_{ik}}\Big).$$

We have by (58) and (60)

$$W(y_1,\ldots,y_k)\,|\,F\Big(\prod_{i=1}^{k}y_i^{\mu_{i1}},\ldots,\prod_{i=1}^{k}y_i^{\mu_{ik}}\Big),$$

by (59) and (61)

$$f(x)=\operatorname{const}LW(x^{v_1},\ldots,x^{v_k}).$$

Since $f(x)$ is irreducible, the last two formulae imply in view of (50)

$$\text{(62)}\qquad f(x)=\operatorname{const}LF_\varrho(x^{v_1},\ldots,x^{v_k})\quad\text{for some } \varrho\leqslant s_1$$

and since $Jf(x^{-1})\neq\pm Jf(x)$ we have by (54) $\varrho\leqslant s$. By (57)

$$\Big(f(x),\prod_{\sigma=s+1}^{s_1}LF_\sigma(x^{v_1},\ldots,x^{v_k})^{e_\sigma}\Big)=1$$

and because of the arbitrariness of $f(x)$

$$\Big(LF(x^{n_1},\ldots,x^{n_k}),\prod_{\sigma=s+1}^{s_1}LF_\sigma(x^{v_1},\ldots,x^{v_k})^{e_\sigma}\Big)=1.$$

Since by (50) and (52)

$$LF(x^{n_1}, \ldots, x^{n_k}) = \text{const} \prod_{\sigma=1}^{s_1} LF_\sigma(x^{v_1}, \ldots, x^{v_k})^{e_\sigma},$$

it follows that

$$LF(x^{n_1}, \ldots, x^{n_k}) = \text{const} \prod_{\sigma=1}^{s} LF_\sigma(x^{v_1}, \ldots, x^{v_k})^{e_\sigma}.$$

Moreover, none of the $LF_\sigma(x^{v_1}, \ldots, x^{v_k})$ $(\sigma \leqslant s)$ is reducible since taking as $f(x)$ any of its irreducible factors we would obtain from (62) a contradiction with (57).

It remains to prove that none of $LF_\sigma(x^{v_1}, \ldots, x^{v_k})$ $(\sigma \leqslant s)$ is constant unless (46) holds with

$$0 < h(\gamma) < \begin{cases} 8\,|F| \exp 9 \cdot 2^{\|F\|-3} & \text{if} \quad k = 2, \\ \exp_{2k-4}(7k\,|F|^{*\|F\|-1} \log \|F\|) & \text{if} \quad k > 2. \end{cases}$$

This follows from Lemma 10 on taking $Q = F_\sigma$, since (38) implies (46) with $\gamma = \beta M^{\Delta}$ and

$$0 < h(\gamma) \leqslant kh(M^{\Delta})h(\beta) \leqslant k(k-1)^{(k-1)/2} h(M)^{k-1} 2\,|P|$$
$$\leqslant 2k^2(k-1)^{(k-1)/2} h(M)^k |F| \leqslant 2k^2(k-1)^{(k-1)/2} |F| \exp 9 k^2 2^{\|F\|-5}.$$

Remark. A comparison of Lemma 12 with the conjecture from [9] shows besides the replacement of K by L the two differences:

it is not assumed that F is irreducible,

it is not assumed that $n_1 > 0, \ldots, n_k > 0$ and it is not asserted that $v_1 \geqslant 0, \ldots, v_k \geqslant 0$ (instead it is asserted that M is triangular).

As to the first difference one may note the fact overlooked in [9] that if F is irreducible all the exponents e_σ in (44) are 1. Indeed, in the notation of the preceding proof $e_\sigma > 1$ implies

$$F_\sigma(y_1, \ldots, y_k) \,|\, \left(P(y_1, \ldots, y_k), \frac{\partial P}{\partial y_1}, \ldots, \frac{\partial P}{\partial y_k}\right)$$

hence

$$\big(JF(x_1, \ldots, x_k), H_1(x_1, \ldots, x_k), \ldots, H_k(x_1, \ldots, x_k)\big) \neq 1.$$

Since $|M| \neq 0$ it follows by the definition of H_i that

$$\left(JF(x_1, \ldots, x_k), x_1 \frac{\partial F}{\partial x_1}, \ldots, x_k \frac{\partial F}{\partial x_k}\right) \neq 1,$$

which for an irreducible F is impossible.

As to the second difference it may be noted that the formulation with the assumption $n_1 \geqslant 0, \ldots, n_k \geqslant 0$ and the assertion $v_1 \geqslant 0, \ldots, v_k \geqslant 0$ (but M not necessarily triangular and $h(M)$ possibly greater) is also true its proof however involves the following theorem of Schmidt [10].

If $\mathfrak{M}$ is a sublattice of the integral k-dimensional lattice and $\mathfrak{M}^+$ consists of all vectors of $\mathfrak{M}$ with nonnegative coordinates then there exists a finite subset $\mathfrak{M}_0$ of $\mathfrak{M}^+$ such that every vector of $\mathfrak{M}^+$ is a linear combination of k vectors of $\mathfrak{M}_0$ with nonnegative integral coefficients.

In the proof of Lemma 5 of [9] the truth of this theorem for $k = 2$ was established together with a bound for the height of the vectors of $\mathfrak{M}_0$ in terms of $\mathfrak{M}$. Such a bound in the general case has been found recently by R. Lee.

Proof of Theorem 2. The theorem is true for $k = 1$ by Lemma 12. Assume that it is true for polynomials in $k-1$ variables and consider $F(x_1, \ldots, x_k)$. By Lemma 12 either there exist a matrix M and a vector v with the properties (42), (43), (45) or we have $\|F\| \geqslant 3$ and there exists a vector γ satisfying (46), (47). In the former case the theorem holds with $r = k$, in the latter case n belongs to the module $\mathfrak{N}$ of integral vectors perpendicular to γ. If $\gamma = [0, \ldots, 0, \gamma_\nu, \ldots, \gamma_k]$ with $\gamma_\nu \neq 0$, $\mathfrak{N}$ contains $k-1$ linearly independent vectors $[1, 0, \ldots, 0], \ldots, [0, \ldots, 1, 0, \ldots, 0]$, $[0, \ldots, \gamma_{\nu+1}, -\gamma_\nu, 0, \ldots, 0], \ldots, [0, \ldots, \gamma_k, 0, \ldots, -\gamma_\nu]$ and by Lemma 6 it has a basis which written in the form of a matrix $\Delta = [\delta_{ij}]_{\substack{i<k \\ j\leqslant k}}$ satisfies

$$h(\Delta) \leqslant (k-1)h(\gamma), \tag{63}$$

$$\text{rank of } \Delta = k-1, \tag{64}$$

$$n = m\Delta, \quad m \text{ integral} \neq 0. \tag{65}$$

Set

$$F'(z_1, \ldots, z_{k-1}) = JF\left(\prod_{i=1}^{k-1} z_i^{\delta_{i1}}, \prod_{i=1}^{k-1} z_i^{\delta_{i2}}, \ldots, \prod_{i=1}^{k-1} z_i^{\delta_{ik}}\right). \tag{66}$$

We have clearly $F'(x^{m_1}, \ldots, x^{m_{k-1}}) \neq 0$,

$$|F'|^* \leqslant 2(k-1)\,|F|^* h(\Delta), \tag{67}$$

and by (8) and (9)

$$\|F'\| \leqslant \max_{0\leqslant\varphi\leqslant 2\pi} |F'(e^{i\varphi_1}, \ldots, e^{i\varphi_{k-1}})|^2 \leqslant \max_{0\leqslant\theta\leqslant 2\pi} |F(e^{i\theta_1}, \ldots, e^{i\theta_k})|^2 \leqslant \|F\|^2. \tag{68}$$

By the inductive assumption there exist an integral matrix $N' = [v'_{it}]_{\substack{i\leqslant r \\ t\leqslant k}}$ and an integral vector $v = [v_1, \ldots, v_r]$ such that

$$h(N') \leqslant \begin{cases} \exp 9(k-1) 2^{\|F'\|-5} & \text{if} \quad k-1 = r, \\ \exp(5 \cdot 2^{\|F'\|^2-4} + 2\,\|F'\| \log |F'|^*) & \text{if} \quad k+r-1 = 3, \\ \exp_{(k-r-1)(k+r-4)}(8(k-1)\,|F'|^{*\|F'\|-1} \log \|F'\|) & \text{otherwise}; \end{cases} \tag{69}$$

(70) $$\text{rank of } N' = r;$$

(71) $$\boldsymbol{m} = \boldsymbol{v}N';$$

$$LF'\Big(\prod_{i=1}^{r} y_i^{v'_{i1}}, \ldots, \prod_{i=1}^{r} y_i^{v'_{ik-1}}\Big) \overset{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} F_\sigma(y_1, \ldots, y_r)^{e_\sigma}$$

implies

(72) $$LF'(x^{m_1}, \ldots, x^{m_{k-1}}) \overset{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s_0} LF_\sigma(x^{v_1}, \ldots, x^{v_r})^{e_\sigma}.$$

Set

(73) $$N = N'\Delta.$$

It follows from (64) and (70) that N is of rank r. By (65) and (71) $\boldsymbol{n} = \boldsymbol{v}N$. By (66) and (73)

$$LF'\Big(\prod_{i=1}^{r} y_i^{v'_{i1}}, \ldots, \prod_{i=1}^{r} y_i^{v'_{i,k-1}}\Big) = LF\Big(\prod_{i=1}^{r} y_i^{v_{i1}}, \ldots, \prod_{i=1}^{r} y_i^{v_{ik}}\Big)$$

and by (65) and (66)

$$JF'(x^{m_1}, \ldots, x^{m_{k-1}}) = JF(x^{n_1}, \ldots, x^{n_k}).$$

In view of (72) it remains to estimate $h(N)$. By (69) and (73)

$$h(N) \leqslant (k-1)^2 h(\gamma) h(N').$$

To proceed further we use the inequalities (47), (67)-(69), $|F|^* \geqslant 2$, $\|F\| \geqslant 3$ and distinguish four cases:

1. $k = 2, r = 1$. Here

$$h(N) \leqslant \max\{120(2|F|^*)^{2\|F\|-1}\log\|F\|, 8|F|\exp 9\cdot 2^{\|F\|-3}\}\exp 9\cdot 2^{\|F\|^2-5}$$
$$\leqslant \exp(5\cdot 2^{\|F\|^2-4} + 2\|F\|\log|F|^*).$$

2. $k = 3, r = 1$. Here we use the inequality

$$22|F|^{*\|F\|-1}\log\|F\| + 5\cdot 2^{\|F\|^4-4} + 2\|F\|^2\log 8|F|^* < \|F\|^2\exp(21|F|^{*\|F\|-1}\log\|F\|)$$

and obtain

$$h(N) \leqslant 4\exp(21|F|^{*\|F\|-1}\log\|F\|)\exp(5\cdot 2^{\|F'\|^2-4} + 2\|F'\|\log|F'|^*)$$
$$< \exp(22|F|^{*\|F\|-1}\log\|F\|)\times$$
$$\times\exp\big(5\cdot 2^{\|F\|^4-4} + 2\|F\|^2\log 8|F|^* + 2\|F\|^2\exp(21|F|^{*\|F\|-1}\log\|F\|)\big)$$
$$< \exp\big(3\|F\|^2\exp(21|F|^{*\|F\|-1}\log\|F\|)\big) < \exp_2(24|F|^{*\|F\|-1}\log\|F\|).$$

3. $k-1 = r > 1$. Here we use the inequality

$$(k-1)^2\exp 9(k-1)2^{\|F\|^2-5} < \exp 11(k-1)2^{\|F\|^2-5} < \exp_2 7k2^{\|F\|-1}$$
$$< \exp_2(7k|F|^{*\|F\|-1}\log\|F\|)$$

and obtain

$$h(N) \leqslant (k-1)^2\exp_{2k-4}(7k|F|^{*\|F\|-1}\log\|F\|)\cdot\exp 9(k-1)2^{\|F\|^2-5}$$
$$\leqslant \exp^2_{2k-4}(7k|F|^{*\|F\|-1}\log\|F\|) < \exp_{2k-4}(8k|F|^{*\|F\|-1}\log\|F\|).$$

4. $k-1 > \max(r, 2)$. Here we use the inequality

$$16k\log\|F\|\big(2k^2|F|^*\exp_{2k-4}(7k|F|^{*\|F\|-1}\log\|F\|)\big)^{\|F\|^2}$$
$$< \big(\exp_{2k-4}(7k|F|^{*\|F\|-1}\log\|F\|)\big)^{2\|F\|^2}$$
$$= \exp_2\big(\exp_{2k-6}(7k|F|^{*\|F\|-1}\log\|F\|) + \log 2\|F\|^2\big)$$
$$< \exp_{2k-4}(7k|F|^{*\|F\|-1}\log\|F\| + 1)$$

and obtain

$$h(N) \leqslant (k-1)^2\exp_{2k-4}(7k|F|^{*\|F\|-1}\log\|F\|)\times$$
$$\times\exp_{(k-r-1)(k+r-4)}\big(8(k-1)|F'|^{*\|F'\|-1}\log\|F'\|\big)$$
$$< \exp_{2k-4}(8k|F|^{*\|F\|-1}\log\|F\|)\times$$
$$\times\exp_{(k-r-1)(k+r-4)}\big(16k\log\|F\|\big(2k^2|F|^*\exp(7k|F|^{*\|F\|-1}\log\|F\|)\big)^{\|F\|^2}\big)$$
$$< \exp_{2k-3}(7k|F|^{*\|F\|-1}\log\|F\| + 1)\times$$
$$\times\exp_{(k-r-1)(k+r-4)+2k-4}(7k|F|^{*\|F\|-1}\log\|F\| + 1)$$
$$< \exp^2_{(k-r)(k+r-3)}(7k|F|^{*\|F\|-1}\log\|F\| + 1)$$
$$< \exp_{(k-r)(k+r-3)}(8k|F|^{*\|F\|-1}\log\|F\|).$$

Proof of Corollary. Let $JF(x) = a_0 + \sum_{j=1}^{k} a_j x^{n_j}$, where $a_j \neq 0$, n_j distinct > 0. Set in Theorem 2

$$F(x_1, \ldots, x_k) = a_0 + \sum_{j=1}^{k} a_j x_j.$$

We have

(74) $$k \leqslant \|F\| - 1 = \|f\| - 1, \quad |F|^* = 2.$$

By Theorem 2, the number l of irreducible factors of $Lf(x)$ equals the number of irreducible factors of

$$LF\Big(\prod_{i=1}^{r} y_i^{v_{i1}}, \prod_{i=1}^{r} y_i^{v_{i2}}, \ldots, \prod_{i=1}^{r} y_i^{v_{ik}}\Big)$$

(in the notation of the theorem), hence $l = 0$ if $\|f\| \leqslant 2$ and $l \leqslant 2rh(N)$ otherwise. Thus if $k \neq 2$ we get from (i) and (74)

$$l \leqslant \max\{2k\exp 9k\cdot 2^{\|F\|-5}, \max_{r<k} 2r\exp_{(k-r)(k+r-3)}(8k|F|^{*\|F\|-1}\log\|F\|)\}$$
$$\leqslant 2\exp_{k^2-3k+2}(k\cdot 2^{\|F\|+2}\log\|F\|) \leqslant 2\exp_{\|f\|^2-5\|f\|+6}\big((\|f\|-1)2^{\|f\|+2}\log\|f\|\big)$$
$$< \exp_{\|f\|^2-5\|f\|+7}(\|f\| + 2).$$

If $k=2$ we have

$$l \leqslant \max\{4\exp 9\cdot 2^{\|f\|-4}, 2\exp(5\cdot 2^{\|f\|^2-4}+2\|f\|\log 2)\} < \exp_{\|f\|^2-5\|f\|+7}(\|f\|+2)$$

except when $\|f\|=3$. However in this case $Jf(x) = \pm x^{n_1} \pm x^{n_2} \pm 1$ has at most one irreducible non-reciprocal factor (see [4] or [13]) and the proof is complete.

§ 4. LEMMA 13. *If $KF(x_1, x_2) = LF(x_1, x_2)$ and $[n_1, n_2] \neq \mathbf{0}$ then either $KF(x^{n_1}, x^{n_2}) = LF(x^{n_1}, x^{n_2})$ or for each zero ξ of $\dfrac{KF(x^{n_1}, x^{n_2})}{LF(x^{n_1}, x^{n_2})}$ the inequality holds*

$$\frac{\max\{|n_1|, |n_2|\}}{(n_1, n_2)} e(\xi, \boldsymbol{Q}(\xi)) \leqslant 120(2|F|^*)^{2\|F\|-1}\log\|F\|.$$

Proof. We can assume $|F| \geqslant 4$ since otherwise

$$KF(x^{n_1}, x^{n_2}) = LF(x^{n_1}, x^{n_2})$$

holds trivially. Set

$$P = F(x_1, x_2), \quad Q_1 = JF(x_1^{-1}, x_2^{-1}), \quad Q_2 = \frac{\partial P}{\partial x_1}, \quad G_i = (P, Q_i),$$

$$T_i = PG_i^{-1}, \quad U_i = Q_i G_i^{-1}, \quad V = \big(LF(x_1, x_2), LF(x_1^{-1}, x_2^{-1})\big).$$

By the assumption $KF(x_1, x_2) = LF(x_1, x_2)$, we have

$$G_1 = \frac{JF(x_1, x_2)}{KF(x_1, x_2)} V(x_1, x_2),$$

(75)

$$T_1 = L(x_1, x_2)V^{-1}, \quad U_1 = L(x_1, x_2)V^{-1}.$$

If ξ is a zero of $\dfrac{KF(x^{n_1}, x^{n_2})}{LF(x^{n_1}, x^{n_2})}$ then ξ is conjugate to ξ^{-1} thus $P(\xi^{n_1}, \xi^{n_2}) = Q_1(\xi^{n_1}, \xi^{n_2}) = 0$. On the other hand, ξ not being a root of unity is not a zero of $\dfrac{JF(x^{n_1}, x^{n_2})}{KF(x^{n_1}, x^{n_2})}$ and we get from (75) either $T_1(\xi^{n_1}, \xi^{n_2}) = U_1(\xi^{n_1}, \xi^{n_2}) = 0$ or $V(\xi^{n_1}, \xi^{n_2}) = 0$.

In the second case (ξ^{n_1}, ξ^{n_2}) is a zero of a certain irreducible factor of $V(x_1, x_2)$, $f(x_1, x_2)$ say. Without loss of generality we may assume $\partial f/\partial x_1 \neq 0$. By the definition of V, it follows that $g(x_1, x_2) = Jf(x_1^{-1}, x_2^{-1})$ divides V and is prime to f. Set

$$P = f^\alpha g^\beta h, \quad \text{where} \quad \alpha\beta > 0, \ (f, g) = (f, h) = (g, h) = 1.$$

We have

$$Q_2 = \frac{\partial P}{\partial x_1} = P\left(\alpha\frac{\partial f/\partial x_1}{f} + \beta\frac{\partial g/\partial x_1}{g} + \frac{\partial h/\partial x_1}{h}\right) \neq 0,$$

$$G_2 = \frac{P}{fgh}\left(\frac{\partial h}{\partial x_1}, h\right), \quad T_2 = \frac{fgh}{(\partial h/\partial x_1, h)},$$

$$U_2 = \alpha\frac{\partial f}{\partial x_1} g\frac{h}{(\partial h/\partial x_1, h)} + \beta f\frac{\partial g}{\partial x_1}\cdot\frac{h}{(\partial h/\partial x_1, h)} + fg\frac{\partial h/\partial x_1}{(\partial h/\partial x_1, h)}.$$

Since $f(\xi^{n_1}, \xi^{n_2}) = g(\xi^{n_1}, \xi^{n_2})$ it follows

$$T_2(\xi^{n_1}, \xi^{n_2}) = U_2(\xi^{n_1}, \xi^{n_2}) = 0.$$

In any case

(76) $$T_i(\xi^{n_1}, \xi^{n_2}) = U_i(\xi^{n_1}, \xi^{n_2}) \quad \text{with suitable } i.$$

Let R_{ij} be the resultant of T_i, U_i with respect to x_j and S_{ij} a nonvanishing minor of Sylvester's matrix of P, Q_i divisible by R_{ij}. Since

$$|P| = |F|, \quad |Q_i| \leqslant |F|, \quad \|P\| = \|F\|, \quad \|Q_i\| \leqslant |F|^2\|F\|$$

we get from Lemma 5

$$|S_{ij}| \leqslant 2|F|^2, \quad \|S_{ij}\| \leqslant (|F|\,\|F\|)^{4|F|} \quad (1 \leqslant i, j \leqslant 2).$$

Set $\boldsymbol{\Omega} = \boldsymbol{Q}(\xi^{n_1}, \xi^{n_2})$. By (76) $|\boldsymbol{\Omega}|$ does not exceed the number of distinct pairs $\langle\eta, \vartheta\rangle$ satisfying $T_i(\eta, \vartheta) = U_i(\vartheta, \eta) = 0$ and by Lemma 4

$$|\boldsymbol{\Omega}| \leqslant |R_i| \leqslant |S_i|.$$

Since $\xi^{(n_1, n_2)} \in \boldsymbol{\Omega}$, it follows

$$|\boldsymbol{Q}(\xi)| \leqslant (n_1, n_2)|\boldsymbol{\Omega}|.$$

Moreover $R_{3-j}(\xi^{n_j}) = 0$, $S_{3-j}(\xi^{n_j}) = 0$ and we get by Lemma 1 with $\boldsymbol{\Omega}_1 = \boldsymbol{Q}(\xi)$

$$\begin{aligned}|n_j|\, e(\xi, \boldsymbol{Q}(\xi)) &\leqslant e(\xi^{n_j}, \boldsymbol{Q}(\xi)) \leqslant (n_1, n_2)e(\xi^{n_j}, \boldsymbol{\Omega}) \\ &\leqslant (n_1, n_2)20|\boldsymbol{\Omega}|^2\log|\boldsymbol{\Omega}|^*\log\|S_{3-j}\| \\ &\leqslant (n_1, n_2)20|S_j|^2\log|S_j|^*\cdot 4|F|\log(|F|\,\|F\|) \\ &\leqslant (n_1, n_2)120(2|F|^*)^{2\|F\|-1}\log\|F\|,\end{aligned}$$

which completes the proof.

Proof of Theorem 3. If $\|F\| \leqslant 2$ then $s = 0$, $KF(x^{n_1}, x^{n_2}) = \text{const}$ and it suffices to take $N = \boldsymbol{I}_2$. Suppose therefore $\|F\| \geqslant 3$ and assume first

$$\frac{\max\{|n_1|, |n_2|\}}{(n_1, n_2)} > 120(2|F|^*)^{2\|F\|-1}\log\|F\|.$$

We apply Lemmata 12 and 13 to polynomial F and vector $[n_1, n_2]$. If $\boldsymbol{M} = [\mu_{ij}]$ is the matrix of Lemma 12 then $[n_1, n_2] = [v_1, v_2]\boldsymbol{M}$. Moreover

$$KF(y_1^{\mu_{11}}y_2^{\mu_{21}}, y_1^{\mu_{12}}y_2^{\mu_{22}}) \overset{\text{can}}{=} \text{const}\prod_{\sigma=1}^{s} F_\sigma(y_1, y_2)^{e_\sigma} \tag{77}$$

implies by Lemma 11

$$LF(y_1^{\mu_{11}}y_2^{\mu_{21}}, y_1^{\mu_{12}}y_2^{\mu_{22}}) \overset{\text{can}}{=} \text{const}\prod_{\sigma=1}^{s_0} F_\sigma(y_1, y_2)^{e_\sigma},$$

where $JF_\sigma(y_1^{-1}, y_2^{-1}) \neq \pm F_\sigma(y_1, y_2)$ for $\sigma \leqslant s_0$ exclusively, and by Lemma 12

$$LF(x^{n_1}, x^{n_2}) = \text{const}\prod_{\sigma=1}^{s_0} LF_\sigma(x^{v_1}, x^{v_2})^{e_\sigma}, \tag{78}$$

the polynomials $LF_\sigma(x^{v_1}, x^{v_2})$ are relatively prime in pairs and either irreducible or constant.

By Lemma 13, $KF(x^{n_1}, x^{n_2}) = LF(x^{n_1}, x^{n_2})$, thus

$$KF_\sigma(x^{v_1}, x^{v_2}) = LF_\sigma(x^{v_1}, x^{v_2}) \quad (\sigma \leqslant s_0)$$

and we get

$$KF(x^{n_1}, x^{n_2}) = \text{const}\prod_{\sigma=1}^{s_0} KF_\sigma(x^{v_1}, x^{v_2})^{e_\sigma}.$$

If none of $LF_\sigma(x^{v_1}, x^{v_2})$ $(\sigma \leqslant s_0)$ is constant we set $\boldsymbol{N} = \boldsymbol{M}$. By (42) and (43), (i) and (ii) hold. As to (iii) it remains to prove $s_0 = s$. Supposing contrarywise that

$$F_s(y_1, y_2) = \pm JF_s(y_1^{-1}, y_2^{-1})$$

we obtain

$$D(z_1, z_2) = JF_s(z_1^{\mu_{22}}z_2^{-\mu_{21}}, z_1^{-\mu_{12}}z_2^{\mu_{11}}) = \pm JF_s(z_1^{-\mu_{22}}z_2^{\mu_{21}}, z_1^{\mu_{12}}z_2^{-\mu_{11}}).$$

On the other hand, by (77), $F_s(y_1, y_2)$ divides $f(y_1^{\mu_{11}}y_2^{\mu_{21}}, y_1^{\mu_{12}}y_2^{\mu_{22}})$ where $f(x_1, x_2)$ is a certain irreducible factor of $KF(x_1, x_2)$. By the assumption $KF(x_1, x_2) = LF(x_1, x_2)$ we have

$$\big(f(x_1, x_2), Jf(x_1^{-1}, x_2^{-1})\big) = 1 \quad \text{and} \quad \big(JF(z_1^{|\boldsymbol{M}|}, z_2^{|\boldsymbol{M}|}), JF(z_1^{-|\boldsymbol{M}|}, z_2^{-|\boldsymbol{M}|})\big) = 1.$$

On substituting $y_1 = z_1^{\mu_{22}}z_2^{-\mu_{21}}$, $y_2 = z_1^{-\mu_{12}}z_2^{\mu_{11}}$ we infer that $D(z_1, z_2)$ divides $JF(z_1^{|\boldsymbol{M}|}, z_2^{|\boldsymbol{M}|})$ and $JF(z_1^{-|\boldsymbol{M}|}, z_2^{-|\boldsymbol{M}|})$, thus $D(z_1, z_2) = \text{const}$ and since the substitution is invertible $(|\boldsymbol{M}| \neq 0)$, $F_s(y_1, y_2) = \text{const}$, a contradiction.

If some $LF(x^{v_1}, x^{v_2})$ is constant then we have by Lemma 10

$$\frac{\max\{|v_1|, |v_2|\}}{(v_1, v_2)} \leqslant 2|F_\sigma| \leqslant 4|F|\,h(\boldsymbol{M}). \tag{79}$$

In this case we set $r = 1$,

$$\boldsymbol{N} = \left[\frac{n_1}{(v_1, v_2)}, \frac{n_2}{(v_1, v_2)}\right]$$

so that (ii) is clearly satisfied. By (42), (43) and (79)

$$h(\boldsymbol{N}) \leqslant 8|F|\,h(\boldsymbol{M})^2 \leqslant 8|F|\exp(9\cdot 2^{\|F\|-3}),$$

thus (i) holds. Finally by (78)

$$KF(x^{n_1/(v_1,v_2)}, x^{n_2/(v_1,v_2)}) = \text{const}\prod_{\sigma=1}^{s_0} KF_\sigma(x^{v_1/(v_1,v_2)}, x^{v_2/(v_1,v_2)})^{e_\sigma},$$

where the polynomials $KF_\sigma(x^{v_1/(v_1,v_2)}, x^{v_2/(v_1,v_2)})$ are relatively prime in pairs and irreducible or constant simultaneously with $KF_\sigma(x^{v_1}, x^{v_2})$. This proves (iii).

Assume now that

$$\frac{\max\{|n_1|, |n_2|\}}{(n_1, n_2)} \leqslant 120(2|F|^*)^{2\|F\|-1}\log\|F\| = m \tag{80}$$

and set

$$F'(x) = JF(x^{n_1/(n_1,n_2)}, x^{n_2/(n_1,n_2)}). \tag{81}$$

Clearly

$$|F'| \leqslant 2|F|\,m$$

and by (8) and (9)

$$\|F'\| \leqslant \max_{0\leqslant\varphi\leqslant 2\pi}|F'(e^{i\varphi})|^2 \leqslant \max_{0\leqslant\vartheta\leqslant 2\pi}|F(e^{i\vartheta_1}, e^{i\vartheta_2})|^2 \leqslant \|F\|^2.$$

Let ξ be a zero of $F'(x)$. If ξ^{-1} is not conjugate to ξ, then by Lemma 1

$$e(\xi, \boldsymbol{Q}(\xi)) \leqslant \tfrac{5}{2}|F'|\log\|F'\| \leqslant 10|F|\,m\log\|F\|.$$

If ξ^{-1} is conjugate to ξ, then ξ is a zero of

$$\frac{KF(x^{n_1/(n_1,n_2)}, x^{n_2/(n_1,n_2)})}{LF(x^{n_1/(n_1,n_2)}, x^{n_2/(n_1,n_2)})}$$

and by Lemma 13

$$e(\xi, \boldsymbol{Q}(\xi)) \leqslant m.$$

In both cases

$$e(\xi, \boldsymbol{Q}(\xi)) \leqslant 600(2|F|^*)^{2\|F\|}\log^2\|F\|, \tag{82}$$

$$\log e(\xi, \boldsymbol{Q}(\xi)) \leqslant 3\|F\|\,|F|^*. \tag{83}$$

Put

(84) $$\nu = \left(n_1, n_2, \max 2^{e(\xi, Q(\xi))-1} e(\xi, Q(\xi))!\right), \qquad (n_1, n_2) = \nu v,$$

where the maximum is taken over all zeros ξ of $F(x)$.

It follows like in the proof of Theorem 1 that

$$KF'(x^{\nu}) \overset{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} F_{\sigma}(x)^{e_{\sigma}}$$

implies

(85) $$KF'(x^{(n_1, n_2)}) \overset{\text{can}}{=} \text{const} \prod_{\sigma=1}^{s} F_{\sigma}(x^{v})^{e_{\sigma}}$$

(since $v > 0$, $KF_{\sigma}(x^{v}) = JF_{\sigma}(x^{v}) = F_{\sigma}(x^{v})$). Set

$$N = \left[\frac{n_1}{(n_1, n_2)}, \frac{n_2}{(n_1, n_2)}\right]\nu.$$

We get from (80), (82), (83) and (84)

$$\begin{aligned} h(N) &\leqslant m \max e(\xi, Q(\xi))^{e(\xi, Q(\xi))} \\ &\leqslant \exp\{3\|F\|\,|F|^{*} + 900\,(2\,|F|^{*})^{2\|F\|+1}\|F\|\log^2\|F\|\} \\ &\leqslant \exp\{500\,(2\,|F|^{*})^{2\|F\|+1}\|F\|^2\}, \end{aligned}$$

thus (i) holds. (ii) is clear from (84). Finally by (81)

$$KF(x^{\nu_{11}}, x^{\nu_{12}}) = KF'(x^{\nu}), \qquad KF(x^{n_1}, x^{n_2}) = KF'(x^{(n_1, n_2)})$$

and (iii) follows from (85).

§ 5. LEMMA 14. *If $k \geqslant 2$, $a_j \neq 0$ $(0 \leqslant j \leqslant k)$ are complex numbers and $M = [\mu_{ij}]$ is an integral nonsingular matrix of degree k then*

$$J\left(a_0 + \sum_{j=1}^{k} a_j \prod_{i=1}^{k} z_i^{\mu_{ij}}\right)$$

is absolutely irreducible.

Proof. We may assume without loss of generality that $|M| > 0$. Suppose that there is a factorization

$$J\left(a_0 + \sum_{j=1}^{k} a_j \prod_{i=1}^{k} z_i^{\mu_{ij}}\right) = T(z_1, \ldots, z_k)\, U(z_1, \ldots, z_k),$$

where $T \neq \text{const}$, $U \neq \text{const}$.

Setting

$$z_i = \prod_{h=1}^{k} y_h^{\mu'_{hi}}, \quad \text{where } [\mu'_{hi}] = |M|\cdot M^{-1}$$

we obtain

(86) $$a_0 + \sum_{j=1}^{k} a_j y_j^{|M|} = T'(y_1, \ldots, y_k)\, U'(y_1, \ldots, y_k),$$

where

$$T' = JT\left(\prod_{h=1}^{k} y_h^{\mu'_{h1}}, \ldots, \prod_{h=1}^{k} y_h^{\mu'_{hk}}\right) \neq \text{const},$$

$$U' = JU\left(\prod_{h=1}^{k} y_h^{\mu'_{h1}}, \ldots, \prod_{h=1}^{k} y_h^{\mu'_{hk}}\right) \neq \text{const}.$$

However (86) is impossible since as follows from Capelli's theorem already

$$a_0 + a_1 y_1^{|M|} + a_2 y_2^{|M|}$$

is absolutely irreducible (cf. [14]).

Remark. The following generalization of the lemma seems plausible.

If $a_j \neq 0$ $(0 \leqslant j \leqslant k)$ are complex numbers and the rank of an integral matrix $[\mu_{ij}]_{\substack{i \leqslant l \\ j \leqslant k}}$ exceeds $(k+1)/2$, then

$$J\left(\sum_{j=0}^{k} a_j \prod_{i=1}^{l} z^{\mu_{ij}}\right)$$

is absolutely irreducible.

Proof of Theorem 4. Set in Lemma 12:

$$F(x_1, \ldots, x_k) = a_0 + \sum_{j=1}^{k} a_j x_j$$

and let M be the matrix of that lemma. Since by Lemma 14

$$JF\left(\prod_{i=1}^{h} y_i^{\mu_{i1}}, \ldots, \prod_{i=1}^{k} y_i^{\mu_{ik}}\right)$$

is irreducible, we conclude that either $LF(x^{n_1}, \ldots, x^{n_k})$ is irreducible or constant or $\gamma n = 0$ with

$$0 < h(\gamma) < \begin{cases} 120\,(2\,|F|^{*})^{2\|F\|-1}\log\|F\| & \text{if } k = 2, \\ \exp_{2k-4}(7k\,|F|^{*\|F\|-1}\log\|F\|) & \text{if } k > 2. \end{cases}$$

If however $LF(x^{n_1}, \ldots, x^{n_k})$ is constant we obtain the relation $\gamma n = 0$ from Lemma 10. Taking into account that $|F|^{*} = 2$, $\|F\| = \sum_{j=0}^{k} a_j^2$, we get the theorem.

Proof of Theorem 5. It follows from Theorem 4 that $L(ax^n+ +bx^m+c)$ is irreducible unless

$$\frac{\max\{n,m\}}{(n,m)} \leqslant 2^{4(a^2+b^2+c^2)+5}\log(a^2+b^2+c^2).$$

On the other hand, by Lemma 13 (with $F(x_1,x_2) = ax_1+bx_2+c$)

$$K(ax^n+bx^m+c) = L(ax^n+bx^m+c)$$

unless

$$\frac{\max\{n,m\}}{(n,m)} \leqslant 120\cdot 4^{2(a^2+b^2+c^2)-1}\log(a^2+b^2+c^2)$$
$$\leqslant 2^{4(a^2+b^2+c^2)+5}\log(a^2+b^2+c^2).$$

This proves the first part of the theorem. To obtain the second part we apply Theorem 3 with $F(x_1,x_2) = ax_1+bx_2+c$. In view of Lemma 14 and the reducibility of $K(ax^n+bx^m+c)$, the matrix $\boldsymbol{N}$ is of rank 1 and we have

$$h(\boldsymbol{N}) \leqslant \exp\{500(2|F|^*)^{2\|F\|+1}\|F\|^2\} \leqslant \exp\left(2^{4(a^2+b^2+c^2)+11}(a^2+b^2+c^2)^2\right).$$

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Note added in proof. The original result of [1] concerning an algebraic integer α of degree n is

$$\overline{|\alpha|} > 1+(40n^2\log n)^{-1} \quad (n>1).$$

This implies the inequality

$$\overline{|\alpha|} > 1+(40n^2\log n^*-1)^{-1}$$

used in the proof of Lemma 1 since $40n^2\log(n^*/n) > 1$ for $n>1$. For completeness we list below the modifications needed in [3] in order to obtain the inequality

$$\overline{|\alpha|} > 1+(5n-1)^{-1}$$

used in the same proof.

Inequality (2.4) should be replaced by

$$1 < \varrho \leqslant 1+\frac{1}{5n-1}$$

(this is permissible since $\varrho = 5n/(5n-1)$ satisfies (2.1)). The right hand side of (3.2) should be replaced by $(\delta e^{1/e})^n$ (this is permissible since $t^{1/t} \leqslant e^{1/e}$ for all $t>0$). Inequality (4.4) and the preceeding formula should be replaced by

$$\delta = \left(1+\frac{1}{5n-1}\right)^2 - 1 = \frac{10n-1}{(5n-1)^2}, \qquad \Pi_1 \leqslant (\delta e^{1/e})^n.$$

The two inequalities following (4.5) should be replaced by

$$\varrho^{2n(n-1)} \leqslant \left(1+\frac{1}{5n-1}\right)^{2n(n-1)} \leqslant e^{2n/5},$$

$$\Pi_1\Pi_2 < (n\delta e^{1/e+2/5})^n < 1 \qquad (n>2).$$

For $n=2$ the lemma is true because then $\overline{|\alpha|} \geqslant \sqrt{2}$.

Corrigenda to [9]

p. 1 line 9. For "$f(x)$" read "$f(x) \neq 0$".
p. 3 lines 12 and 11 should read "and their totally complex quadratic extensions (in the latter case the condition $JF(y,z) \neq \pm\overline{JF(y^{-1},z^{-1})}$ should be replaced by $JF(y,z) \neq \mathrm{const}\,\overline{JF(y^{-1},z^{-1})}$".
p. 10 line 13. For "$F(x)$" read "$F(x) \neq 0$".
p. 11 lines 7–8. For "$G(y,z), H(y,z)$" read "$G(y,z) \neq 0, H(y,z) \neq 0$".
p. 23 formula (77). For "$KF(x^n, x)$" read "$KF(x^n, x^m)$".

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Approximate functional equation for Hecke's L-functions of quadratic field

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Introduction

1. The aim of the present paper is to prove an approximate functional equation for the Hecke's L-functions $\zeta(s, \chi)$ of any quadratic field K. That equation being merely an auxiliary result(¹) we will confine ourselves to proving it merely on the line $\sigma = \frac{1}{2}$ in the plane of complex numbers $s = \sigma + it$. Having such a very limited purpose in proving the result, we shall not give here a full account of the existing papers about approximate functional equations in general, since none of them would do just as well for the applications which we have in view(²).

In 1961 Linnik ([10], § 40) proved a shortened functional equation for the Dirichlet L-function $L(s, \chi)$ with a primitive character $\chi \bmod D$ on the line $\sigma = \frac{1}{2} + it$ with $t \ll 1$ and D unbounded(³). Using the incomplete Γ-function Lavrik [8] proved the analogous result for all s in the strip $0 < \sigma < 1$. He gave [9] also the corresponding result for Hecke's L-functions with Grössencharakter of imaginary quadratic field. But if the functional equation contains a higher power of Γ-function than the first one, his method does not give satisfactory results, since then the corresponding residue sums do not represent familiar functions.

In the present paper(⁴) we shall prove the following

(¹) Which will be used in a later paper for the proof of a sieve theorem of Bombieri's type (see [1], Theorem 4) but for the set of primes which are representable by a given quadratic form.

(²) The result of Lavrik [9] (for example) concerns merely the imaginary quadratic field and the simplest case (out of three possible cases) in the real quadratic field (see further §§ 5 and 6).

(³) With the restriction $\sigma = 1/2$, $t \ll 1$ Linnik's method is applicable to Hecke's L-functions of any algebraic field. See further § 11.

(⁴) A short description of the method and results of the present paper has been given in [4].