4.3. Theorem. *Suppose $\{a_k\}$ is generated by a sieve which satisfies the following conditions:*

(a) *For each $k > 1$, $\Delta_k = o(a(k) - a(k-1))$.*

(b) $a(k) \sim s_k(1) \sim c(a_k)^a(\log a_k)^b$ *for $1 < a < e$ and $c > 0$.*

Then $a_k \sim k \log k$.

It should be pointed out that the theorem above cannot yield a proof of the prime number theorem since we know that the first number eliminated at the kth sieving is $(p_k)^2$. This fact together with the second condition in the above theorem immediately implies the prime number theorem. It would be interesting to know whether $\Delta_k = o(a(k) - a(k-1))$ holds for the prime sieve. Since $a(k) = \frac{1}{2}(a_k)^2/\log(a_k)$ and $a_k \asymp k \log k$, we are asking whether $\Delta_k = o(k \log k)$. This question has already been posed by Buschman [2] and some computational evidence made by the author seems to indicate that the condition holds.

Bibliography

[1] W. E. Briggs, *Prime-like sequences generated by a sieve process*, Duke Math. Journ. 30 (1963), pp. 297–312.

[2] R. G. Buschman, *A uniformity condition for sieving prime-like sequences*. (Submitted for publication.)

[3] — and M. C. Wunderlich, *Sieve-generated sequences with translated intervals*, Canad. Journ. Math. 19 (1967), pp. 559–570.

[4] — — *Sieves with generalized intervals*, Boll. Un. Mat. Ital. 21 (1967), pp. 362–367.

[5] P. Erdös and E. Jabotinsky, *On sequences generated by a sieving process*, Nederl. Akad. Wetensch. Indag. Math. 61 (1958), pp. 115–128.

[6] M. C. Wunderlich, *Sieve-generated sequences*, Canad. Journ. Math. 18 (1966), pp. 291–299.

[7] — *Sieving procedures on digital computers*, J. Assoc. Comput. Mach. 14 (1967), pp. 10–19.

[8] — and W. E. Briggs, *Second and third term approximations of sieve-generated sequences*, Ill. Journ. Math. 10 (1966), pp. 694–700.

[9] — *Sieve-generated sequences with a limiting distribution property*, Journ. London Math. Soc. 43 (1968), pp. 339–346.

NORTHERN ILLINOIS UNIVERSITY
De Kalb, Illinois

Reçu par la Rédaction le 6. 8. 1968

ACTA ARITHMETICA
XVI (1969)

On a question related to diophantine approximation

by

Donald L. Goldsmith (Cambridge)

1. Introduction. In an extension of a result of Cassels [1], Davenport [4] proved the following theorem on simultaneous diophantine approximation. Let $\lambda_q^{(1)}, \ldots, \lambda_q^{(k)}$ $(q = 1, \ldots, r)$ be r sets of k real numbers. Then there exist continuum-many sets of real numbers $a_1, \ldots, a_k$ such that

$$\max_{1 \leqslant j \leqslant k} \|(a_j + \lambda_q^{(j)})u\|^k > C/u \tag{1.1}$$

for every integer $u > 0$, and for $q = 1, \ldots, r$, where C is a positive constant depending on r and k, and $\|x\|$ represents the distance from x to the nearest integer.

As was also noted in [4], relation (1.1) has a simple geometrical interpretation. Let L_q $(q = 1, \ldots, r)$ be r lines through the origin in $(k+1)$-dimensional space defined by the equations

$$x_j - \lambda_q^{(j)} x_0 = 0 \quad (j = 1, \ldots, k), \tag{1.2}$$

and suppose that we surround each of these lines L_q by a tube

$$|x_j - \lambda_q^{(j)} x_0| < \min(1, |x_0|^{-1/k}) \quad (j = 1, \ldots, k). \tag{1.3}$$

Then relation (1.1) implies that there exist continuum-many lattices with no point (except the origin O) in any of the tubes. In fact, we may define the lattices by

$$C^{1/(k+1)} x_0 = u_0, \quad C^{1/(k+1)} x_j = u_j - a_j u_0 \quad (j = 1, \ldots, k). \tag{1.4}$$

Now by calling upon a standard transference principle (see, for example, [2], chapter 5, section 2), Davenport showed that (1.1) is equivalent to

$$\Big\| \sum_{j=1}^{k} (a_j + \lambda_q^{(j)}) u_j \Big\| > C_1 (\max_j |u_j|)^{-k}, \tag{1.5}$$

for some constant $C_1 > 0$, and all sets of k integers $u_1, \ldots, u_k$, not all 0. Relation (1.5) has a geometric interpretation dual to that of (1.1). Namely, if P_q $(q = 1, \ldots, r)$ are r hyperplanes in $(k+1)$-space defined by the equations

$$x_0 + \lambda_q^{(1)} x_1 + \ldots + \lambda_q^{(k)} x_k = 0, \tag{1.6}$$

and if we place a layer around each of these hyperplanes:

$$|x_0 + \lambda_q^{(1)} x_1 + \ldots + \lambda_q^{(k)} x_k| < (1 + \max_j |x_j|)^{-k}, \tag{1.7}$$

then there exist continuum-many lattices with no points except the origin in any of these layers. Here we may define the lattices by

$$C_1^{1/(k+1)} x_0 = u_0 + a_1 u_1 + \ldots + a_k u_k, \quad C_1^{1/(k+1)} x_j = u_j \quad (j = 1, \ldots, k). \tag{1.8}$$

Since the two types of lattice, (1.4) and (1.8), given by Davenport are inconsistent, the question was asked in [4] whether, given any finite set of lines and hyperplanes through O, there was a lattice that would simultaneously avoid the tubes and layers around them. We will show here that such a lattice does exist, in fact there are continuum-many of them.

We would like to thank Professor Davenport for a number of very useful suggestions.

2. The main theorem. We will state and prove our principal result first for 3-dimensional space, since this will contain all the essentials of the general case. The general case will be presented in Section 4.

THEOREM 1. *Let r lines L_q, defined by the equations*

$$y - \lambda_q x = 0, \quad z - \mu_q x = 0 \quad (q = 1, \ldots, r), \tag{2.1}$$

and s planes P_t, defined by

$$x + \theta_t y + \varphi_t z = 0 \quad (t = 1, \ldots, s), \tag{2.2}$$

be given, each passing through O. Then there exists a lattice Λ and a positive number ϱ such that every point (x, y, z) of Λ other than O satisfies

(1) $\max(|y - \lambda_q x|, |z - \mu_q x|) > \varrho \min(1, |x|^{-1/2})$ $(q = 1, \ldots, r)$,

(2) $|x + \theta_t y + \varphi_t z| > \varrho (1 + \max\{|x|, |y|, |z|\})^{-2}$ $(t = 1, \ldots, s)$.

In fact, the set of all such lattices has the cardinal of the continuum.

Proof. If

$$a \equiv (a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}, a_{13}, a_{23}, a_{33})$$

is a point in 9-dimensional space for which the three vectors

$$(a_{1j}, a_{2j}, a_{3j}) \quad (j = 1, 2, 3),$$

are independent, then we may associate with a the lattice $\Lambda = \Lambda(a)$ of points (x, y, z) given by

$$\begin{aligned} x &= a_{11} u + a_{12} v + a_{13} w, \\ y &= a_{21} u + a_{22} v + a_{23} w, \\ z &= a_{31} u + a_{32} v + a_{33} w, \end{aligned} \tag{2.3}$$

as u, v, w assume all integer values. Moreover, about each such point a is a neighborhood $N(a)$ in 9-space in which the correspondence between points and lattices is one-to-one. That is, each point a' in $N(a)$ corresponds to a basis of a lattice Λ, and no two points a', a'' in $N(a)$ correspond to bases of the same lattice. Conversely, a lattice Λ given by (2.3) defines a point a in 9-space such that $\Lambda = \Lambda(a)$.

We will denote by $T(a)$ the linear transformation (2.3) with matrix $[a_{ij}]$, and write $\Delta = \Delta(a)$ for $\det a_{ij}$.

We define the distance between two points a, a' to be

$$|a - a'| = \max_{i,j} |a_{ij} - a'_{ij}|.$$

With the point a in 9-space we associate also the *adjoint point* a^*, defined by

$$a^*_{ij} = \frac{A_{ij}}{\Delta(a)},$$

where A_{ij} is the co-factor of a_{ij}. If the coordinates of a, three-by-three, determine three independent vectors, then the same is true also for a^*. The lattices $\Lambda(a), \Lambda(a^*)$ are then polar lattices, and $T(a), T(a^*)$ are polar transformations. We will denote the function from a to a^* by π.

For the moment, we reduce our problem to one involving only lines by replacing each of the planes P_t by its polar reciprocal L_t^*, the line through the origin and perpendicular to P_t. Let L_t^* be given by the equations

$$y - \lambda_t^* x = 0, \quad z - \mu_t^* x = 0. \tag{2.4}$$

We begin the search for a lattice Λ satisfying (1) and (2) by choosing a lattice Λ_0, a point a_0 such that $\Lambda_0 = \Lambda(a_0)$, and a (small) positive number δ_0 so that the following conditions are satisfied. We will impose further restrictions on δ_0 as the proof progresses.

(I) $\Delta(a_0) \neq 0$ and δ_0 is small enough so that the correspondence between points a and lattices $\Lambda(a)$ described earlier is one-to-one for all a in the hypercube $\mathscr{C}_0 = \mathscr{C}_0(a_0, 2\delta_0)$ of center a_0 and edge length $2\delta_0$. In fact, δ_0 is small enough so that there exists a number $D > 0$ such that $|\Delta(a)| > D$ for all a in $\mathscr{C}_0$ and all a in $\pi(\mathscr{C}_0)$.

(II) There exist positive constants K, K^* such that (i) if $\mathscr{C}(a, l)$ is any hypercube in $\mathscr{C}_0$, we may extract from $\pi(\mathscr{C})$ a hypercube $\mathscr{D}(a^*, l')$ such that $l' = K^* l$, and similarly (ii) if $\mathscr{D}(a^*, l')$ is any hypercube in $\pi(\mathscr{C}_0)$, we may extract a hypercube $\mathscr{C}(a, l)$ from $\pi^{-1}(\mathscr{D})$ such that $l = Kl'$.

(III) There exists a positive number H such that the set of determinants

$$\begin{vmatrix} a_{2i} - \lambda a_{1i} & a_{3i} - \mu a_{1i} \\ a_{2j} - \lambda a_{1j} & a_{3j} - \mu a_{1j} \end{vmatrix} \tag{2.5}$$

are all in absolute value greater than H, where (i, j) may take the values $(1, 2), (1, 3), (2, 3)$, where $a = (a_{ij})$ is any point in $\mathscr{C}_0$ or in $\pi(\mathscr{C}_0)$, and where λ, μ runs over the $r+s$ pairs of numbers λ_q, μ_q and λ_t^*, μ_t^* given in (2.1) and (2.4).

Condition I implies that the Jacobian of π is not zero in $\mathscr{C}_0$. Condition II is merely a restatement of the fact that π is approximately linear around a_0 (this follows from I). Without loss of generality we may assume that none of the lines L_q or L_t^* lies in the y, z-plane. Condition III then has the effect of not permiting any of the basis vectors of $\Lambda(a)$, for $a \in \mathscr{C}_0$ or $a \in \pi(\mathscr{C}_0)$, to be too close to any L_q or L_t^*, and, in fact, of not permitting the plane determined by any pair of basis vectors of $\Lambda(a)$ to be too close to any L_q or L_t^*.

The scheme of the proof is as follows. We will construct two nested sequences of hypercubes in 9-space, $\mathscr{C}_n(a_n, l_n)$ and $\mathscr{D}_n(a_n^*, l_n')$, with

$$\begin{gathered} l_n \equiv 2\delta_n = l_0 R^{-3n/2}, \quad l_n' \equiv 2\delta_n' = l_0' R^{-3n/2}, \\ l_n = K l_n', \quad \max(l_0, l_0') < 2K_1^{1/2} R^{-3/2}, \end{gathered} \tag{2.6}$$

where $R > 1$ and K_1 are constants which will be chosen later. Moreover, for any a in $\mathscr{C}_n$, no point of $\Lambda(a)$ (except O) which is the image under $T(a)$ of an integral point (u, v, w) with $\max(|u|, |v|, |w|) < R^n$ will fall in any of the tubes

$$|y - \lambda_q x| < \varrho_1 \min(1, |x|^{-1/2}), \quad |z - \mu_q x| < \varrho_1 \min(1, |x|^{-1/2}) \tag{2.7}$$

$q = 1, \ldots, r)$ around the lines L_q. ϱ_1 is a fixed small positive number which will be chosen later, and which will depend only on Λ_0. Similarly, for every a in $\mathscr{D}_n$, no point of $\Lambda(a)$ except O which arises from one of these integral points will fall in any of the tubes

$$|y - \lambda_t^* x| < \varrho_1 \min(1, |x|^{-1/2}), \quad |z - \mu_t^* x| < \varrho_1 \min(1, |x|^{-1/2}) \tag{2.8}$$

$(t = 1, \ldots, s)$ around the lines L_t^*. The intersection of the sequence of hypercubes $\{\mathscr{C}_n\}$ will yield a point a, and the intersection of $\{\mathscr{D}_n\}$ will define a point a^* (the point defined by $\bigcap \mathscr{D}_n$ will, indeed, be the adjoint of the point given by $\bigcap \mathscr{C}_n$, since the centers of $\mathscr{C}_n$ and $\mathscr{D}_n$ are adjoints

for every n). $\Lambda(a)$ will have no point except O in the tubes (2.7) around the lines L_q, and $\Lambda(a^*)$ will avoid the tubes (2.8) around the lines L_t^*. We will conclude the proof by using the relation between polar reciprocal bodies and polar lattices to show that $\Lambda(a)$ also has no point other than O in layers about the planes P_t.

LEMMA. *Suppose* $\bar{a} \in \mathscr{C}_0$ $[\bar{a} \in \pi(\mathscr{C}_0)$, resp.$]$ *and suppose that* U *is a positive number. Let* L *be any of the lines* L_q *or* L_t^*; *say* L *is defined by the equations* $y - \lambda x = 0$, $z - \mu x = 0$. *Then there exist constants* A, B, C, *not all zero, such that if* $\Lambda(a)$ *is any lattice for which*

(a) $a \in \mathscr{C}_0$ $[a \in \pi(\mathscr{C}_0)$, resp.$]$, $|a - \bar{a}| < K_1^{1/2} U^{-3/2}$,

(b) $T(a)$ *carries some integer point* (u, v, w) $(\neq O)$ *to* L, *where* $\max(|u|, |v|, |w|) < U$,

then the coordinates of a *satisfy*

$$\begin{aligned} &A[(a_{22} - \lambda a_{12})(a_{33} - \mu a_{13}) - (a_{23} - \lambda a_{13})(a_{32} - \mu a_{12})] + \\ &+ B[(a_{23} - \lambda a_{13})(a_{31} - \mu a_{11}) - (a_{21} - \lambda a_{11})(a_{33} - \mu a_{13})] + \\ &+ C[(a_{21} - \lambda a_{11})(a_{32} - \mu a_{12}) - (a_{22} - \lambda a_{12})(a_{31} - \mu a_{11})] = 0. \end{aligned} \tag{2.9}$$

Remark. A, B, C will depend on $\bar{a}$, U, and L, but not on $\Lambda(a)$ or on (u, v, w). The lemma really makes a statement about a collection of points which are close together. In the proof of Theorem 1, the role of $\bar{a}$ will be played by a_n and a_n^*, the centers of $\mathscr{C}_n$ and $\mathscr{D}_n$. The constant K_1 is defined by

$$K_1 = H 2^{-3} 3^{-2} \{1 + \max(|\lambda|, |\mu|)\}^{-2},$$

where λ, μ runs over the set of pairs λ_q, μ_q and λ_t^*, μ_t^* which define L_q and L_t^*. K_1 depends only on Λ_0 and on the lines L_q and L_t^*.

Proof of the lemma. For simplicity, let us assume first that L is the x-axis and $\bar{a} \in \mathscr{C}_0$. Suppose that $\Lambda(a), \Lambda(a'), \Lambda(a'')$ are three lattices, with a, a', a'' in $\mathscr{C}_0$, such that $|a - \bar{a}| < \delta$, and similarly for a' and a'', where $\delta < K_1^{1/2} U^{-3/2}$, and such that $T(a), T(a'), T(a'')$ carry the integral points (u, v, w), (u', v', w'), (u'', v'', w''), respectively, to L, where $\max(|u|, |v|, |w|) < U$, and similarly for (u', v', w') and (u'', v'', w'').

Let us write

$$G = \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix}.$$

We have then

$$G \begin{vmatrix} 1 & a_{21} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & a_{23} & a_{33} \end{vmatrix} = \begin{vmatrix} u & 0 & 0 \\ u' & \beta_{21}' u' + \beta_{22}' v' + \beta_{23}' w' & \beta_{31}' u' + \beta_{32}' v' + \beta_{33}' w' \\ u'' & \beta_{21}'' u'' + \beta_{22}'' v'' + \beta_{23}'' w'' & \beta_{31}'' u'' + \beta_{32}'' v'' + \beta_{33}'' w'' \end{vmatrix}, \tag{2.10}$$

where $\beta'_{ij} = a_{ij} - a'_{ij}$, $\beta''_{ij} = a_{ij} - a''_{ij}$. It follows (see condition III) that $|G| H \leqslant 2^3 3^2 \delta^2 U^3$, and therefore that $|G| < 1$. But the entries of G are all integers, so G must be an integer. Hence $G = 0$. We conclude that all integral points (u, v, w), with $\max(|u|, |v|, |w|) < U$, which are mapped to L by some $T(a)$ for which $a \in \mathscr{C}_0$ and $|a - \bar{a}| < K_1^{1/2} U^{-3/2}$, satisfy a relation

$$Au + Bv + Cw = 0, \tag{2.11}$$

with A, B, C not all zero. In other words, all such integer points lie on a plane through the origin. A, B, C do not depend on a or on (u, v, w).

Since $T(a)$ carries (u, v, w) to the x-axis, we have the relations

$$\begin{aligned} 0 &= a_{21} u + a_{22} v + a_{23} w, \\ 0 &= a_{31} u + a_{32} v + a_{33} w. \end{aligned} \tag{2.12}$$

From (2.11) and (2.12) we have

$$A(a_{22}a_{33} - a_{23}a_{32}) + B(a_{23}a_{31} - a_{21}a_{33}) + C(a_{21}a_{32} - a_{22}a_{31}) = 0, \tag{2.13}$$

which is (2.9) in the special case $\lambda = \mu = 0$.

For the general case when L is given by $y - \lambda x = 0$, $z - \mu x = 0$, we replace the left-hand side of (2.10) by

$$G \begin{vmatrix} 1 & a_{21} - \lambda a_{11} & a_{31} - \mu a_{11} \\ 0 & a_{22} - \lambda a_{12} & a_{32} - \mu a_{12} \\ 0 & a_{23} - \lambda a_{13} & a_{33} - \mu a_{13} \end{vmatrix}$$

and proceed as before. That proves the lemma.

Returning to the proof of Theorem 1, we will show by induction that we may construct the sequences of hypercubes $\mathscr{C}_n$, $\mathscr{D}_n$ with the required properties (i.e., those given in the sketch of the proof). For the case $n = 0$, we already have a hypercube $\mathscr{C}_0(a_0, l_0)$. From $\pi(\mathscr{C}_0)$ extract a hypercube $\mathscr{D}_0(a_0^*, l'_0)$, and from $\pi^{-1}(\mathscr{D}_0)$ extract a hypercube $\mathscr{C}_0^{(1)}(a_0, l_0^{(1)})$, with $l_0^{(1)} = Kl'_0$, and such that $\max(l_0^{(1)}, l'_0) < 2K_1^{1/2} R^{-3/2}$. We now assume that $\mathscr{C}_0 = \mathscr{C}_0^{(1)}$ and $l_0 = l_0^{(1)}$. This is permissible since a reduction in the size of l_0 will not affect any of our previous work. Since there are no integer points (u, v, w) with $0 < \max(|u|, |v|, |w|) < R^0$, the properties required of $\mathscr{C}_0$ and $\mathscr{D}_0$ are satisfied vacuously.

Let us suppose then that $\mathscr{C}_n(a_n, l_n)$ and $\mathscr{D}_n(a_n^*, l'_n)$ have been constructed satisfying (2.6), and such that no $T(a)$, for $a \in \mathscr{C}_n$ or $a \in \mathscr{D}_n$, carries an integer point (u, v, w), with

$$1 \leqslant \max(|u|, |v|, |w|) < R^n, \tag{2.14}$$

into any of the tubes (2.7) or (2.8), respectively. Consider the set of all a in $\mathscr{C}_n$ for which $T(a)$ carries an integer point satisfying

$$R^n \leqslant \max(|u|, |v|, |w|) < R^{n+1} \tag{2.15}$$

to the line L_q. From the lemma, applied with $\bar{a} = a_n$ and $U = R^{n+1}$ (it is here that we need $\max(l_0, l'_0) < 2K_1^{1/2} R^{-3/2}$), we see that these a satisfy relation (2.9) for some A_q, B_q, C_q not all zero, with $\lambda = \lambda_q$, $\mu = \mu_q$.

Denote by σ_q, where $q = 1, \ldots, r$, the linear transformation that takes the point $(a_{11}, a_{21} - \lambda_q a_{11}, a_{31} - \mu_q a_{11}, a_{12}, a_{22} - \lambda_q a_{12}, a_{32} - \mu_q a_{12}, a_{13}, a_{23} - \lambda_q a_{13}, a_{33} - \mu_q a_{13})$ to the point $a = (a_{ij})$. Define σ^* $(t = 1, \ldots, s)$ analogously. Suppose $L_q = L_1$. The transformation $\pi\sigma_1$ carries $\mathscr{C}_n$ to a slightly distorted parallelepiped and takes the surface (2.9) (with $A, B, C, \lambda, \mu = A_1, B_1, C_1, \lambda_1, \mu_1$, resp.) to a plane. Hence we may choose a parallelepiped $\mathscr{P}$ from $\pi\sigma_1(\mathscr{C}_n)$, and choose from $\mathscr{P}$ an "octant" $\mathscr{P}'$ that avoids the plane. We may then choose a hypercube $\mathscr{C}_{n,1}(a_{n,1}, l_{n,1})$ from $\sigma_1^{-1}\pi^{-1}(\mathscr{P}')$. If l_0 is small enough, this can be done so that $l_{n,1} \geqslant 2^{-1-1/r} l_n$. No a in $\mathscr{C}_{n,1}$ satisfies (2.9) with $A, B, C, \lambda, \mu = A_1, B_1, C_1, \lambda_1, \mu_1$, resp.

We repeat the process, transforming $\mathscr{C}_{n,1}$ by $\pi\sigma_2$, extracting an octant to avoid the corresponding plane, and transforming back to get $\mathscr{C}_{n,2}$. Then we transform $\mathscr{C}_{n,2}$ by $\pi\sigma_3$, and so forth. After repeating this operation r times, we get a hypercube $\mathscr{C}_{n,r}(a_{n,r}, l_{n,r})$ with the property that if $a \in \mathscr{C}_{n,r}$, then a does not satisfy (2.9) with $A, B, C, \lambda, \mu = A_q, B_q, C_q, \lambda_q, \mu_q$, resp., for $1 \leqslant q \leqslant r$, and so $T(a)$ does not map any integral points satisfying (2.15) onto any of the lines $L_1, \ldots, L_r$.

(IV) We will assume that l_0 is small enough so that this process may be carried out with

$$l_{n,r} = 2^{-r-1} l_n.$$

Next we map $\mathscr{C}_{n,r}$ to the adjoint space, and we extract from $\pi(\mathscr{C}_{n,r})$ a hypercube $\mathscr{D}_{n,r}(a_{n,r}^*, l'_{n,r})$, where $l'_{n,r} = K^* l_{n,r}$. We now carry out the construction of $\mathscr{D}_{n,r+1}, \ldots, \mathscr{D}_{n,r+s}$, where $\mathscr{D}_{n,r+j} \supset \mathscr{D}_{n,r+j+1}$ $(j = 0, \ldots, s-1)$, using the transformations $\pi\sigma_1^*, \ldots, \pi\sigma_s^*$ and avoiding the appropriate planes. We have finally a hypercube $\mathscr{D}_{n,r+s}(a_{n,r+s}^*, l'_{n,r+s})$ where

(IV′) we assume that l_0 is small enough so that

$$l'_{n,r+s} = 2^{-s-1} l'_{n,r}.$$

(Assumptions IV and IV′ are possible since the transformations σ_q, σ_t^* are linear and, as noted before, π is approximated by a linear transformation in a small neighborhood of a_0.) For any a in $\mathscr{D}_{n,r+s}$, the corresponding $T(a)$ will not map any integral point satisfying (2.15) onto any of the lines $L_1^*, \ldots, L_s^*$. Finally, we extract from $\pi^{-1}(\mathscr{D}_{n,r+s})$ a hypercube $\mathscr{C}_{n,r+s}(a_{n,r+s}, l_{n,r+s})$, where $l_{n,r+s} = Kl'_{n,r+s}$. We have then

$$l_{n,r+s} = 2^{-(r+s+2)} KK^* l_n, \qquad l'_{n,r+s} = 2^{-(r+s+2)} KK^* l'_n.$$

We have shown that $T(\alpha)$ maps no integral point satisfying (2.15) onto any of the lines L_q if α is in $\mathscr{C}_{n,r+s}$, and none onto the lines L_t^* if α is in $\mathscr{D}_{n,r+s}$. In order that the lattices should have no points in the tubes surrounding these lines we will remove a border from $\mathscr{C}_{n,r+s}$ and $\mathscr{D}_{n,r+s}$.

Let K_2 be a positive constant, depending only on α_0 (for δ_0 small) such that if $|\alpha-\alpha_0|<\delta_0$ and if $T(\alpha)$ carries (u,v,w) to (x,y,z), then $x^2+y^2+z^2>K_2^2(u^2+v^2+w^2)$, and similarly if $T(\alpha)$ takes (u,v,w) to (x,y,z) where $|\alpha-\alpha_0^*|<\delta_0'$. Also, let K_3 be a positive constant depending on the lines L_q and L_t^* such that, if (x,y,z) lies in one of the tubes (2.7) or (2.8), then $|x|>K_3^2(x^2+y^2+z^2)^{1/2}$ (recall that we have assumed none of the lines L_q or L_t^* lies in the y,z-plane).

We construct $\mathscr{C}_{n+1}$ by removing from $\mathscr{C}_{n,r+s}$ a border of thickness β_n, and $\mathscr{D}_{n+1}$ by removing from $\mathscr{D}_{n,r+s}$ a border of thickness β_n', where

$$\beta_n=\varrho_1(K+1)K_2^{-1/2}K_3^{-1}R^{-3n/2},\qquad \beta_n'=K^{-1}\beta_n.$$

We have to show that this construction of the sequence of hypercubes $\mathscr{C}_n,\mathscr{D}_n$ is possible (roughly speaking, that the border is not too large to allow us to iterate the process), and that if α,α' are in $\mathscr{C}_{n+1},\mathscr{D}_{n+1}$, resp., then $\Lambda(\alpha),\Lambda(\alpha')$ avoid the tubes (2.7), (2.8) resp. (in other words, the border is large enough).

l_0 and l_0' have already been chosen to satisfy various conditions. To show that the construction of the hypercubes is possible, we must show that R and ϱ_1 may be chosen, independent of n, so that (2.6) holds. We need $l_{n+1}=l_0R^{-3(n+1)/2}$. Since

$$l_n=l_0R^{-3n/2}\quad\text{and}\quad l_{n+1}=2^{-(r+s+2)}KK^*l_n-2\varrho_1(K+1)K_2^{-1/2}K_3^{-1}R^{-3n/2},$$

we take

$$R^{3/2}=\frac{2^{r+s+3}}{KK^*},\qquad \varrho_1=\frac{KK^*K_2^{1/2}K_3l_0}{(K+1)2^{r+s+4}}. \tag{2.16}$$

(The order in which the constants are chosen is: first R, as in (2.16), then l_0 and l_0' to satisfy I–IV$'$ and $\max(l,l_0')<2K_1^{1/2}R^{-3/2}$, then ϱ_1 as in (2.16).) It is easy to verify that (2.6) is satisfied.

Finally, we must show that $\mathscr{C}_{n+1}$ and $\mathscr{D}_{n+1}$ carry the inductive property; that is, if α is in $\mathscr{C}_{n+1}$, then $T(\alpha)$ does not carry any integral point (u,v,w) satisfying (2.15) into any of the tubes (2.7), with an analogous result for α in $\mathscr{D}_{n+1}$. Suppose, on the contrary, that $T(\alpha)$ does carry such an integer point to (x,y,z), where (x,y,z) is within the tube (2.7) around L_q. Since $(u^2+v^2+w^2)^{1/2}\geqslant R^n$, it follows that $(x^2+y^2+z^2)^{1/2}>K_2R^n$, and so the point (x,y,z) differs from the point (x,λ_qx,μ_qx) on L_q by at most $\varrho_1K_2^{-1/2}K_3^{-1}R^{-n/2}$ with respect to each of the y- and z- coordinates. Thus, since $\max(|u|,|v|,|w|)\geqslant R^n$, we need vary the components α_{ij} of α by no more than $\varrho_1K_2^{-1/2}K_3^{-1}R^{-3n/2}$ to find a point α' such that $T(\alpha')$ carries (u,v,w) to L_q. But this is not possible, for α' would still be within the border around $\mathscr{C}_{n+1}$.

The reasoning for $\mathscr{D}_{n+1}$ is the same.

3. Completion of the proof. In the preceding section we constructed two nested sequences of hypercubes, $\mathscr{C}_n(\alpha_n,l_n)$, $\mathscr{D}_n(\alpha_n^*,l_n')$, with $l_n,l_n'\to0$. The two points α,α^* defined by their respective intersections are, as was noted earlier, adjoint points. Moreover, $T(\alpha)$ carries no integral point except O into any of the tubes (2.7), and $T(\alpha^*)$ carries none into the tubes (2.8). We will conclude the proof by demonstrating that $\Lambda(\alpha)$ has no point except O in any of the layers

$$|x+\theta_ty+\varphi_tz|<\varrho\{1+\max(|x|,|y|,|z|)\}^{-2} \tag{3.1}$$

around the planes P_t $(t=1,\ldots,s)$ for some suitably small positive number ϱ.

For simplicity, let us suppose that P_t is the y,z-plane, so that L_t^* is the x-axis. We would then like to show that $\Lambda(\alpha)$ has no points (except O) in the layer

$$|x|<\eta\{1+\max(|x|,|v|,|z|)\}^{-2}$$

for some $\eta>0$, or, equivalently, in the layer

$$|x|<\eta'(1+y^2+z^2)^{-1} \tag{3.2}$$

for some positive η'. It is, therefore, sufficient to show that there exists $\eta'>0$ such that, for every $W>1$, all points of $\Lambda(\alpha)$ other than O satisfy

$$y^2+z^2<W^2\Rightarrow|x|>\eta'W^{-2}. \tag{3.3}$$

Relation (3.3) will hold if there exists $\delta>0$ such that, for every $V>1$,

$$\delta^{-4/3}V^4x^2+\delta^{2/3}V^{-2}(y^2+z^2)>\delta^{2/3}. \tag{3.4}$$

The set of points defined by (3.4) is simply the exterior of the ellipsoid

$$\delta^{-4/3}V^4x^2+\delta^{2/3}V^{-2}(y^2+z^2)=1 \tag{3.5}$$

after shrinking by a factor $\delta^{1/3}$. So it is sufficient to show that δ exists such that, for every $V>1$, the ellipsoid (3.5) has first minimum $\gamma_1>\delta^{1/3}$ with respect to the lattice $\Lambda(\alpha)$.

The polar reciprocal of the ellipsoid (3.5) is the ellipsoid

$$\delta^{4/3}V^{-4}X^2+\delta^{-2/3}V^2(Y^2+Z^2)=1. \tag{3.6}$$

Suppose that $\gamma_1,\gamma_2,\gamma_3$ are the successive minima of the ellipsoid (3.5) with respect to the lattice $\Lambda(\alpha)$, and that ν_1,ν_2,ν_3 are the successive minima of (3.6) with respect to $\Lambda(\alpha^*)$. The numbers $\gamma_1,\gamma_2,\gamma_3,\nu_1,\nu_2,\nu_3$

depend on V. However, since the volume of the ellipsoids (3.5) and (3.6) is independent of V, we conclude from the well known results of Minkowski and Mahler (see, for example, [3], pages 218, 219), that there exist positive constants A_1, A_2, A_3, A_4, which depend on the dimension of the space (and, in the case of A_1 and A_2, on $\Lambda(\alpha)$ and $\Lambda(\alpha^*)$, hence ultimately on α_0), but not on V, such that

$$(3.7)\qquad \begin{gathered} A_1 \leqslant \gamma_1\gamma_2\gamma_3 \leqslant A_2, \quad A_1 \leqslant \nu_1\nu_2\nu_3 \leqslant A_2, \\ A_3 \leqslant \gamma_1\nu_3 \leqslant A_4, \quad A_3 \leqslant \gamma_2\nu_2 \leqslant A_4, \quad A_3 \leqslant \gamma_3\nu_1 \leqslant A_4. \end{gathered}$$

It follows then that $\gamma_1 \geqslant \nu_1^2 A_3/A_2$, so it is sufficient to show that $\nu_1 \geqslant \delta^{1/6}(A_2/A_3)^{1/2}$. This means that the ellipsoid (3.6), after having been shrunk by a factor $\delta^{1/6}(A_2/A_3)^{1/2}$, should contain no points of $\Lambda(\alpha^*)$ other than O. In other words, it is sufficient that for every $V > 1$ all points of $\Lambda(\alpha^*)$ except O should satisfy

$$(3.8)\qquad \delta^{4/3}V^{-4}X^2 + \delta^{-2/3}V^2(Y^2+Z^2) > \delta^{1/3}A_2/A_3.$$

Relation (3.8) will be satisfied if there exists $\delta > 0$ such that for every $V > 1$, and for every point ($\neq O$) of $\Lambda(\alpha^*)$, we have

$$(3.9)\qquad |X| < \delta^{-1/2}V^2(A_2/A_3)^{1/2} \Rightarrow Y^2+Z^2 > \delta V^{-2}(A_2/A_3).$$

Now we know already that $\Lambda(\alpha^*)$ has no point except O in a tube around the X-axis. Suppose this tube is given by

$$(3.10)\qquad Y^2+Z^2 < \varrho_2/(1+|X|)$$

for some sufficiently small ϱ_2. This means that for some small ϱ_3, for every point of $\Lambda(\alpha^*)$ except O and every $0 < W < \varrho_3(A_3/A_2)$ we have

$$(3.11)\qquad Y^2+Z^2 < W^2 \Rightarrow |X| > \varrho_3 W^{-2}.$$

This is just relation (3.9) with $\delta = \varrho_3^2(A_3/A_2)^3$, $V = \delta^{1/2}W^{-1}(A_2/A_3)^{1/2}$. Hence $\Lambda(\alpha)$ avoids a layer of the form (3.2) around the y,z-plane for some $\eta' > 0$.

If L_t^* is not the x-axis, rotate L_t^* to coincide with the x-axis, repeat the above procedure using the rotated images of $\Lambda(\alpha)$ and $\Lambda(\alpha^*)$, then rotate L_t^* back to its original position. Thus $\Lambda(\alpha)$ avoids a layer around each of the planes P_t. For the value of ϱ in the statement of the theorem we may take any positive number less than ϱ_1 and less than the values of η that arise in the above process from each of the lines L_t^*.

To show that there are in fact continuum-many such $\Lambda(\alpha)$, we note that we could have replaced $\mathscr{C}_n$ and $\mathscr{D}_n$ by any one of their octants (at the cost of reducing the value of ϱ), and so had several choices at each step in our construction of $\Lambda(\alpha)$.

4. Extension to $k+1$ dimensions ($k \geqslant 1$). The extension of Theorem 1 to higher dimensions offers no difficulty. We associate with Λ a point $\alpha = (a_{ij})$ in $(k+1)^2$-dimensional space. The construction of the hypercubes $\mathscr{C}_n$ and $\mathscr{D}_n$ proceeds as before; now $l_n = l_0 R^{-(k+1)n/k}$. Again we get two polar lattices $\Lambda(\alpha)$ and $\Lambda(\alpha^*)$. The relations analogous to (3.3) and (3.4) are

$$(4.1)\qquad x_1^2+\ldots+x_k^2 < W^2 \Rightarrow |x_0| > \eta' W^{-k},$$

$$(4.2)\qquad \delta^{-2k/(k+1)}V^{2k}x_0^2 + \delta^{2/(k+1)}V^{-2}(x_1^2+\ldots+x_k^2) > \delta^{2/(k+1)},$$

and we continue in a straightforward manner to prove

THEOREM 2. *Let r lines L_q, defined by the equations*

$$(4.3)\qquad x_j - \lambda_q^{(j)}x_0 = 0 \quad (j = 1, \ldots, k;\ q = 1, \ldots, r),$$

and s planes P_t, defined by

$$(4.4)\qquad x_0 + \mu_t^{(1)}x_1 + \ldots + \mu_t^{(k)}x_k = 0 \quad (t = 1, \ldots, s),$$

be given in $(k+1)$-dimensional space. Then there exists a $(k+1)$-dimensional lattice Λ and a positive number ϱ such that every point $(x_0, \ldots, x_k)$ of Λ other than the origin satisfies

(1) $\max\limits_j(|x_j - \lambda_q^{(j)}x_0|) > \varrho\min(1, |x_0|^{-1/k})$ $(q = 1, \ldots, r)$,

(2) $|x_0 + \mu_t^{(1)}x_1 + \ldots + \mu_t^{(k)}x_k| > \varrho(1+\max\limits_j|x_j|)^{-k}$ $(t = 1, \ldots, s)$.

In fact, the set of all such lattices has the cardinal of the continuum.

We are thus able to construct a lattice which simultaneously avoids tubes and layers of appropriate thickness around 1-dimensional and k-dimensional subspaces, respectively. The question of finding a lattice that avoids subspaces of intermediate dimension remains open.

5. Infinitely many lines and planes. We close by showing that with only minor modifications of the preceding argument we can prove the analogs of Theorems 1 and 2 for a denumerably infinite set of lines and planes passing through the origin. For simplicity we will outline the required changes only for the three-dimensional case, the extension to higher dimensions being straightforward.

THEOREM 3. *Let $L_1, L_2, \ldots$ be a sequence of lines through the origin; say L_q $(q = 1, 2, \ldots)$ is defined by*

$$(5.1)\qquad y - \lambda_q x = 0, \quad z - \mu_q x = 0.$$

Also, let $P_1, P_2, \ldots$ be a sequence of planes through the origin; say P_q $(q = 1, 2, \ldots)$ is defined by

$$(5.2)\qquad x + \theta_q y + \varphi_q z = 0.$$

Then there exists a sequence of positive constants ϱ_q $(q = 1, 2, \ldots)$ and continuum-many lattices Λ, such that every point (x, y, z) of Λ satisfies

(1) $\max(|y-\lambda_q x|, |z-\mu_q x|) > \varrho_q \min(1, |x|^{-1/2})$ $(q = 1, 2, \ldots)$,

(2) $|x+\theta_q y+\varphi_q z| > \varrho_q(1+\max\{|x|, |y|, |z|\})^{-2}$ $(q = 1, 2, \ldots)$.

Proof. We begin the proof as in Theorem 1, associating lattices Λ with points a in 9-space, and defining $T(a)$, $\Delta(a)$, $|a-a'|$ and a^* as before. We replace the planes P_q by their polar reciprocal lines L_q^* and suppose that L_q^* is defined by

$$y-\lambda_q^* x = 0, \quad z-\mu_q^* x = 0. \tag{5.3}$$

Next we choose a lattice Λ_0, a point a_0 in 9-space such that $\Lambda_0 = \Lambda(a_0)$, and a small positive number δ_0 to satisfy conditions (I) and (II) and also conditions (IV) and (IV′) with $r = s = 1$. (We do not assume condition (III) since this would impose a needless restriction on the lines L_q and L_q^*.) We suppose also that a positive constant $M = M(a_0)$ is given such that

$$|a_{ij}| \leqslant M \tag{5.4}$$

for all a in $\mathscr{C}_0$ and in $\pi(\mathscr{C}_0)$.

The scheme of the proof is as follows. We will construct two nested sequences of hypercubes in 9-space, $\mathscr{C}_n(a_n, l_n)$ and $\mathscr{D}_n(a_n^*, l_n')$, with

$$l_n = 2\delta_n = l_0 R^{-3n/2}, \quad l_n' = 2\delta_n' = l_0' R^{-3n/2}, \quad l_n = K l_n', \tag{5.5}$$

and

$$\max(l_0, l_0') < 2K_4^{1/2} R^{-3/2}, \tag{5.6}$$

where $R > 1$ will be chosen later and $K_4 = D/3^{10}M$. K_4 plays the role that K_1 played in Theorem 1, but depends only on Λ_0, not on any of the lines L_q or L_q^*. Now for any positive integer n we may write

$$n = 2^{q-1}(1+2m),$$

where $q = q(n) \geqslant 1$ and $m = m(n) \geqslant 0$ are integers uniquely determined by n. Using a modification of the Lemma of Section 2, we will construct the hypercubes $\mathscr{C}_n$ and $\mathscr{D}_n$ so that, if $a \in \mathscr{C}_n$, $T(a)$ does not map into the tube

$$|y-\lambda_q x| < \bar{\varrho}_q \min(1, |x|^{-1/2}), \quad |z-\mu_q x| < \bar{\varrho}_q \min(1, |x|^{-1/2}) \tag{5.7}$$

$(q = q(n))$ around L_q any integer point (u, v, w) satisfying

$$R^{n-2^{q(n)}} \leqslant \max(|u|, |v|, |w|) < R^n, \tag{5.8}$$

and similarly if $a \in \mathscr{D}_n$, then $T(a)$ will not carry any integer point satisfying (5.8) into the tube

$$|y-\lambda_q^* x| < \bar{\varrho}_q \min(1, |x|^{-1/2}), \quad |z-\mu_q^* x| < \bar{\varrho}_q \min(1, |x|^{-1/2}) \tag{5.9}$$

$(q = q(n))$ around L_q^*. The constants $\bar{\varrho}_q$ $(q = 1, 2, \ldots)$ will be chosen later. In this way we will determine two adjoint points a and a^*, such that $\Lambda(a)$ will have no points in the tubes (5.7) around the lines L_q $(q = 1, 2, \ldots)$ and $\Lambda(a^*)$ will have no points in the tubes (5.9) around the lines L_q^* $(q = 1, 2, \ldots)$. The Theorem then follows from the method of Section 3.

The modification of the Lemma which we require to construct $\mathscr{C}_n$ and $\mathscr{D}_n$ as described above is simply a replacement of K_1 by K_4 in the condition $|a-\bar{a}| < K_1^{1/2} U^{-3/2}$ in the hypothesis of the Lemma. The proof of the Lemma in the new form follows the original proof quite closely, except that we replace the determinant

$$\begin{vmatrix} 1 & a_{21}-\lambda a_{11} & a_{31}-\mu a_{11} \\ 0 & a_{22}-\lambda a_{12} & a_{32}-\mu a_{12} \\ 0 & a_{23}-\lambda a_{13} & a_{33}-\mu a_{13} \end{vmatrix}$$

by the determinant

$$\begin{vmatrix} a_{11} & a_{21}-\lambda a_{11} & a_{31}-\mu a_{11} \\ a_{12} & a_{22}-\lambda a_{12} & a_{32}-\mu a_{12} \\ a_{13} & a_{23}-\lambda a_{13} & a_{33}-\mu a_{13} \end{vmatrix} = |a_{ij}|,$$

and also we assume that $|\lambda| < 2$, $|\mu| < 2$ (this can always be accomplished by an appropriate interchange of the axes).

The construction of the hypercubes now offers no difficulty. We proceed as before and (with the previous notation) obtain two hypercubes

$$\mathscr{C}_{n-1,2}(a_{n-1,2}, l_{n-1,2}) \subset \mathscr{C}_{n-1} \quad \text{and} \quad \mathscr{D}_{n-1,2}(a_{n-1,2}^*, l_{n-1,2}') \subset \mathscr{D}_{n-1},$$

where

$$l_{n-1,2} = 2^{-4} K K^* l_{n-1}, \quad l_{n-1,2}' = 2^{-4} K K^* l_{n-1}'.$$

No $T(a)$, for $a \in \mathscr{C}_{n-1,2}$, carries an integer point satisfying (5.8) onto $L_{q(n)}$, and similarly no $T(a)$, for $a \in \mathscr{D}_{n-1,2}$, carries such an integer point onto $L_{q(n)}^*$. In order to form $\mathscr{C}_n$ we remove from $\mathscr{C}_{n-1,2}$ a border of thickness β_{n-1}, and to form $\mathscr{D}_n$ we remove from $\mathscr{D}_{n-1,2}$ a border of thickness $\beta_{n-1}' = K^{-1}\beta_{n-1}$. For the value of β_{n-1} we take

$$\beta_{n-1} = \bar{\varrho}_q (K+1) K_2^{-1/2} M_q^{-1} R^{3(2^q-n)/2} \quad (q = q(n)), \tag{5.10}$$

where M_q $(q = 1, 2, \ldots)$ is a positive constant depending on L_q and L_q^* such that if a point (x, y, z) of $\Lambda(a)$ or $\Lambda(a^*)$ lies in the tube (5.7) or (5.9), resp., then

$$|x| \geqslant M_q^2 (x^2+y^2+z^2)^{1/2}. \tag{5.11}$$

(M_q may be taken positive since without loss of generality we may assume that none of the lines L_q or L_q^* lies in the y, z-plane.) It is easy to verify that the hypercubes $\mathscr{C}_n$ and $\mathscr{D}_n$ have the required properties if we let

$$R^{3/2} = \frac{2^5}{KK^*}, \qquad \bar{\varrho}_q = \frac{(KK^*)^{2^q} K_2^{1/2} M_q l_0}{(K+1)2^{5 \cdot 2^q+1}}. \tag{5.12}$$

As we noted earlier, the Theorem now follows from the work in Section 3, for the avoidance of a tube around L_q^* by $\Lambda(a^*)$ is equivalent to the avoidance of a layer about P_q by $\Lambda(a)$.

References

[1] J. W. S. Cassels, *On a result of Marshall Hall*, Mathematika 3 (1956), pp. 109–110.
[2] — *An Introduction to Diophantine Approximation*, Cambridge 1957.
[3] — *An Introduction to the Geometry of Numbers*, Berlin 1959.
[4] H. Davenport, *A Note on Diophantine Approximation*, Studies in Mathematical Analysis and Related Topics, pp. 77–81, Stanford University Press, 1962.

Reçu par la Rédaction le 10. 8. 1968

ACTA ARITHMETICA
XVI (1969)

The average order of two arithmetical functions

by

M. M. Dodson (Auckland)

Let $F(k)$ be an arithmetical function of the positive integral variable k. If there is a simple function of k, $f(k)$ say, such that

$$\sum_{k\leqslant N} F(k) \sim \sum_{k\leqslant N} f(k),$$

then we say that $f(k)$ is the *average order* of $F(k)$. In this paper we establish an asymptotic expression for the sum $\sum_{k\leqslant N} f(k)$ when $F(k)$ satisfies certain conditions. By considering two special cases we obtain the average order of two arithmetical functions. First we show that the average order of the function $\Gamma^*(k)$, introduced by Davenport and Lewis in their work on homogeneous additive equations [3], is $\frac{\pi^2 k^2}{6\log k}$. Then we show that the average order of the function $\Gamma(k)$, introduced by Hardy and Littlewood in their work on Waring's Problem [5], is $\frac{5\pi^2 k}{12\log k}$. We make use of a result in Sieve Theory on the distribution of primes and the underlying idea is that, with a permissible error, the values of k for which the function $F(k)$ is large have a simple distribution.

We begin with some notation and lemmas. Throughout this paper, k will denote a positive integer, N a sufficiently large positive integer and p a prime. We shall always write $r = [(\log N)^2]$, the integral part of $(\log N)^2$. Also we shall always write $d = (k, p-1)$, the highest common factor of k and $p-1$.

Now for any given prime p, we can express the positive integer k as

$$k = p^v dm = p^v \frac{p-1}{t} m, \tag{1}$$

where k is divisible by p^v but not by p^{v+1} and where $d = (k, p-1)$ and $t = \frac{p-1}{d}$. Thus in the representation (1) of an integer k, we have $(m, p) = 1$ and $(t, m) = \left(\frac{p-1}{d}, m\right) = 1$.