replacing z^γ and $c(z)$ bounded on each product of finite angular sectors. Finally in each case

$$z^{-\gamma} z_k^{-(s+3/2)} \left[\left(z_k^{s+1} r_2(z_k) \right) *_k f(z) \right]$$

is bounded on every product of finite angular sectors. This proves Theorem III.

References

[1] Ch. F. Osgood, *A method in diophantine approximation*, Acta Arith. 12 (1966), pp. 111–128.

[2] — *A method in diophantine approximation (II)*, Acta Arith. 13 (1967), pp. 383–393.

[3] — *A method in diophantine approximation (III)*, this volume, pp. 5–22.

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A general class of sieve generated sequences

by

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There have been a number of recent investigations (see bibliography) into the density of sequences of integers which are generated by a sieve process. The sieve was always set up to be stochastically similar to the sieve of Eratosthenes, but with the exception of Buschman's [2] recent formulation of the sieve, none of the sieves were stated with enough generality to include the sieve of Eratosthenes. Thus, the theorems which were obtained were only of intrinsic interest, and did not make any real progress toward a new sieve proof of the prime number theorem if such a proof is indeed possible. In this paper, the author describes a sieve process in a very general context so that the prime number sieve as well as the lucky number type sieves can be described. Conditions are then obtained which imply that the sequence generated is prime-like, that is, the sequence $\{a_n\}$ satisfies $a_n \sim n \log n$.

1. The sieve process. The sieve process which generates the sequence $A = \{a_n\}$ can be completely described by a nested sequence $A^{(1)} \supset A^{(2)} \supset A^{(3)} \ldots$ where each $A^{(j)}$ is itself a sequence of positive integers which we will denote by $\{a_k^{(j)}\}$. We will take $A^{(1)}$ to be the sequence of all integers greater than 1 so that $a_k^{(1)} = k+1$. $A = \{a_n\}$ is then the set theoretic intersection of the $A^{(j)}$. For each $n \geqslant 1$, we will let the sequence $\{s_n(k)\}$ describe the elements eliminated at the nth sieving in the following way: Let

$$a^{(n)}_{s_n(1)} < a^{(n)}_{s_n(2)} < a^{(n)}_{s_n(3)} < \ldots$$

be the elements contained in $A^{(n)}$ but not in $A^{(n+1)}$. Thus the sequences $\{s_n(k)\}$ completely determine the sieve process.

We will furthermore assume the following conditions:

(a) $s_n(1) > n$.

(b) For each n, $s_n(k) \sim k a_n$.

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The first property assures us that at the nth sieving, a_n, is defined (and is, in fact, $a_n^{(n)}$). The second property is the one that is characteristic of the sieve of Eratosthenes. Indeed when sieving with p_n, $1/p_n$ of the elements left are sieved out.

The lucky number sieves which Briggs studied in [1] are easily described within the framework of this definition. Using his notation,

$$s_j(k) = j+(k-1)a_j+r_j$$

where r_j is an integer satisfying $1 \leqslant r_j \leqslant a_j$. The sequences discussed in [6] are obtained by letting

$$s_j(k) = j+(k-1)a_j+r_j(k) \quad \text{where} \quad 1 \leqslant r_j(k) \leqslant a_k.$$

The sieve of Eratosthenes is more difficult to describe. Since all even numbers >2 are eliminated at the first sieving, $s_1(k) = 1+2k$. When sieving with 3, we still have the simple formulation $s_2(k) = 2+3k$. For $j>2$ however, the description of $s_j(k)$ becomes more difficult. $\{s_3(k)\} = \{10, 13, 20, 23, 30, 33, \ldots\}$, and $s_4(k)$ consists of all the numbers $4+n$ where n ranges over all positive integers congruent to 12, 19, 23, 30, 34, 41, 53, and 56 modulo 56. To describe $s_5(k)$, we need 48 congruence classes and to describe $s_6(k)$ we need 480. In general, to describe $s_j(k)$, we need to enumerate $\prod_{n=1}^{j-1}(p_n-1)$ congruence classes modulo $p_j\prod_{n=1}^{j-1}(p_n-1)$. The author enumerated these up to $j = 7$.

2. A formula for a_n. From past experience, there seems to be two properties of the $s_k(n)$ which determine the asymptotic character of a_n. One is the size of $s_1(n)$ or the size of the gap between a_n and the first element sieved out, and the other is the uniformity of the sieving beyond the gap. In general, the larger the gap, the less uniform the sieving need be beyond the gap to obtain prime-like sequences. To measure these two properties, we will first assume that an integer-valued function $a(n)$ exists which is increasing and for which $s_n(1) \geqslant a(n) \geqslant 1$. Clearly $a(n) \geqslant n$. We next measure the uniformity of the sieving beyond $a(n)$ by letting

$$\delta_n(x) = N_n(x) - \frac{x-a(n)}{a_n}$$

where $N_n(x)$ is the number of elements of the sequence $\{s_n(k)\}$ which do not exceed x. Following Buschman's [2] terminology, we call $\delta_n(x)$ the *discrepancy function*. In an ideally uniform sieve (as in Briggs [1]) the function $\delta_n(x)$ will "sawtooth" between 0 and 1 as x increases beyond $a(n)$. For $x < a(n)$, the discrepency does not measure uniformity but its value can be computed exactly.

2.1. LEMMA. *If $x < a(n)$, then*

$$\delta_n(x) = -\frac{x}{a_n}+\frac{a(n)}{a_n}.$$

Proof. $N_n(x) = 0$.

We now set up the usual recursive method. Let $R_n(x)$ be the number of elements of $A^{(n)}$ which do not exceed x. Then we write for all x and all n

$$(1)\qquad R_{n+1}(x) = R_n(x)-N_n\big(R_n(x)\big) = R_n(x)-\left(\frac{R_n(x)-a(n)}{a_n}\right)-\delta_n\big(R_n(x)\big)$$
$$= R_n(x)\left(1-\frac{1}{a_n}\right)-\left(\delta_n\big(R_n(x)\big)-\frac{a(n)}{a_n}\right).$$

We now iterate, letting

$$\sigma_n = \prod_{k=1}^{n}\left(1-\frac{1}{a_k}\right)$$

and obtain

$$(2)\qquad R_{n+1}(x) = \sigma_n R_1(x)-\sum_{k=1}^{n}\frac{\sigma_n}{\sigma_k}\left(\delta_k\big(R_n(x)\big)-\frac{a(k)}{a_k}\right).$$

Now $R_{n+1}(a_n+1) = n$ or $n+1$, and $R_1(a_n+1) = a_n$. Thus by letting $x = a_n+1$ in (2) we obtain

$$n = \sigma_n a_n-\varepsilon_n-\sum_{k=1}^{n}\frac{\sigma_n}{\sigma_k}\left(\delta_k\big(R_k(a_n+1)\big)-\frac{a(k)}{a_k}\right)$$

where $\varepsilon_n = 0$ or 1. We have proved the following:

2.2. LEMMA.

$$(3)\qquad \sigma_n a_n = n+E_n(a_n+1)$$

where

$$(4)\qquad E_n(a_n+1) = E_n = \varepsilon_n+\sum_{k=1}^{n}\frac{\sigma_n}{\sigma_k}\left(\delta_k\big(R_k(a_n+1)\big)-\frac{a(k)}{a_k}\right)$$

where $\varepsilon_n = 0$ or 1.

(Note: The ε_n which appears above in the expression for E_n can be easily eliminated by stipulating that $s_1(k) = 2k$ or $s_1(k) = 2k+1$. This would remove the possibility that consecutive elements of A are consecutive integers, and so $R_{n+1}(a_n+1) = n$. However the existence of the ε_n does not cause any difficulty in what follows.)

2.3. THEOREM. *If we let* $D_n = \dfrac{1}{1+(E_n/n)}$, *then*

$$a_n = \frac{n}{D_n}\left(\sum_{k=1}^{n}\frac{D_k}{k}\right)+\frac{n}{D_n}. \tag{5}$$

Proof. From lemma 2.2

$$\frac{1}{\sigma_k a_k} = \frac{1}{k+E_k} = \frac{D_k}{k}.$$

Thus, since $\dfrac{1}{\sigma_k a_k} = \dfrac{1}{\sigma_k} - \dfrac{1}{\sigma_{k-1}}$, we have

$$\sum_{k=1}^{n}\frac{D_k}{k} = \sum_{k=1}^{n}\frac{1}{\sigma_k a_k} = \frac{1}{\sigma_n} - \frac{1}{\sigma_0} = \frac{1}{\sigma_n} - 1. \tag{6}$$

But since $\sigma_n a_n = n/D_n$, the result follows.

3. The small gap. We will first restrict our attention to the case where the gap and the discrepancy are both small. Assume therefore in what follows that

$$a(k) = k \tag{7}$$

and

$$\delta_k(x) = O(1). \tag{8}$$

This of course rules out the sieve of Eratosthenes, but we will return to the "large gap" later in the paper.

3.1. LEMMA. $E_n(x) = O(n)$ *for all* x.

Proof. This follows directly from the definition of $E_n(x)$.

3.2. LEMMA. *There exists a constant* c_1 *for which*

$$a_n > c_1 n\log n.$$

Proof. Let A and B be two positive constants such that

$$-An < \sum_{k=1}^{n}\frac{\sigma_n}{\sigma_k}\left(\delta_k\big(R_k(x)\big) - \frac{k}{a_k}\right) + \varepsilon_n < Bn. \tag{9}$$

From 2.2, we can write

$$\sigma_k a_k = k + E_k(a_n+1) < (B+1)k$$

and

$$\frac{1}{\sigma_k a_k} > \frac{1}{(B+1)k}$$

and summing from 2 to n,

$$\frac{1}{\sigma_n} - \frac{1}{\sigma_1} = \sum_{k=1}^{n}\frac{1}{\sigma_k a_k} > \left(\frac{1}{B+1}\right)\log n.$$

Thus for n sufficiently large,

$$\frac{1}{\sigma_n} > \left(\frac{1}{B+1}\right)\log n. \tag{10}$$

Let $C = [A+1]$. Then

$$Cn \leqslant R_{n+1}(a_{Cn}+1)$$

and using (2) we have

$$Cn \leqslant R_{n+1}(a_{Cn}+1) = \sigma_n a_{Cn} - E_n(a_{Cn}+1) < (B+1)a_{Cn}/\log n + An.$$

And so

$$a_{Cn} > n\log n\left(\frac{C-A}{B+1}\right).$$

Now let t be any integer such that $1 \leqslant t < C$. Then

$$\begin{aligned} Cn - t \leqslant R_{n+1}(a_{Cn-t}+1) &= \sigma_n a_{Cn-t} - E_n(a_{Cn-t}) \\ &< (B+1)a_{Cn-t}/\log n + An \end{aligned} \tag{11}$$

so that

$$a_{Cn-t} > \left(\frac{C-A}{B+1}\right) n\log n - t\log n. \tag{12}$$

(11) and (12) prove the lemma for n sufficiently large and the constant can be adjusted to prove it for all n.

We would like to have a method which would for specific examples of a sieve of this nature, enable us to determine whether E_n/n is asymptotic to a constant and if so, determine what that constant is. From 2.3, the sequence would be prime-like if the constant is not -1. To do this we will introduce some additional terminology. E_n and hence a_n is completely determined by $n-1$ sievings. We shall divide them into two categories in the following way. Since the sequence $k+a_k$ is increasing and the sequence $R_k(a_n)$ is non-increasing, a unique $t = t(n)$ exists for which $t+a_t \leqslant R_t(a_n)$ and $t+1+a_{t+1} > R_{t+1}(a_n)$. We will call the first t sievings *low order sievings* and the remainder $n-1-t$ sievings *high order sievings*. Of course, most of the numbers less than a_n are eliminated by the low order sievings, but we shall prove that the number of elements less than a_n eliminated by high order sievings is asymptotic to E_n. Let $\eta_k(n) = \eta_k$ be the number of elements less than a_n eliminated at the kth sieving. In view of some previous terminology, $\eta_k = N_k\big(R_k(a_n)\big)$.

3.3. LEMMA. *If $k > t$, $\eta_k = O(1)$, or the number of elements less than a_n eliminated by high order sievings is bounded.*

Proof. Clearly, η_k is less than the number of j for which $s_k(j) < k + a_k$ which can be written as $N_k(k+a_k)$, so letting $x_k = k + a_k$, we have

$$\delta_k(x_k) = N_k(x_k) - \frac{x_k - a(k)}{a_k} \geqslant \eta_k - \frac{k + a_k - k}{a_k} = \eta_k - 1$$

and the lemma follows from (8).

3.4. LEMMA. *There exists a constant c_2 such that*

$$t(n) < c_2 n/\log n.$$

Proof. From 3.3, $R_t(a_n) = O(n)$. Thus

$$t + c_1 t \log t < t + a_t < c_2' n.$$

3.5. THEOREM. *E_n is asymptotic to the number of elements less than a_n eliminated by high order sievings.*

Proof. Write

$$(13)\qquad E_n = \sum_{k=1}^{c_2 n/\log n} \frac{\sigma_n}{\sigma_k}\left(\delta_k\big(R_k(a_n+1)\big) - \frac{k}{a_k}\right) + $$

$$+ \sum_{n \geqslant k > c_2 n/\log n} \frac{\sigma_n}{\sigma_k}\left(\delta_k\big(R_k(a_n+1)\big) - \frac{k}{a_k}\right)$$

$$= o(n) + \sum_{n \geqslant k > c_2 n/\log n} \frac{\sigma_n}{\sigma_k}\left(\eta_k(n) - \frac{R_k(a_n+1) - k}{a_k} - \frac{k}{a_k}\right)$$

$$= o(n) + \sum_{n \geqslant k > c_2 n/\log n} \frac{\sigma_n}{\sigma_k}\left(\eta_k(n) - \frac{R_k(a_n+1)}{a_k}\right).$$

Since $k > c_2 n/\log n$ and $a_i > c_1 i \log i$, it is easy to verify that for k in this range

$$(14)\qquad \frac{\sigma_n}{\sigma_k} = \prod_{i=k+1}^{n}\left(1 - \frac{1}{a_i}\right) = 1 + o(1).$$

Thus

$$(15)\qquad E_n = o(n) + \big(1 + o(1)\big)\left(\sum_{n \geqslant k > c_2 n/\log n} \eta_k(n) - \sum_{n \geqslant k > c_2 n/\log n} \frac{R_k(a_n+1)}{a_k}\right).$$

It is easy to verify that

$$(16)\qquad \sum_{n \geqslant k > c_2 n/\log n} \frac{R_k(a_n+1)}{a_k} = o(n)$$

since for k in this range, $R_k(a_n+1) = O(n)$ and $a_k > c_1 k \log k$. Also it is clear that

$$\sum_{k=t+1}^{[c_2 n/\log n]} \eta_k(n) = o(n)$$

since for each k in that range, $\eta_k(n)$ is bounded.

Thus,

$$E_n = \big(1 + o(1)\big)\sum_{k=t+1}^{n} \eta_k(n)$$

which is the theorem.

We can apply this theorem to the following example. We let $s_1(k) = 2 + k$ and for $n \neq 1$, we let $\{s_n(k)\}$ be the sequence

$$\{n+1, n+2, n+3, \ldots, n+M, n+M+a_n, n+M+2a_n, \ldots\}$$

where M is any fixed positive integer. In other words, at every sieving but the first, we eliminate the M elements immediately following a_n and then every a_nth element thereafter. (The exception made for the first sieving was for technical reasons which facilitated machine computation of these numbers.) If $M = 1$, we know from a theorem of W. E. Briggs [1] that the sequence is prime-like. In fact

$$a_n = n\log n + \frac{n}{2}(\log\log n)^2 - (\gamma + \log 2)n\log\log n + o(n\log\log n).$$

However, when M is large, the sequences generated appear to be very thin. For example, for $M = 10$, the sequence begins like $\{2, 3, 25, 59, \ldots\}$. Of course one would expect this tendency to be countered by the fact that the sieving numbers are so large. Figure 1 illustrates this point. For several values of M, the ratio $a_n/n\log n$ is graphed for n ranging between 3 and 100. Table 1 illustrates the behavior of $\pi_n(x)$ which is the number of elements in the sequence which do not exceed x. If the sequence is prime-like, $\pi_M(x) \sim x/\log x$ [1].

We can apply the previous theorem to show that these sequences are all prime-like. $\delta(x)$ clearly lies between $M - \varepsilon$ and $M - 1 - \varepsilon$ where $\varepsilon \to 0$ as $k \to \infty$. Also, for all but at most one high order sieving, M elements less than a_n are eliminated, so as n becomes large, E_n tends to M_n. Thus D_n is asymptotic to a constant and so from 2.3, $a_n \sim n\log n$.

(1) The computing for this paper was done on the IBM 7044 computer at the State University of New York at Buffalo. Funds for computing time were supplied by NSF grant GP-5675 and NIH grant FR-00126. The methods used to compute the sequences are completely described in [7]. Bit representation was used with two levels of tagging. The machine time for each program was roughly one hour.

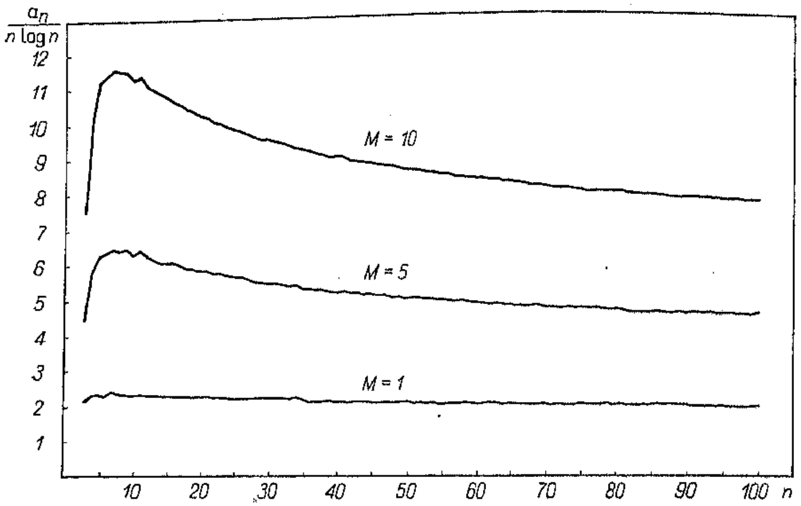

Fig. 1

TABLE I

	$M = 1$		$M = 2$		$M = 5$		$M = 10$	
x	$\pi_1(x)$	$\frac{\pi_1(x)}{x/\log x}$	$\pi_2(x)$	$\frac{\pi_2(x)}{x/\log x}$	$\pi_5(x)$	$\frac{\pi_5(x)}{x/\log x}$	$\pi_{10}(x)$	$\frac{\pi_{10}(x)}{x/\log x}$
100,000	6,812	.784	5,751	.662	3,884	.4471	2,738	.3152
200,000	12,966	.791	11,040	.674	7,559	.461	4,886	.298
300,000	18,918	.795	16,183	.680	11,163	.4692	7,255	.3049
400,000	24,748	.798	21,239	.6849	14,721	.4747	9,606	.3097
500,000	30,484	.800	26,214	.6879	18,248	.4789	11,942	.3134
600,000	36,152	.802	31,144	.6906	21,749	.4822	14,268	.3163
700,000	41,760	.802	36,043	.6930	25,230	.4850	16,584	.3188
800,000	47,323	.804	40,888	.6947	28,692	.4874	18,892	.3209
900,000	52,842	.805	45,711	.6963	32,141	.4896	21,194	.3228
1,000,000	58,336	.806	50,509	.6978	35,573	.4914	23,490	.3245
1,100,000	63,789	.807	55,282	.6991	38,996	.4931	25,780	.3260
1,200,000	69,228	.808	60,031	.7002	42,408	.4947	28,065	.3273

This approach is admittedly not widely applicable. It would be more desirable to obtain conditions on the sieving sequences $s_n(k)$ which imply that the sieved sequence a_k is prime-like. We find that such a result can be obtained by imposing conditions on the distribution of the rational numbers $n/s_n(k)$ in the unit interval. The precise statement reads as follows:

3.6. Theorem. *Let $N_n(x)$ count the number of $s_k(j)$, $k \leqslant n$, $j = 1, 2, \ldots$ for which $k/s_k(j) \geqslant x$ where $0 < x < 1$. If*

$$\mu(x) = \lim_{n\to\infty} \frac{N_n(x)}{n}$$

exists and satisfies the Lipschitz condition in $(0, 1)$, then D_n is asymptotic to a constant and hence $\{a_n\}$ is prime-like.

Proof. The proof uses techniques similar to those which the author used in [9] where the sieve was much more special. We begin by defining a sequence of polygonal functions $P_n(x)$, $n = 1, 2, \ldots$ and for x contained in a subset of $(0, 1)$. Let $x_k = k/n$ for $k = 1, 2, \ldots, n$. Let $P_n(x_k) = R_k(a_n)/n$ whenever $x_k > t(n)$ and if $x_i < z < x_{i-1}$, we let

$$P_n(z) = P_n(x_i) + (x_i - z)[P_n(x_{i-1}) - P_n(x_i)]/n.$$

Since the total number of sievings eliminated by high order sievings is $R_t(a_n) - n$ and in view of 3.5, we can prove our theorem by showing that the polygonal functions $P_n(x)$ converge point-wise to a unique function $f(x)$ as $n \to \infty$. (Actually we need point-wise convergence only for $x = 0$.) Since $t(n) = o(n)$, the domain of definition of $P_n(x)$ tends to $[0, 1]$ as $n \to \infty$.

We achieve this by defining upper and lower bounds for $P_n(x)$ which themselves are polygonal functions. For each $m = 1, 2, \ldots$, let $A_m(1) = B_m(1) = 1$. Let $x_k = k/m$ and assuming that $A_m(x_i)$ and $B_m(x_i)$ is defined for $1 \leqslant i \leqslant n$,

$$A_m(x_{i-1}) = A_m(x_i) + \frac{1}{m}\mu\left(\frac{i-1}{mA_m(x_i)+\beta}\right),$$

$$B_m(x_{i-1}) = B_m(x_i) + \frac{1}{m}\mu\left(\frac{i}{mB_m(x_i)}\right)$$

where $\beta = \max\{\eta_k(m);\, k > t(n),\, m = 1, 2, \ldots\}$. (See 3.3.) As before we define the functions linearly between consecutive x_i.

It is not difficult to show that

$$B_n(x) \leqslant B_{nm}(x) \leqslant A_{nm}(x) \leqslant A_m(x)$$

for all n, m so that $B_n(x) \leqslant A_m(x)$ for any $x \in [0, 1]$. We complete the proof of the theorem by first showing that $A_m(x)$ and $B_m(x)$ do bound $P_n(x)$ and then showing that $\lim_{m\to\infty}(A_m(x) - B_m(x)) = 0$. We need a fundamental lemma.

Lemma. *Suppose $0 < x < y \leqslant 1$ and that a u and v exist such that $u < P_n(y) < v$ for sufficiently large n. Then if $\varepsilon > 0$, there exists an $N(\varepsilon)$*

such that

(a) $P_n(x) > u+(y-x)\mu\left(\frac{y}{u}\right)-\varepsilon$,

(b) $P_n(x) < v+(y-x)\mu\left(\frac{x}{v+\beta(y-x)}\right)+\varepsilon$

for all $n > N(\varepsilon)$.

Proof. For the purposes of this lemma, let $S_k(x)$ be the counting function of the sieving sequence $s_k(n)$ and let $\gamma_{k,n} = S_{k-1}(R_{k-1}(a_n))$. If we consider the set of integers k for which $x < k/n \leqslant y$, we clearly have for any such k,

$$P_n\left(\frac{k-1}{n}\right) = P_n\left(\frac{k}{n}\right)+\frac{\gamma_{k,n}}{n}.$$

Thus

$$P_n(x) = P_n(y)+\frac{1}{n}\sum_{x<k/n\leqslant y}\gamma_{k,n}+o(n) > u+\frac{\sigma_n(x,y)}{n}+o(n) \tag{17}$$

where we are letting

$$\sigma_n(x,y) = \sum_{x<k/n\leqslant y}\gamma_{k,n}.$$

Now

$$\gamma_{k,n} = S_{k-1}(R_{k-1}(a_n)) = S_{k-1}\left(nP_n\left(\frac{k-1}{n}\right)\right)$$

and since $(k-1)/n < y$, we have

$$nP_n\left(\frac{k-1}{n}\right) > \frac{(k-1)P_n(y)}{y} > \frac{(k-1)u}{y}$$

for n large. So,

$$\gamma_{k,n} > S_{k-1}\left(\frac{u(k-1)}{y}\right)$$

and

$$\begin{aligned}\sigma_n(x,y) &= \sum_{x<k/n\leqslant y} S_{k-1}(R_{k-1}(a_n)) \\ &> \sum_{k<ny} S_{k-1}\left(\frac{u(k-1)}{y}\right)-\sum_{k<nx} S_{k-1}\left(\frac{u(k-1)}{y}\right).\end{aligned} \tag{18}$$

From our definition of $\mu(x)$, we see that

$$\mu(x) = \lim_{n\to\infty}\frac{N_n(x)}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{k\leqslant n} S_k\left(\frac{k}{x}\right).$$

Thus taking limits of (18) as $n\to\infty$,

$$\liminf_{n\to\infty}\sigma_n(x,y) > n(y-x)\mu(y/u)$$

which in view of (17) proves part (a). To prove part (b), we note that $(k-1)/n > x+o(n)$ and write

$$nP_n\left(\frac{k-1}{n}\right) < \frac{(k-1)P_n(x)}{x}+o(n).$$

Now $P_n(x) < u+\beta(y-x)$ so we have

$$nP_n\left(\frac{k-1}{n}\right) < \frac{(k-1)(u+\beta(y-x))}{x}+o(n).$$

The rest of the proof goes exactly like part (a).

We now can verify that $B_m(x)$ and $A_m(x)$ form lower and upper bounds for $P_n(x)$ in the following sense: If ε is any positive number and m any positive integer, there exists an $N(\varepsilon, m)$ such that for all $n > N(\varepsilon, m)$

$$B_m(x)-\varepsilon < P_n(x) < A_m(x)+\varepsilon.$$

To verify this we let $x_k = k/m$ where $k = 1, 2, \ldots, m$ and assume that for any $\varepsilon_k > 0$, n can be chosen sufficiently large so that

$$P_n(x_k) < A_m(x_k)+\varepsilon_k.$$

We will show that the same assertion can be made for x_{k-1}. Applying the argument m times establishes the assertion. Apply the upper bound of the previous lemma with $x = x_{k-1}$, $y = x_k$, $v = A_m(x_k)+\varepsilon_k$. Thus for any $\varepsilon' > 0$, there exists $N(\varepsilon')$ such that

$$\begin{aligned}P_n(x_{k-1}) &< A_m(x_k)+\varepsilon_k+\frac{1}{m}\mu\left(\frac{x_{k-1}}{A_m(x_k)+\beta/m}\right)+\varepsilon' \\ &= A_m(x_k)+\frac{1}{m}\mu\left(\frac{k-1}{mA_m(x_k)+\beta+\varepsilon_k}\right)+\varepsilon_k+\varepsilon'.\end{aligned}$$

Now for arbitrary $\varepsilon_{k-1} > 0$, we can choose ε_k and ε' sufficiently small and n sufficiently large so that

$$P_n(x_{k-1}) < A_m(x_k)+\frac{1}{m}\mu\left(\frac{k-1}{mA_m(x_k)+\beta}\right)+\varepsilon_{k-1} = A_m(x_{k-1})+\varepsilon_{k-1}.$$

(We have made use of the uniform continuity of μ.) Since the upper bound holds on the x_k, it holds for all x and the assertion is verified. The lower bound is established in a similar manner.

All that remains is to show that

$$\lim_{m\to\infty}(A_m(x)-B_m(x)) = 0.$$

Let $\delta_m(k) = A_m(x_k)-B_m(x_k)$ and choose ε to be an arbitrary positive number. Let $\lambda = K\varepsilon/(e^K-1)$ where K is the Lipschitz constant for μ.

Choose m sufficiently large so that by virtue of uniform continuity of μ, whenever

$$|x-y| < \beta/m, \quad |\mu(x)-\mu(y)| < \lambda.$$

Letting $j \leqslant m$, we have

$$\delta_m(j-1)-\delta_m(j) = \frac{1}{m}\left[\mu\left(\frac{j-1}{m(A_m(x_j)+\beta)}\right)-\mu\left(\frac{j}{mB_m(x_j)}\right)\right]$$

$$= \frac{1}{m}\left[\mu\left(\frac{j-1}{mA_m(x_j)+\beta}\right)-\mu\left(\frac{j}{mA_m(x_j)}\right)+\mu\left(\frac{j}{mA_m(x_j)}\right)-\mu\left(\frac{j}{mB_m(x_j)}\right)\right]$$

$$< \frac{1}{m}\left\{\left[\mu\left(\frac{j-\beta}{mA_m(x_j)}\right)-\mu\left(\frac{j}{mA_m(x_j)}\right)\right]+\left[\mu\left(\frac{j/m}{A_m(x_j)}\right)-\mu\left(\frac{j/m}{B_m(x_j)}\right)\right]\right\}$$

$$< \frac{1}{m}\left[\lambda+\frac{K_j}{m}\big(A_m(x_j)-B_m(x_j)\big)\right] < \frac{1}{m}\big(\lambda+K\delta_m(j)\big).$$

(We used the fact that each of the $A_m(x_j), B_m(x_j) \geqslant 1$.) We now finish the proof exactly as in [9].

$$\delta_m(j-1) < \delta_m(j)\left(1+\frac{K}{m}\right)+\frac{\lambda}{m},$$

$$\delta_m(j-1)+\frac{\lambda}{m} < \left(\delta_m(j)+\frac{\lambda}{K}\right)\left(1+\frac{K}{m}\right) < \left(\delta_m(j)+\frac{\lambda}{K}\right)e^{K/m}.$$

Iterating the above yields

$$\delta_m(j-1)+\frac{\lambda}{K} < \left(\delta_m(1)+\frac{\lambda}{K}\right)e^{K(m-j+1)/m} < \frac{\lambda}{K}e^{K}.$$

For arbitrary j,

$$\delta_m(j) < \frac{\lambda}{K}(e^k-1) = \varepsilon.$$

It is easy to verify that the example cited on page 47 satisfies the hypothesis of the theorem. In fact, $N_n(x)$ will tend to Mn for all $x>0$ as n becomes large so that $\mu(x) = M$. It is easy, however, to describe a sieve for which 3.6 does not apply as stated. Let $\{s_n(k)\}$ be the sequence

$$\{2n+1, 2n+2, \ldots, 2n+M, n+M+a_n, n+M+2a_n\}$$

where M is a fixed positive integer. Here,

$$\mu(x) = \begin{cases} M; & 0 \leqslant x < 1/2, \\ 0; & 1/2 \leqslant x \end{cases}$$

and it is certainly not continuous. However, all the constructions in the proof of Theorem 3.6 used local properties of μ and if there are a finite number of discontinuities, the constructions can be carried out piecewise. The following theorem is thus true:

3.7. THEOREM. *If $\mu(x)$ is continuous and satisfies the Lipschitz condition on all but a finite number of points in $(0,1)$, then $\{a_n\}$ is prime-like.*

It should be pointed out that hypotheses of 3.6 are sufficiently strong to imply that D_n is asymptotic to a constant. Yet, 2.3 implies that a prime-like sequence can be obtained when D_n is a slowly oscillating function (such as $\sin(\log\log\log\log x)$). It would be interesting to know how the uniformity condition in the hypothesis of 3.6 could be weakened to allow such a slowly oscillating D_n. It would also be of interest and probably easier to construct a set of sieving sequences which would generate a prime-like $\{a_n\}$ and an oscillating D_n.

4. The large gap. In this section, we will study the effect on the sieve when $a(k)$ is of a higher order of magnitude than k. This has already been done by Wunderlich and Buschman [3] for a less general class of sieves and recently Buschman [2] has applied these same methods to a class of sieves which included the sieve of Eratosthenes. Since the theorems in this section are analogs of these previous theorems, the proofs will be abbreviated or even omitted.

We will first obtain a Čebyshev-like theorem. To obtain the theorem, we must impose reasonable upper bounds on $a(k)$, $s_k(1)$, and $\delta_k(x)$. We will restrict our attention to the sieves for which $\delta_k(x)$ is bounded for fixed k (this includes the sieve of Eratosthenes) which permits us to make the following definition.

4.1. DEFINITION. $\Delta_k = \sup_x \delta_k(x)$.

4.2. THEOREM. *Suppose $\{a_k\}$ is generated by a sieve which satisfies the following conditions:*

(a) *For each $k>1$ and for some $\varepsilon>0$,*

$$|\Delta_k| < (\tfrac{1}{2}-\varepsilon)\big(a(k)-a(k-1)\big).$$

(b) *There exists a constant r such that*

$$\frac{\log a(k)}{\log k} < r \quad \textit{for all } k.$$

(c) *There exists a positive number T such that*

$$s_n(1) < a(n)^T.$$

Then there exist two constants c_1 and c_2 such that

$$c_1 n\log n < a_n < c_2 n\log n \quad \textit{for all } n.$$

Remark. Although no lower bound on $a(k)$ is given, the theorem is of no use for $a(k) = k$, for then $|\Delta_k| < \frac{1}{2}$, a condition impossible to fulfill.

Proof. We first define $q(n) = q$ to be the largest k such that

$$s_k(1) < R_k(a_n+1).$$

Thus $R_q(a_n+1) = n$, so q satisfies

$$s_q(1) < n < s_{q+1}(1).$$

We will now iterate in (2) not from 1 to n, but from 1 to q, obtaining

$$n = \sigma_q a_n - \varepsilon_n - \sum_{k=1}^{q} \frac{\sigma_q}{\sigma_k}\left(\delta_k\big(R_k(a_n+1)\big) - \frac{a_k}{a_k}\right) = \sigma_q a_n - E_q(n).$$

We now proceed to establish upper and lower bounds on $E_q(n) = E_q$. First,

$$(19)\qquad \left|\sum_{k=1}^{q} \frac{\sigma_q}{\sigma_k}\Big(\delta_k\big(R_k(a_n+1)\big)\Big)\right|$$

$$< \sum_{k=1}^{q} |\Delta_k| < (\tfrac{1}{2}-\varepsilon)\sum_{k=2}^{q}\big(a(k)-a(k-1)\big) + \Delta_1$$

$$< (\tfrac{1}{2}-\varepsilon)a(q) + \Delta_1 < (\tfrac{1}{2}-\varepsilon)s_q(1) + \Delta_1 < (\tfrac{1}{2}-\varepsilon)n + \Delta_1.$$

So,

$$(20)\qquad \sigma_n a_n < \sigma_q a_n = n + E_q < n + (\tfrac{1}{2}-\varepsilon)n + o(n) < \tfrac{3}{2}n.$$

Thus,

$$\frac{1}{\sigma_n} - 1 = \sum_{k=1}^{n} \frac{\sigma_1}{\sigma_k a_k} > \frac{3}{2}\sum_{k=1}^{n}\frac{1}{k} > \frac{3}{2}\log n.$$

To bound E_q from below, we write

$$E_q(a_n+1) = \sum_{k=1}^{q}\frac{\sigma_q}{\sigma_k}\left(\delta_k\big(R_k(a_n+1)\big) - \frac{\sigma_k}{a_k}\right) > -(\tfrac{1}{2}-\varepsilon)n - \Delta_1 - \sum_{k=1}^{q}\frac{a_k}{a_k}.$$

Clearly $a_k > (2-\delta)k$ for any $\delta > 0$ for $k > N_\delta$ since 2 is the first sieving element. Thus

$$E_q(a_n+1) > -(\tfrac{1}{2}-\varepsilon)n - \Delta_1 - \frac{1}{(2-\delta)}\sum_{k=1}^{q}\frac{a_k}{k}$$

$$> -(\tfrac{1}{2}-\varepsilon)n - \Delta_1 - \frac{q}{2-\delta}\cdot\frac{a_q}{q} > -(\tfrac{1}{2}-\varepsilon)n - \Delta_1 - \frac{n}{(2-\delta)}$$

so by choosing δ sufficiently small (in view of ε) we can obtain

$$E_q(a_n+1) > -n(1-\varepsilon_1) \quad \text{for some } \varepsilon_1.$$

Now since $\log a(n)/\log n < r$ and $s_n(1) < a_n^T$, we obtain

$$\frac{\log s_n(1)}{\log n} < R$$

for some constant R. Thus we can easily show that

$$\frac{\log n}{\log q} < R\big(1+o(1)\big).$$

Thus

$$n = \sigma_q a_n - E_q(n) < \frac{2a_n}{3\log n} + n(1-\varepsilon)$$

which yields

$$(21)\qquad a_n > c_1 n\log n.$$

We can establish an upper bound on a_n by first obtaining by straightforward calculation

$$\frac{\sigma_n}{\sigma_q} = \prod_{k=q+1}^{n}\left(1-\frac{1}{a_k}\right) > c_3 > 0$$

by using (21) and the fact that $\log a_n/\log n < R$. Thus

$$\frac{1}{\sigma_n a_n} = \frac{1}{\sigma_q a_n}\cdot\frac{1}{\sigma_n/\sigma_q} < \frac{1}{c_3}\left(\frac{1}{n-E_q(a_n+1)}\right) < \frac{1}{\varepsilon c_3 n} = \frac{c_4}{n}.$$

And by summing the above, we obtain

$$\frac{1}{\sigma_n} - 1 < c_4\log n$$

and by multiplying the above by (20) we get

$$a_n < c_2 n\log n.$$

We will now strengthen the hypotheses of the previous theorem in order to imply that the generated sequence is prime-like. If we require $|\Delta_k| = o\big(a(k)-a(k-1)\big)$ we see from (19) that

$$E_q(a_n+1) = o(n).$$

We will also require

$$s_n(1) \sim a(n) \sim c(a_n)^a(\log a_n)^b$$

and we are in a position to apply the iteration procedure described in detail in [3]. We will only state the theorem.

4.3. THEOREM. *Suppose $\{a_k\}$ is generated by a sieve which satisfies the following conditions:*

(a) *For each* $k > 1$, $\Delta_k = o(a(k) - a(k-1))$.

(b) $a(k) \sim s_k(1) \sim c(a_k)^a (\log a_k)^b$ *for* $1 < a < e$ *and* $c > 0$.

Then $a_k \sim k \log k$.

It should be pointed out that the theorem above cannot yield a proof of the prime number theorem since we know that the first number eliminated at the kth sieving is $(p_k)^2$. This fact together with the second condition in the above theorem immediately implies the prime number theorem. It would be interesting to know whether $\Delta_k = o(a(k) - a(k-1))$ holds for the prime sieve. Since $a(k) = \frac{1}{2}(a_k)^2/\log(a_k)$ and $a_k \asymp k\log k$, we are asking whether $\Delta_k = o(k \log k)$. This question has already been posed by Buschman [2] and some computational evidence made by the author seems to indicate that the condition holds.

Bibliography

[1] W. E. Briggs, *Prime-like sequences generated by a sieve process*, Duke Math. Journ. 30 (1963), pp. 297–312.

[2] R. G. Buschman, *A uniformity condition for sieving prime-like sequences*. (Submitted for publication.)

[3] — and M. C. Wunderlich, *Sieve-generated sequences with translated intervals*, Canad. Journ. Math. 19 (1967), pp. 559–570.

[4] — — *Sieves with generalized intervals*, Boll. Un. Mat. Ital. 21 (1967), pp. 362–367.

[5] P. Erdös and E. Jabotinsky, *On sequences generated by a sieving process*, Nederl. Akad. Wetensch. Indag. Math. 61 (1958), pp. 115–128.

[6] M. C. Wunderlich, *Sieve-generated sequences*, Canad. Journ. Math. 18 (1966), pp. 291–299.

[7] — *Sieving procedures on digital computers*, J. Assoc. Comput. Mach. 14 (1967), pp. 10–19.

[8] — and W. E. Briggs, *Second and third term approximations of sieve-generated sequences*, Ill. Journ. Math. 10 (1966), pp. 694–700.

[9] — *Sieve-generated sequences with a limiting distribution property*, Journ. London Math. Soc. 43 (1968), pp. 339–346.

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On a question related to diophantine approximation

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1. Introduction. In an extension of a result of Cassels [1], Davenport [4] proved the following theorem on simultaneous diophantine approximation. Let $\lambda_q^{(1)}, \ldots, \lambda_q^{(k)}$ $(q = 1, \ldots, r)$ be r sets of k real numbers. Then there exist continuum-many sets of real numbers $a_1, \ldots, a_k$ such that

$$\max_{1 \leqslant j \leqslant k} \|(a_j + \lambda_q^{(j)})u\|^k > C/u \tag{1.1}$$

for every integer $u > 0$, and for $q = 1, \ldots, r$, where C is a positive constant depending on r and k, and $\|x\|$ represents the distance from x to the nearest integer.

As was also noted in [4], relation (1.1) has a simple geometrical interpretation. Let L_q $(q = 1, \ldots, r)$ be r lines through the origin in $(k+1)$-dimensional space defined by the equations

$$x_j - \lambda_q^{(j)} x_0 = 0 \quad (j = 1, \ldots, k), \tag{1.2}$$

and suppose that we surround each of these lines L_q by a tube

$$|x_j - \lambda_q^{(j)} x_0| < \min(1, |x_0|^{-1/k}) \quad (j = 1, \ldots, k). \tag{1.3}$$

Then relation (1.1) implies that there exist continuum-many lattices with no point (except the origin O) in any of the tubes. In fact, we may define the lattices by

$$C^{1/(k+1)} x_0 = u_0, \quad C^{1/(k+1)} x_j = u_j - a_j u_0 \quad (j = 1, \ldots, k). \tag{1.4}$$

Now by calling upon a standard transference principle (see, for example, [2], chapter 5, section 2), Davenport showed that (1.1) is equivalent to

$$\Big\| \sum_{j=1}^{k} (a_j + \lambda_q^{(j)}) u_j \Big\| > C_1 (\max_j |u_j|)^{-k}, \tag{1.5}$$

for some constant $C_1 > 0$, and all sets of k integers $u_1, \ldots, u_k$, not all 0. Relation (1.5) has a geometric interpretation dual to that of (1.1). Namely,