Now we can apply the Corollary to (20) and the objects $\overline{D}'$, $N'$, $z_0'$, $z_1'$, and $\mathscr{C}'$ where

$$\overline{D}' = \overline{D} - z_1 = \{z - z_1 \mid z \in \overline{D}\},$$

$N'$ is a sufficiently small neighborhood of zero, $z_0' = z_1' = 0$, and $\mathscr{C}'$ is the translation of $\mathscr{C}$. Thus the conclusion of Proposition I holds here but for possibly new values of $l$ and $d$.

### References

[1] Ch. F. Osgood, *A method in diophantine approximation*, Acta Arith. 12 (1966), pp. 111–128.

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# A method in diophantine approximation (IV)*

by

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**Introduction.** In this paper we shall extend to functions of $n > 1$ complex variables some of the results of Part III of this series of papers (referred to below as Part III).

DEFINITION. By $Z, Q, Q(i)$, and $C$ we shall mean, respectively, the integers, the rational numbers, the Gaussian rational numbers, and the complex numbers. Throughout this paper $t$ and $z$ will stand for $n$-tuples of complex numbers.

By $D^h$ we shall mean $\frac{\partial^{h_1}}{\partial z_1^{h_1}} \cdots \frac{\partial^{h_n}}{\partial z_n^{h_n}}$, where each $h_k$ $(1 \leqslant k \leqslant n)$ is a non-negative integer. Analogously we define $D^j$ and $D^\infty$. We define $D^{h+j}$ to be $D^h \cdot D^j$. We use $D_k$ to denote $\frac{\partial}{\partial z_k}$ and $\theta$ to denote $\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_n}$. By $|h|$ or $|j|$ we shall mean $\max\{h_k \mid 1 \leqslant k \leqslant n\}$ or $\max\{j_k \mid 1 \leqslant k \leqslant n\}$, respectively.

If for some positive integer $N$, $g(z)$ belongs to the $N$ by $N$ matrices over $Q[i, z]$, then by $\deg_k g(z)$ we mean $\min_i \{i \geqslant 0 \mid D_k^{i+1} g(z) \equiv 0\}$.

We define a norm, $\| \ \|$, on matrices over the complex numbers by letting $\|\text{matrix}\|$ denote the maximum of the absolute values of the entries of the matrix.

For each $1 \leqslant k \leqslant n$ choose $X_k$ to be a bounded starshaped region about zero in $C$ which shall remain fixed throughout this paper. Let $L_k$ and $Y_k$ denote the Riemann surfaces generated by $\log z_k$ over $C$ and $X_k$ respectively. Note zero is not in either $L_k$ or $Y_k$. We shall regard $C - \{0\}$ and $X_k - \{0\}$ as being embedded in $L_k$ and $Y_k$ respectively. Set $L = \prod_{k=1}^{n} L_k$, $Y = \prod_{k=1}^{n} Y_k$ and $X = \prod_{k=1}^{n} (X_k - \{0\})$. Then $X \subset Y \subset L$. From now on any

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function of $z$ will be assumed to be analytic on $Y$ unless the contrary is explicity stated.

By a *finite angular sector in* $Y_k$ we mean any set of the form

$$\{z_k|\ z_k \in Y_k \text{ and } -\infty < \gamma_k < \arg z_k < \delta_k < +\infty\}.$$

By any *finite angular sector in* $Y$ we mean a cartesian product of finite angular sectors in the $Y_k$.

We say that a function $f(z)$ *has property* $\mathrm{A}_k$ $(1 \leqslant k \leqslant n)$ if it is analytic on $X_k \times \big(\prod_{i\neq k} Y_i\big)$. We say that $f(z)$ *has property* $\mathrm{B}_k$ $(1 \leqslant k \leqslant n)$ if it is bounded on every subset of $Y$ of the form a finite angular sector in $Y_k$ cartesian product a closed polydisk in $\prod_{i\neq k} Y_i$. We say that $f(z)$ *has property* C if

$$f(z) = \sum_{l=1}^{n} g_l(z)$$

where each $g_l(z)$ has property $\mathrm{A}_l$. We say that $f(z)$ *has the stronger property* $\mathrm{C}_k$ (order $j_k$) if for some non-negative integer $j_k$

$$f(z) = D_k^{j_k} \sum_{l=1}^{n} g_l(z),$$

and each $g_l(z)$ has both properties $\mathrm{A}_l$ and $\mathrm{B}_k$.

If $f(z)$ has property $\mathrm{B}_k$ let $E_k f(z)$ denote

$$\int_0^{z_k} f(z_1, \ldots, z_{k-1}, t_k, z_{k+1}, \ldots, z_n)\,\partial t_k$$

where the path of integration is the half open ray in $Y_k$ from $z_k$ to (but not including) zero. Let $E$ denote $\prod_{k=1}^{n} E_k$. For any $t \in Y$ and all $f(z)$ (analytic on $Y$ by our convention) let $E_{k,t} f(z)$ denote

$$\int_{t_k}^{z_k} f(z_1, \ldots, z_{k-1}, t_k, z_{k+1}, \ldots, z_n)\,\partial t_k$$

and $E_t$ denote $\prod_{k=1}^{n} E_{k,t}$.

As the reader may verify properties $\mathrm{A}_k$, $\mathrm{B}_k$, and $\mathrm{C}_k$ order zero are preserved under the action of $E_k$; properties $\mathrm{B}_k$ and $\mathrm{C}_k$ order zero are preserved under the action of $D_l$ if $l \neq k$ (use the integral representation, integrating about the perimeter of an appropriate closed disk in $Y_l$); properties $\mathrm{A}_k$, C, and $\mathrm{C}_k$ are preserved under the action of each $D_l$. If $f(z)$ takes values in $C^N$ for some $N \geqslant 1$, we say that it has property $\mathrm{A}_k$, $\mathrm{B}_k$, C, $\mathrm{C}_k$, or $\mathrm{C}_k$ order $j_k$ if each of its component functions has the respective property. We note that for each $1 \leqslant k \leqslant n$ properties $\mathrm{A}_k$ and $\mathrm{B}_k$ agree in $Y_k$ with the definitions of A and B given in Part III (where we worked in $C^1$).



DEFINITIONS. Let $M$ denote the set of all scalar valued functions $m(z)$ which are component functions of some $N$ by 1 matrix valued function $\overline{m}(z)$ analytic on $Y$ which satisfies a set of $n$ simultaneous linear vector partial differential equations of the form

$$\text{(1)} \qquad \overline{m}(z) = \sum_j g_j^{(p)}(z) D^j \overline{m}(z) + c^{(p)}(z) \qquad (1 \leqslant p \leqslant n),$$

where each $g_j^{(p)}(z)$ belongs to the $N$ by $N$ matrices over $Q[i, z]$; each $c^{(p)}(z)$ is an $N$ by 1 vector valued function with property C; for every $0 \leqslant l, p \leqslant n$ $\big(j_p - \deg_p g_j^{(p)}(z)\big) j_l^{-1} \geqslant n+1$, if $l \neq p$; and each $\deg_l g_j^{(p)}(z) \leqslant j_l$. (Our convention is that $c/0 = +\infty$ if $c > 0$ and, of course, $0/0$ is undefined.) We call $\overline{m}(z)$ a "carrier" of $m(z)$. If $Q(i)$ is replaced by any finite extension field of $Q(i)$ in the above definition we obtain the same set of functions. To see this choose a basis $1 = \omega_0, \ldots, \omega_s$ of the extension field over $Q(i)$ and replace (1) by a similar system of equations (with coefficient matrices having entries belonging to $Q[i, z]$) in the column vector valued function $\tilde{m}(z)$, where $\big(\tilde{m}(z)\big)^T = \big(\omega_0 \overline{m}(z), \ldots, \omega_s \overline{m}(z)\big)$.

Let $M_k$ be the set of all $m(z)$ in $M$ possessing a carrier $\overline{m}(z)$ with property $\mathrm{B}_k$ and a set of corresponding $c^{(p)}(z)$ each having property $\mathrm{C}_k$. One may verify that applying $D_l$ to a set of equations of type (1) gives a set of equations of type (1) in $D_l \overline{m}(z)$. (Use $\deg_l g_j^{(p)}(z) \leqslant j_l$ here.) Thus if $m(z) \in M$ then $D^h m(z) \in M$ for each $h$.

EXAMPLE. Consider the function

$$F(z) = \sum_{\alpha \geqslant 1, \beta \geqslant 1} \frac{z_1^{4\alpha} z_2^{5\beta}}{(4\alpha)!(5\beta)!(\alpha-1)!(\beta-1)!(\alpha+\beta-2)!}.$$

Set

$$G(z) = F(z)(\log z_1)(\log z_2).$$

It is true then that

$$\text{(i)} \qquad G(z) = D_1^4(\tfrac{1}{4} z_1 D_1 - 1)(\tfrac{1}{4} z_1 D_1 + \tfrac{1}{5} z_2 D_2 - 2) G(z) + c^{(1)}(z)$$

and

$$\text{(ii)} \qquad G(z) = D_2^5(\tfrac{1}{5} z_2 D_2 - 1)(\tfrac{1}{4} z_1 D_1 + \tfrac{1}{5} z_2 D_2 - 2) G(z) + c^{(2)}(z),$$

where $c^{(1)}(z)$ and $c^{(2)}(z)$ are the terms obtained when the respective differential operators partially differentiate the $(\log z_1)(\log z_2)$ factor at least once, with respect to either variable. As may easily be verified both $c^{(1)}(z)$ and $c^{(2)}(z)$ have properties C, $\mathrm{C}_1$, and $\mathrm{C}_2$ for any allowable $X_1$ and $X_2$; further $G(z)$ has properties $\mathrm{B}_1$ and $\mathrm{B}_2$ for all allowable $X_1$ and $X_2$.

The required inequalities hold in equations (i) and (ii), so $G(z)$ belongs to both $M_1$ and $M_2$. We shall take up this example again after presenting the theorems of this paper.

DEFINITION. Let $R'_k$ $(1 \leqslant k \leqslant n)$ denote the set of all functions $r(z_k)$ which have property $A_k$ and which satisfy an ordinary linear homogeneous differential equation with coefficients in $Q[i, z_k]$ that has a regular singular point at infinity. (In Part III where we had $n = 1$, $R'_1$ was denoted by $R'_Y$.) Again regular points are considered regular singular.

By Theorem I of Part III $R'_k$ is a ring under addition of functional values and convolution product $(*_k)$ where

$$r_1(z_k) *_k r_2(z_k) = \int_0^{z_k} r_1(z_k - t_k) r_2(t_k) \partial t_k,$$

and the path of integration is the ray from zero to $z_k$. Also $R'_k$ is a ring under addition and multiplication of functional values. Further one may replace $Q(i)$ in the definition of $R'_k$ by any finite extension field of $Q(i)$ and obtain the same set of functions. (For proofs see Part III.)

Notation. Let $R^*_k$ denote $R'_k(\delta)$ where $\delta$ is a formal multiplicative identity.

THEOREM I. *Each set $M_k$, for $1 \leqslant k \leqslant n$, is a unitary $R^*_k$ module under addition of functional values and $*_k$ product.*

DEFINITIONS. Suppose that $f(z)$ is analytic on $Y$. Let $\Delta_k f(z)$ be defined on $Y$ by $\int (D_k f(z)) \partial z_k$ where the path of integration is any path lying in $X_k$, which is closed at $z_k$, and which winds once about zero in the positive direction. Let

$$\Delta f(z) = \Delta_n \ldots \Delta_1 f(z) \equiv \int \ldots \int \partial f(t) \partial t_1 \ldots \partial t_n$$

for the appropriate set of paths. (Note that the order in which the $\Delta_k$ operate is immaterial.) It is immediate that $\Delta E_t f(z)$ is independent of $t \in Y$. The reader may verify that if $f(z)$ has property C then $\Delta f(z) \equiv 0$.

For each positive integer $v$ let $(p_0, \ldots, p_v, q)$ denote a general $v+2$ tuple of Gaussian integers with $q \neq 0$. Pick $m(z) \in M$ and $\alpha$ any algebraic point $(\alpha_1, \ldots, \alpha_n)$ in $Y$.

THEOREM II. *There exist a positive integer $v$ and a positive real number $c$ that either*

(i) $$\Delta E^u_t m(\alpha) = 0$$

*for all sufficiently large positive integers $u$, or*

(ii) $$\max_{0 \leqslant l \leqslant v} \{|\theta^l \Delta m(\alpha) - p_l q^{-1}|\} \geqslant |q|^{-c}$$

*for all sufficiently large $|q|$.*



*Further if $f(z) \in M$ we may choose one ordered pair $(v, c)$ which suffices in* (ii) *above for each $m(z)$ of the form $D^h f(z)$.*

Naturally case (i) above is not very interesting. We present in Theorem III two different sets of assumptions on $f(z) \in M$ which make option (i) of Theorem II impossible for any $m(z)$ of the form $D^h f(z)$.

Notation. Below let

$$z^\gamma = \prod_{k=1}^n z_k^{\gamma_k} \quad \text{and} \quad z^{\delta_i} = \prod_{k=1}^n z_k^{\delta_{i,k}}.$$

THEOREM III. (i) *Suppose that*

$$f(z) = c(z) + z^\gamma \Big(\varepsilon + \sum_i z^{\delta_i} w_i(z)\Big)$$

*is an element of $M$, where $c(z)$ has property* C; *$\varepsilon$ is a nonzero complex number; the above sum is finite; each $w_i(z)$ is bounded on every cartesian product of finite angular sectors in the different $Y_k$; the $\gamma_k$ are each non-negative, non-integral, real numbers; and the $\delta_{i,k}$ are each non-negative real numbers such that every $\sum_{k=1}^n \delta_{i,k}$ is positive. Then case* (ii) *of Theorem* II *holds for all $m(z)$ of the form $D^h f(z)$.*

(ii) *Suppose that*

$$f(z) = c(z) + z^\gamma \Big(\varepsilon \log z_1 \ldots \log z_n + \sum z^{\delta_i} w_i(z)\Big)$$

*is an element of $M$; that $c(z)$ has property* C, *$\varepsilon \neq 0$ is a complex number; the above sum is finite; each $w_i(z)$ is bounded on every cartesian product of finite angular sectors; the $\gamma_k$ are positive integers; and the $\delta_{i,k}$ are non-negative real numbers such that every $\sum_{k=1}^n \delta_{i,k}$ is positive. Then case* (ii) *of Theorem* II *holds for any $m(z)$ of the form $D^h f(z)$.*

(iii) *If in* (i) *or* (ii) *of this theorem $c(z) = \sum_{l=1}^n g_l(z)$ where each $g_l(z)$ has property $A_l$ and is bounded on each cartesian product of finite angular sectors, then case* (ii) *of Theorem* II *holds for every $m(z)$ belonging to $D^h(R_1^* *_1 \ldots *_{n-1} R_n^* *_n f(z))$ for any $h$.*

EXAMPLE (continued). We see that

$$\Delta G(z) = (2\pi i)^2 F(z).$$

Also

$$G(z) = \frac{z_1^4}{4!} \cdot \frac{z_2^5}{5!} \big(\log z_1 \log z_2 + z_1 w_1(z) + z_2 w_2(z)\big)$$

where $w_1(z)$ and $w_2(z)$ are bounded in any product of finite angular sectors Since $c(z) \equiv 0$ we have also that each element of $D^j\big(R_1^* *_1 F(z) *_2 R_2^*\big)$

is a $\Delta m(z)$ for which case (ii) of Theorem II holds. By Lemma IV (b) of Part III

$$D^h D_1^4 D_2^5 (R_1^* *_1 F(z) *_2 R_2^*) = D^h (R_1^* *_1 D_1^4 D_2^5 F(z) *_2 R_2^*)$$

$$= D^h \left( R_1^* *_1 \left( \sum_{\alpha,\beta\geq 0} \frac{z_1^{4\alpha} z_2^{5\beta}}{(4\alpha)!\,(5\beta)!\,\alpha!\,\beta!\,(\alpha+\beta)!} \right) *_2 R_2^* \right),$$

so case (ii) of Theorem II holds for each of these $\Delta m(z)$.

**Section I.** We shall show that each $M_k$ is closed under $+$. Suppose that $m_1(z)$ and $m_2(z)$ belong to $M_k$ and have carriers $\overline{m}_1(z)$ and $\overline{m}_2(z)$, respectively. Let $\overline{m}_1(z) \dotplus \overline{m}_2(z)$ denote a row vector valued function whose component functions are all possible sums of the form, a component of $\overline{m}_1(z)$ plus a component of $\overline{m}_2(z)$. It will now follow that, where the column vector $\overline{m}(z)$ is defined (using block notation) by

$$(\overline{m}(z))^T = (\overline{m}_1(z)^T, \overline{m}_2(z)^T, \overline{m}_1(z) \dotplus \overline{m}_2(z)) \qquad (T \text{ denotes transpose}),$$

$\overline{m}(z)$ satisfies a set of equations of type (1). To see this simply write out simultaneously, in terms of matrices multiplying derivatives of $\overline{m}(z)$, the equations of type (1) satisfied by $\overline{m}_1(z)$ and $\overline{m}_2(z)$ as well as the equations for each component of $\overline{m}_1(z) \dotplus \overline{m}_2(z)$ (obtained by addition). Thus $M_k$ is closed under $+$.

LEMMA I. *Let* $1 \leq k \leq n$.

(i) *If* $g(z_k)$ *has property* $A_k$ *and* $f(z)$ *has property* $B_k$, *then* $g(z_k) *_k f(z)$ *is defined and has property* $B_k$.

(ii) *If* $g(z_k)$ *has property* $A_k$ *and* $f(z)$ *has both properties* $B_k$ *and* $A_l$ *(for any* $1 \leq l \leq n$*), then* $g(z_k) *_k f(z)$ *has property* $A_l$.

(iii) *If* $g(z_k)$ *has property* $A_k$ *and* $f(z)$ *has property* $C_k$ *order zero, then* $g(z_k) *_k f(z)$ *has property* $C_k$ *order zero.*

Proof. (i) In Part III (see Observations in the Introduction) we showed this result when $n$ equaled one. Thus $g(z_k) *_k f(z)$ is analytic in $z_k$ on $Y$. Since

$$\int_0^{z_k} g(z_k - t_k) f(z_1, \ldots, z_{k-1}, t_k, z_{k+1}, \ldots, z_n)\, \partial t_k$$

converges uniformly if $(z_1, \ldots, \hat{z}_k, \ldots, z_n)$ belongs to any closed polydisk in $\prod_{i\neq k} Y_i$ it follows that the convolution product is analytic in $z$ on $Y$. That the product has property $B_k$ follows immediately.

(ii) For $l = k = 1$ this was shown in the Introduction to Part III under Observations. By this previous result and by uniform convergence in $(z_1, \ldots, \hat{z}_k, \ldots, z_n)$ we are through if $l = k$. If $l \neq k$ we need to know



that properties $B_k$ and $A_l$ together imply that $f(z)$ is bounded on any set of the form (a finite angular sector in $Y_k$) $\times$ (a closed disk about zero in $X_l$) $\times$ (any closed polydisk in $\prod_{i\neq k,l} Y_i$). The desired result for $l \neq k$ would then follow by uniform convergence. We see this needed consequence of properties $B_k$ and $A_l$ by choosing a path $\gamma = \gamma(t)$ in $X_l - \{0\} \subset Y_l$ of the form $r\exp(2\pi i t_l)$, $0 \leq t_l < 1$, where $r$ is a positive real number larger than the radius of our previously given disk in $X_l$, and writing $f(z)$ as

$$(2\pi i)^{-1} \int_\gamma f(z_1, \ldots, z_{l-1}, t_l, z_{l+1}, \ldots, z_n)(z_l - t_l)^{-1} dt_l.$$

Now a finite number of closed disks in $Y_l$ cover the locus of $\gamma(t)$ so we easily obtain our upper bound.

(iii) If $f(z) = \sum_{l=1}^{n} g_l(z)$ where each $g_l(z)$ has properties $A_l$ and $B_k$ then

$$g(z_k) *_k f(z) = \sum_{l=1}^{n} (g(z_k) *_k g_l(z))$$

where each $g(z_k) *_k g_l(z)$ has properties $A_l$ and $B_k$ by parts (i) and (ii) above.

LEMMA II. *Suppose that* $g(z_k)$ *has property* $A_k$ *and* $f(z)$ *has property* $B_k$. *Then:*

(i) $z_k(g(z_k) *_k f(z)) = (z_k g(z_k)) *_k f(z) + g(z_k) *_k (z_k f(z))$;

(ii) $D_k(g(z_k) *_k f(z)) = (D_k g(z_k)) *_k f(z) + g(0) f(z)$, *which equals*

$$g(z_k) *_k (D_k f(z)) + g(z_k) f(0),$$

*if* $D_k f(z)$ *has property* $B_k$; *and*

(iii) $D_k z_k (g(z_k) *_k f(z)) = (D_k z_k g(z_k)) *_k f(z) + g(z_k) *_k (D_k z_k f(z))$ *if* $D_k f(z)$ *has property* $B_k$.

Proof. This is Lemma IV of Part III in slightly different notation.

After introducing some additional notation we shall state a Proposition which will be shown to imply Theorem I.

Notation. Let $m(z)$ be an element of $M_k$ with a carrier $\overline{m}(z)$ which has property $B_k$ and satisfies equations (1) where each $c^{(p)}(z)$ has property $C_k$ order $\beta$. Let $b = \max\{\beta, \max\{j_k \mid j \text{ appears in } (1)\}\}$. Now $b > 1$, since if $l \neq k$,

$$(j_k - \deg_k g^{(k)}(z)) j_l^{-1} > n+1 \geq 2.$$

Let $r(z_k)$ denote an element of $R_k'$. From Lemma I of Part III we recall that for some non-negative integer $a$ and collection of $p_j(D_k)$ in $Q[i, D_k]$, with $\deg p_j(D_k) \leq j$ for each $j$, we have

$$(2) \qquad (D_k z_k)^a r(z_k) = \sum_{j=1}^{a} p_j(D_k)(D_k z_k)^{a-j} r(z_k).$$

Note that operating on (2) with $D_k z_k$ gives another equation of type (2) only with $a+1$ replacing $a$. Therefore we shall assume $a \geqslant 1$ in what follows.

PROPOSITION. *Suppose,* (i) $\overline{m}(z)$ *vanishes at* $z_k = 0$ *to the order* $b$; (ii) $D_k^b \overline{m}(z)$ *has property* $B_k$; (iii) *each of the* $c^{(p)}(z)$ *has property* $C_k$ *order zero; and* (iv) *the inequalities* $\big(j_p - \deg_p g_j^{(p)}(z)\big)(j_l + a - 1)^{-1} > n+1$ *hold if* $p \neq l$; *then* $r(z_k) *_k m(z)$ *is in* $M_k$ *and there exists* $\overline{m}(z)^*$, *a carrier of* $r(z_k) *_k m(z)$, *such that* $D^b \overline{m}(z)^*$ *has property* $B_k$.

Proof that the Proposition implies Theorem I. Suppose that $m(z)$ belongs to $M_k$ and has carrier $\overline{m}(z)$ which satisfies equations (1). For each $1 \leqslant p \leqslant n$ we write equation $p$ of (1) in the form $\overline{m}(z) = \varphi^{(p)} \overline{m}(z) + c^{(p)}(z)$ where $\varphi^{(p)}$ is the appropriate differential operator and, of course, each $c^{(p)}(z)$ has property $C_k$. Then for each $q \geqslant 1$

$$\overline{m}(z) = (\varphi^{(p)})^q \overline{m}(z) + c_q^{(p)}(z) \qquad (1 \leqslant p \leqslant n)$$

for an appropriate set of $c_q^{(p)}(z)$ with property $C_k$ order $\beta q$. If for the operators $\varphi^{(p)}$, $\big(j_p - \deg_p g_j^{(p)}(z)\big) j_l^{-1} \geqslant \mu > n+1$ (when $p \neq l$) and $\deg_l g_j^{(p)}(z) \leqslant j_l$ (always) then the corresponding inequalities hold for the $(\varphi^{(p)})^q$, and we have for each term in $(\varphi^{(p)})^q$ that

$$(\text{degree in } D_k - \text{degree in } z_p) \geqslant q.$$

(To see this latter statement note that we must have $\big(i_p - \deg_p g_j^{(p)}(z)\big) \geqslant 1$ in (1) for each $1 \leqslant p \leqslant n$.) Let $a$ be as in equation (2). Then in the set of equations

$$\overline{m}(z) = (\varphi^{(p)})^q \overline{m}(z) + c_q^{(p)}(z) \qquad (1 \leqslant p \leqslant n)$$

we have for each term of each $(\varphi^{(p)})^q$ that if $l \neq p$

$$(\text{degree in } D_p - \text{degree in } z_p)(\text{degree in } D_l + a - 1)^{-1} \geqslant \big(\mu^{-1} + (a-1)q^{-1}\big)^{-1} > n+1$$

when $q$ is sufficiently large. Therefore, without loss of generality, we may assume that in (1) we have

$$\big(j_p - \deg_p g_j^{(p)}(z)\big)(j_l + a - 1)^{-1} > n+1 \qquad \text{when} \qquad p \neq l.$$

Since each $c^{(p)}(z)$ has property $C_k$ order $\beta \leqslant b$ there exist a set of $f^{(p)}(z)$ with property $C_k$ order zero such that each $c^{(p)}(z) = D_k^b f^{(p)}(z)$. Integrating each term of equations (1) with respect to $z_k$ and using integration by parts repeatedly we obtain

$$(3) \qquad \big(E_k \overline{m}(z)\big) = \sum_j \Big[\sum_{l \geqslant 0} (-1)^l \big(D_k^l g_j^{(p)}(z)\big) D^j E_k^l \big(E_k \overline{m}(z)\big)\Big] + D^{b-1}\big(f^{(p)}(z) + E_k^{b-1} g^{(p)}(z)\big) \qquad (1 \leqslant p \leqslant n),$$



where each $g^{(p)}(z)$ is independent of $z_k$ and, therefore, has properties $A_k$ and $B_k$, hence property $C_k$ order zero. Then $D_k^{b-1}\big(f^{(p)}(z) + E_k^{b-1} g^{(p)}(z)\big)$ has property $C_k$ order $b-1$ for each $1 \leqslant p \leqslant n$. After simplification we see that (3) is a system of equations of type (1) in $E_k \overline{m}(z)$ satisfying $b \geqslant \max\{j_k \mid j \text{ appears in } (3)\}$. We may continue the above procedure until we obtain a set of equations of type (1) in $E_k^b \overline{m}(z)$ where $b \geqslant \max\{j_k \mid j$ appears in any of those equations$\}$; $E_k^b \overline{m}(z)$ vanishes to the order $b$ at $z_k = 0$; $D_k^b\big(E_k^b \overline{m}(z)\big)$ has property $B_k$; and each of the inhomogeneous terms of these equations has property $C_k$ order zero. Applying the Proposition we obtain the result that $r(z_k) *_k \big(E_k^b m(z)\big)$ is in $M_k$ with a carrier $\overline{m}(z)^*$ such that $D_k^b \overline{m}(z)^*$ has property $B_k$. From Lemma II part (ii), it follows that

$$D_b^k\big(r(z_k) *_k E_k^b m(z)\big) = r(z_k) *_k m(z)$$

which belongs to $M_k$, since $M$ is closed under differentiation. This completes the proof.

Proof of the Proposition. By $\big((D_k z_k)^\gamma r(z_k)\big) *_k \overline{m}(z)$ we shall mean the row vector of all convolution products of the form $(D_k z_k)^\gamma r(z_k)$ times a component of $\overline{m}(z)$ for each $0 \leqslant \gamma \leqslant a-1$. Let $\big(D_k^\varepsilon r(0)\big) \overline{m}(z)$ denote the row vector of all ordinary products of the form $D_k^\varepsilon r(0)$ times a component of $\overline{m}(z)$ for $0 \leqslant \varepsilon < a+b$. Using matrix block notation, we consider the column vector $\overline{m}(z)^*$ defined by

$$\big(\overline{m}(z)^*\big)^T = \Big(\big((D_k z_k)^\gamma r(z_k)\big) *_k \overline{m}(z), \big(D^\varepsilon r(0)\big) \overline{m}(z)\Big)$$

which we must show satisfies a system of equations of type (1). We need only use our equations (1) in order to rewrite the bottom block of rows $n$ different times (for $1 \leqslant p \leqslant n$) as a sum of terms of the proper sort for a system of equations of type (1) in $\overline{m}(z)^*$.

Now to consider the top block of rows. Without loss of generality we choose $\big((D_k z_k)^\gamma r(z_k)\big) *_k m(z)$ for some $0 \leqslant \gamma < a$, as a typical term and must rewrite it $n$ times as a sum of terms of the proper sort in $\overline{m}(z)^*$. First convolute, using $*_k$, the function $(D_k z_k)^\gamma r(z_k)$ term by term with each of the $n$ scalar equations that we have in which $m(z)$ occurs on the left-hand side. The resulting inhomogeneous terms all have property $C_k$, since the $c^{(p)}(z)$ all have property $C_k$ order zero, by Lemma I (iii). We observe that all $D_l$'s and $z_l$'s in the second factor may be brought outside of the convolution products, if $l \neq k$. By Lemma II (i) each $z_k$ in the second factor of the $*_k$ product may either be brought out of the convolution product or placed immediately to the left of $(D_k z_k)^\gamma r(z_k)$ inside of the convolution product. By Lemma II (ii) and our assumption that $\overline{m}(z)$ vanishes to the order $b \geqslant \max\{j_k \mid j \text{ appears in } (1)\}$ at $z_k = 0$ we see that every $D_k$ appearing next to an element of $\overline{m}(z)$ in a convolution product may be taken outside of the product. Now the first factor of a typical convolution product is of the form $z_k^\lambda(D_kz_k)^\gamma r(z_k)$ where $0\leqslant\lambda\leqslant b$ and $0\leqslant\gamma<a$. (Recall that $\deg_l g_j^{(p)}(z)\leqslant j_l\leqslant b$ for each $l, j$, and $p$.) Also, since $\deg_l g_j^{(p)}(z)\leqslant j_l$, there must exist at least $\lambda$ more factors of $D_k$ than $z_k$ outside of the convolution product. Using $D_kz_k = z_kD_k+1$ repeatedly we may suppose that each factor of $D_k$ outside of the convolution product appears to the right of each factor of $z_k$ and there still must be at least $\lambda$ more factors of $D_k$ than $z_k$ outside of the convolution product. By Lemma II part (ii) and the fact that $z_k^\lambda$ vanishes to the order $\lambda$ at $z_k=0$ we may move $D_k^\lambda$ inside the first factor. Now $D_k^\lambda z_k^\lambda(D_kz_k)^\gamma r(z_k)$ equals a sum of terms of the form $(D_kz_k)^\varphi r(z_k)$ where $0\leqslant\varphi\leqslant\lambda+\gamma<a+b$. If $0\leqslant\varphi<a$ then, as we shall show, we have a proper sort of differential operator acting on a component of $\overline{m}(z)^*$ and need proceed no further. Clearly $\varphi$ lying in this range makes the convolution product a component of $\overline{m}(z)^*$; if we lump together those factors of $D_l$ and $z_l$ inside of the convolution product and those factors outside of the convolution product, we see that degree in $D_l$ $(l\neq k)$, degree in $z_l$ $(l\neq k)$, and (degree in $D_k$ — degree in $z_k$) have remained unchanged during the above manipulations while degree in $D_k$ has been possibly decreased. Recall now that we convolute equations from (1) with $(z_kD_k)^\gamma r(z_k)$ where $\gamma<a$; thus, degree in $D_k$ outside of the convolution product (i.e. $j_k$) lies, inclusively, between zero and its previous value plus $a-1$. Now looking outside of the convolution product, (degree in $D_k$ — degree in $z_k$), degree in $D_l$ $(l\neq k)$, and degree in $z_l$ $(l\neq k)$ are unchanged. Then by our assumption that in (1)

$$\big(j_p-\deg_p g_j^{(p)}(z)\big)(j_l+a-1)^{-1}>n+1 \quad \text{if} \quad l\neq p$$

we have that the required inequality

$$(\text{degree in } D_p - \text{degree in } z_p)(\text{degree in } D_l)^{-1}>n+1, \quad \text{if} \quad l\neq p,$$

holds here.

If $\varphi\geqslant a$, then, by using an equation of type (2) for $r(z_k)$ with $(D_kz_k)^\varphi r(z_k)$ appearing on the left-hand side and afterwards other equations with lesser powers of $(D_kz_k)$ appearing on the left-hand side, we can express $(D_kz_k)^\varphi r(z_k)$ as a sum of terms of the form $D_k^\delta(D_kz_k)^\zeta r(z_k)$ where $0\leqslant\zeta<a$ and $0\leqslant\delta\leqslant\varphi-\zeta<a+b-\zeta\leqslant a+b$. If $\delta=0$, this puts $\zeta$ in the desired range. Suppose $\delta\geqslant 1$, then by Lemma II part (ii) we may bring $D_k^\delta$ outside of $\big(D_k^\delta(D_kz_k)^\zeta r(z_k)\big)*_k m(z)$ if we include "correcting terms" of the form $\big(D_k^\varepsilon r(0)\big)D^{\delta-1-\varepsilon}m(z)$, where $0\leqslant\varepsilon\leqslant\delta-1<a+b$ (neglecting both coefficients in $Q(i)$ and the differential operator which appeared in front of $\big(D_k^\delta(D_kz_k)^\zeta r(z_k)\big)*_k m(z)$). For the terms where $D_k^\delta$ has been brought outside we have that the degree in $D_k$ outside of the convolution product is at most $a-1$ larger than in the original term from (1); (degree in $D_k$ — degree in $z_k$) has not decreased; and both (degree in $D_l$) and (degree in $z_l$), $l\neq k$, are unchanged. Thus the terms are all right.

The "correcting terms" have degree in $D_l$ and degree in $z_l$ unchanged from the original term in (1) if $l\neq k$, while (degree in $D_l$ — degree in $z_k$) has increased by at least $\delta-1-\varepsilon\geqslant 0$, which is all to the good, and degree in $D_k$ has increased by at most $a-1-\varepsilon\leqslant a-1$. Thus we are through.

**Section II.**

Proof of Theorem II. Suppose that $f(z)$ belongs to $M$ and has carrier $\bar f(z)$. Choose $t\in Y$. One may check by differentiating that the following equation holds for a set of $g^{(p)}(z)$ independent of $z_k$

$$(4)\qquad \big(E_{k,t}\bar f(z)\big)=\sum_j\Big[\sum_{l\geqslant 0}(-1)^l\big(D_k^l g_j^{(p)}(z)\big)D^jE_{k,t}^l\big(E_{k,t}\bar f(z)\big)\Big]+ \\ +\big(E_{k,t}c^{(p)}(z)+g^{(p)}(z)\big).$$

Now each $E_{k,t}c^{(p)}(z)$ has property C, since property C is preserved under $E_{k,t}$ (use uniform convergence) and each $g^{(p)}(z)$ has property C (since it is independent of $z_k$). (The reader may wish to compare equations (3) with equations (4). The fact that $\bar f(z)$ is not assumed to have property $\mathrm{B}_k$ necessitates working with $E_{k,t}$ not $E_k$.)

Equations (4) are a set of equations of type (1) in $E_{k,t}\bar f(z)$. Evidently we can apply $E_{k,t}^u$ where $u$ is any positive integer and obtain a system of equations of type (1) in $E_{k,t}^u\bar f(z)$. We write out the actual equation obtained so as to demonstrate that the coefficient functions are matrices over $Q[i, z, u]$. Let

$$\binom{a}{b}=\frac{a!}{b!(a-b)!}.$$

Then

$$\big(E_{k,t}^u\bar f(z)\big)=\sum_j\Big[\sum_{l\geqslant 0}(-1)^l\binom{u+l-1}{l}\big(D_k^l g_j^{(p)}(z)\big)D^jE_{k,t}^l\big(E_{k,t}^u\bar f(z)\big)\Big]+c_u^{(p)}(z)$$

$(1\leqslant p\leqslant n)$ where each $c_u^{(p)}(z)$ has property C. Obviously then we can write equations of type (1) in

$$E_t^u\bar f(z)=\prod_{k=1}^{n}E_{k,t}^u\bar f(z)$$

with coefficients in the matrices over $Q[i, z, u]$. Next applying $D^\omega$ where $\omega_1,\ldots,\omega_n$ are non-negative integers we obtain equations of type (1) in $D^\omega E_t^u\bar f(z)$ with coefficients in the matrices over $Q[i, z, \omega, s]$. Let

$$\Gamma=\max\{|j|\mid j \text{ appears in this equation}\}\geqslant 1.$$

We define $T$ to be $\theta^{n\Gamma}$ and $T^{-1}$ to be $E_t^{n\Gamma}$. Setting $u=n\Gamma s$ above we have above a system of equations of type (1) in $D^\omega T^{-s}\bar f(z)$ with coefficients in the matrices over $Q[i, z, \omega, s]$. Recall $\Delta c(z) = 0$ if $c(z)$ has property $C$. Applying $\Delta$ term by term above we obtain a *homogeneous* set of equations of type (1) in $\Delta D^{\omega}T^{-s}\bar{f}(z)$ with coefficients in the matrices over $Q[i, z, \omega, s]$. (Also we note that $\Delta T^{-1}D^{\omega} = \Delta D^{\omega}T^{-1}$ and $\Delta T^{-1}T = \Delta TT^{-1} = \Delta$.)

We shall demonstrate shortly that for each non-negative integer $l$, $\Delta D^{\omega}T^{-(s+l)}$ (any component of $\bar{f}(z)$), may be expressed as a linear combination over the Noetherian ring $Q[i, \omega, s, z]$ of the objects $D^{j}\Delta D^{\omega}T^{-s}$ (some component of $\bar{f}(z)$), where $0 \leqslant |j| < n\Gamma$. It will then follow by the ascending chain condition on finitely generated modules over a Noetherian ring that for some non-negative integer $\gamma$ there exists an equation of the form

$$\Delta D^{\omega}T^{-(s+\gamma)}f(z) = \sum_{l=1}^{\gamma} p_l(\omega, s, z)\Delta D^{\omega}T^{-(s+\gamma-l)}f(z), \tag{5}$$

where each $p_l(\omega, s, z)$ belongs to $Q[i, \omega, s, z]$.

We shall next demonstrate a slightly stronger statement than we actually need; namely, that for each non-negative integer $l$ the quantity

$$\Delta D^{\omega+j}T^{-(s+l)}\bar{f}(z), \quad 0 \leqslant |j| < n\Gamma,$$

may be expressed as a linear combination (over the matrices with entries in $Q[i, \omega, s, z]$) of the different

$$\Delta D^{\omega+j}T^{-s}\bar{f}(z), \quad \text{where} \quad 0 \leqslant |j| < n\Gamma.$$

If we obtain the desired identity for $l = 1$, then this identity plus the one obtained from it by substituting $s+1$ for $s$ throughout enable one to handle the case when $l = 2$, etc. Therefore, without loss of generality, we need only treat the case where $l$ equals one. We shall show that

$$\Delta D^{\omega+j}T^{-(s+1)}\bar{f}(z), \quad 0 \leqslant |j| < n\Gamma,$$

may be expressed as a linear combination (over the matrices with entries in $Q[i, \omega, s, z]$) of the

$$\Delta D^{\omega+j}T^{-(s+1)}\bar{f}(z), \quad \text{where} \quad n\Gamma \leqslant |j| < 2n\Gamma.$$

We consider our equations of type (1) in $\Delta D^{\omega}T^{-s}\bar{f}(z)$. Substitute $\omega+j$ for $\omega$ and $s+1$ for $s$ in these equations. Now choose the equations where $p = 1$. This gives us a way of expressing $\Delta D^{\omega+j}T^{-(s+1)}\bar{f}(z)$ as a linear combination of terms of the form $\Delta D^{\omega+j}T^{-(s+1)}\bar{f}(z)$, for a new collection of $j$, where (i) no $j_l$ has decreased, (ii) $0 <$ the increase in $j_1 \leqslant \Gamma$, and (iii) the (increase in $j_1$) (the increase in $j_l)^{-1} > n+1$, if $l \neq 1$. (Use the definition of $\Gamma$ and the inequalities satisfied by any equation of type (1).) Using equations of this type repeatedly we may write our original term as a linear combination of terms where

$n\Gamma \leqslant j_1 < (n+1)\Gamma$ and $j_l < n\Gamma+(n+1)(n+1)^{-1}\Gamma = (n+1)\Gamma$, if $l \neq 1$.



Now use our equations with $p = 2$ to obtain a linear combination of terms such that $n\Gamma \leqslant j_2 < (n+1)\Gamma$, $n\Gamma \leqslant j_1$, and $j_l < (n+2)\Gamma$, if $1 \leqslant l \leqslant n$. In general, before stage $p$ we have $n\Gamma \leqslant j_l$ if $l < p$ and each $j_k < (n+p-1)\Gamma$ for $1 \leqslant k \leqslant n$. We obtain after the $p$th stage, $n\Gamma \leqslant j_l$ if $l \leqslant p$ and each $j_k < (n+p)\Gamma$ for $1 \leqslant k \leqslant n$. Setting $p = n$ we have our desired identity. Thus (5) holds.

If we substitute $a$ for $z$ in (5), and substitute $h+(\gamma+\delta)(1, \ldots, 1)$ for $\omega$ we obtain, for nonnegative $s$, $\delta$, and $h_k$

$$\Delta T^{-s}\theta^{\delta}D^{h}f(a) = \sum_{l=1}^{\gamma} q_l(s, h, \delta)\Delta T^{-(s-l)}\theta^{\delta}D^{h}f(a). \tag{6}$$

(Recall $\Delta\theta^{\delta}D^{h}T^{-s} = \Delta T^{-s}\theta^{\delta}D^{h}$.)

We wish to show that without loss of generality we may take the $q_l(s, h, \delta)$ to be in $Q[i, s, h, \delta]$. Since $[Q(i, a):Q(i)] < \infty$ it follows from (6) that the various $\Delta T^{-s}\theta^{\delta}D^{h}f(a)$ belong to a finitely generated module over the Noetherian ring $Q[i, s, h, \delta]$; therefore, an equation of type (6) holds but with the $q_l(s, h, \delta)$ in $Q[i, s, h, \delta]$. We shall assume in what follows that the $q_l(s, h, \delta)$ in (6) belong to $Q[i, s, h, \delta]$.

Suppose case (i) of Theorem II does not hold for $D^{h}f(z)$. Then there exists $0 \leqslant \delta < n\Gamma$ such that an infinite number of the $\Delta T^{-s}\theta^{\delta}D^{h}f(a)$ are nonzero. Set $d = \max_{l \leqslant \gamma}\{(\deg_s q_l(s, h, \delta))l^{-1}\}$. Replace $\gamma$, the upper bound of summation in (6), by $\mu = \mu(\delta, h)$ where $q_{\mu}(s) \not\equiv 0$ as a function of $s$. Writing $q_l(s)$ for each $q_l(s, h, \delta)$, $1 \leqslant l \leqslant \mu$, we have from (6)

$$\Delta T^{-s}\theta^{\delta}D^{h}f(a) = \sum_{l=1}^{\mu} q_l(s)\Delta T^{-(s-l)}\theta^{\delta}D^{h}f(a) \quad \text{where} \quad q_{\mu}(s) \not\equiv 0.$$

We wish to show that for some $K_1(h, \gamma)$ which may be taken to be larger than or equal to one

$$|\Delta T^{-s}\theta^{\delta}D^{h}f(a)| \leqslant \big(K_1(h, \gamma)s^{-1}\big)^{n^2\Gamma s}$$

for each $s \geqslant 1$. Since $\Delta T^{-s}\theta^{\delta}D^{h}f(a)$ is independent of the choice of $t$ we set $t = a$. For an appropriate $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_n)$ lying over $a$ in $Y$

$$|\Delta T^{-s}\theta^{\delta}D^{h}f(a)|$$
$$= \big((sn\Gamma-1)!\big)^{-n}\Big|\int_{a_1}^{\tilde{a}_1}\ldots\int_{a_n}^{\tilde{a}_n}[(a_1-t_1)\ldots(a_n-t_n)]^{sn\Gamma-1}\theta^{\delta}D^{h}f(t)\,\partial t_1\ldots\partial t_n\Big|$$
$$\leqslant \big(K_1(h, \gamma)s^{-1}\big)^{n^2\Gamma s},$$

if $s \geqslant 1$, for some appropriate $K_1(h, \gamma)$. Theorem II now follows immediately from the next lemma, whose hypotheses we next give.

We suppose that $F(s)$ is a function from the integers to the $n$ by 1 matrices over $C$; that $F(s)$ is nonzero for an infinite number of positive integral $s$; that

$$F(s) = \sum_{j=1}^{\mu} q_j(s)F(s-j)$$

where each $q_j(s)$ is an $n$ by $n$ matrix with entries in $Q[i, s]$, and the determinant of $q_\mu(s) \not\equiv 0$; that $d = \max_j \{(\deg q_j(s))j^{-1}\}$; and that if $s \geqslant 1$, $\|F(s)\| \leqslant (K_1 s^{-1})^{\Lambda s}$ for some $K_1 > 1$ and $\Lambda > 0$. (Recall $\|\text{matrix}\|$ is the maximum of the absolute values of its entries.) Let $P_k$ $(0 \leqslant k \leqslant \mu-1)$ denote an $n$ by 1 vector of Gaussian integers and $q$ denote a nonzero Gaussian integer.

Lemma III. *For each $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that*

$$\max_{0 \leqslant k \leqslant \mu-1} \|F(-k) - P_k q^{-1}\| \geqslant c(\varepsilon)|q|^{-(1+d/\Lambda+\varepsilon)}.$$

Proof. We shall first show that without loss of generality we may assume that $q_\mu(s)$ is nonsingular if $s \geqslant 0$. Clearly if $s$ is larger than some appropriately chosen $s_0 \geqslant -1$ then $q_\mu(s)$ is nonsingular. If we substitute $s-(s_0+1)$ for $s$ above then the conclusion of the lemma holds, but for the function $F(s_0+1+s)$. This easily implies the desired conclusion for the function $F(s)$, if we use the relation

$$F(s) = \sum_{j=1}^{\mu} q_j(s)F(s-j)$$

repeatedly to express $F(s_0+1), \ldots, F(s_0-\mu+2)$ as linear combinations of $F(0), F(-1), \ldots, F(-\mu+1)$ over $Q[i]$.

Set $W'(s)^T = (F(s)^T, \ldots, F(s-\mu+1)^T)$. We have then

$$W'(s) = \varphi'(s)W(s-1)$$

for some $\mu m$ by $\mu m$ matrix with entries in $Q[i, s]$. Choose a positive integer $K_2$ such that $K_2\varphi'(s)$ has entries in $Z[i, s]$ ($Z$ denotes the integers). Then set $W(s) = K_2^s W'(s)$ and $\varphi(s) = K_2^s \prod_{j=1}^{s} \varphi'(j)$, so we have, if $s \geqslant 0$,

$$W(s) = \varphi(s)W(0).$$

The entries in $\varphi(s)$ are in $Q[i, s]$ and $\varphi(s)$ is nonsingular if $s \geqslant 0$. Now $\|\varphi(s)\| \leqslant (K_1' s)^{ds}$ for some $K_1' > 1$, as can be seen by recursively substituting for $F(s-j)$ in the relation $F(s) = \sum_{j=1}^{\mu} q_j(s)F(s-j)$ to obtain the relation $W(s) = \varphi(s)W(0)$. Without loss of generality we may choose a new $K_1 > 1$ such that

$$\|W(s)\| < (K_1 s^{-1})^{\Lambda s} \quad \text{if} \quad s \geqslant 1, \quad \text{and} \quad \|\varphi(s)\| \leqslant (K_1 s)^{ds}.$$



Define $P$ by $P^T = (P_0^T, \ldots, P_{\mu-1}^T)$. We shall show that

$$\|W(0) - Pq^{-1}\| \geqslant |q|^{-(1+d/\Lambda)-\varepsilon} \quad \text{for all sufficiently large } |q|.$$

We write

$$W(s) = \varphi(s)\big(W(0) - Pq^{-1}\big) + \varphi(s)Pq^{-1},$$

so

$$\varphi(s)\big(W(0) - Pq^{-1}\big) = -\varphi(s)Pq^{-1} - W(s)$$

which implies that

$$\|\varphi(s)\big(W(0) - Pq^{-1}\big)\| \geqslant \|\varphi(s)Pq^{-1}\| - \|W(s)\|.$$

Now if $P \neq 0$, $\|\varphi(s)Pq^{-1}\| \geqslant |q|^{-1}$, since $\varphi(s)$ is nonsingular for all $s \geqslant 0$ and the entries of $P$ are Gaussian integers. Thus

$$\|\varphi(s)\big(W(0) - Pq^{-1}\big)\| \geqslant |q|^{-1} - (K_1 s^{-1})^{\Lambda s}, \quad \text{if} \quad s \geqslant 1.$$

Choose $s_1$ sufficiently large that $(K_1 s_1^{-1})^{\Lambda s_1} < |2q|^{-1}$, and take it to be the first such positive integer. Since $K_1 > 1$ we have $s_1 > 1$ and

(9) $$(K_1 s_1^{-1})^{\Lambda s_1} < |2q|^{-1} \leqslant (K_1 s_1^{-1})^{\Lambda(s_1-1)}.$$

Observe that $-\Lambda s_1(\log s_1 - \log K_1)$ is asymptotically equal to

$$-\Lambda(s_1-1)[\log(s_1-1) - \log K_1].$$

Thus given $\varepsilon_1 > 0$ there exists a positive integer $N$ such that if $s_1 \geqslant N \geqslant 1$ we have

$$(K_1 s_1^{-1})^{\Lambda s_1} \geqslant \big(K_1(s_1-1)^{-1}\big)^{\Lambda(1+\varepsilon)(s_1-1)},$$

and, from (9) and our choice of $s_1$,

(10) $$(K_1 s_1^{-1})^{\Lambda s_1} \geqslant |2q|^{-(1+\varepsilon_1)}.$$

If $s_1 \leqslant K_1$ then (9) can not hold. Thus $s_1 > K_1 > 1$. For $s_1 > K_1$ the extreme left-hand side of (9) decreases in a strictly monotone manner to zero as $s_1 \to \infty$. Therefore, if we restrict ourselves to values of $q$ which satisfy $(K_1 N^{-1})^{\Lambda N} > |2q|^{-1}$ we must have $s_1 > N$ and may use (10). Also we require that $K_1^2 < (K_1^{-1}s_1)^{s_1}$. Now we have

(i) $$\|\varphi(s_1)\big(W(0) - Pq^{-1}\big)\| \geqslant |2q|^{-1};$$

and

(ii) $$\|\varphi(s_1)\| \leqslant (K_1 s_1)^{ds_1} \leqslant [K_1^2(K_1^{-1}s_1)]^{\theta_1\theta_2},$$

where $\theta_1 = \Lambda s_1(1+\varepsilon_1)^{-1}$ and $\theta_2 = d(1+\varepsilon_1)\Lambda^{-1}$, which along with (10) gives

(iii) $$\|\varphi(s_1)\| \leqslant |2q|^{\theta_2(1+\varepsilon_1)} \leqslant |2q|^{\theta_3},$$

where $\theta_3 = d\Lambda^{-1}(1+\varepsilon_1)^2$. Thus

$$\text{(iv)} \qquad \|W(0)-Pq^{-1}\| > |2n\mu q|^{-\theta_4},$$

where $\theta_4 = 1+d\Lambda^{-1}(1+\varepsilon_1)^2$, if $P \neq 0$ and $|q|$ is sufficiently large. By hypothesis there exists some $s \geqslant 0$ such that $F(s) \neq 0$; hence, $W(0) \neq 0$, so

$$\|W(0)-0q^{-1}\| = \|W(0)\| > 0.$$

Now setting $d\Lambda^{-1}(1+\varepsilon_1)^2 = d\Lambda^{-1}+\varepsilon$ inequality (iv) is seen to imply the conclusion of the lemma.

**Section III.**

Proof of Theorem III. (i) We shall assume the negation of the desired statement and obtain a contradiction. Since $\Lambda D^h E_t^u c(z) \equiv 0$ we may assume $c(z) \equiv 0$ in what follows with no loss of generality. Recall

$$\Lambda E_t^u D^h f(z) = \Lambda D^h E_t^u f(z)$$

is independent of the choice of $t \in Y$. Since

$$f(z) = z^{\gamma}\Big(\varepsilon + \sum_i z^{\delta_i} w_i(z)\Big)$$

we have, using the hypotheses, that $f(z)$ is bounded on

$$X = \prod_{k=1}^{n} (X_k - \{0\}) \subset Y.$$

We restrict $t$ to $X$ and let $t \to (0, \ldots, 0)$. This defines $E^u f(z)$ as an improper integral and, by continuity, we have

$$\Lambda E_t^u D^h f(z) \equiv \Lambda D^h E_t^u f(z) \equiv \Lambda D^h E^u f(z).$$

Under our assumption above we must have that

$$\lim_{u\to\infty}(\alpha_1^{h_1-u} \ldots \alpha_n^{h_n-u})\Gamma(\gamma_1+u-h_1+1)\big(\Gamma(\gamma_1+1)\big)^{-1}\times$$
$$\ldots\times\Gamma(\gamma_n+u-h_n+1)\big(\Gamma(\gamma_n+1)\big)^{-1}\Lambda D^h E^u f(\alpha) = 0$$

or

$$\text{(11)} \qquad \Big(\prod_{i=1}^{n} (1^{\gamma_i}-1)\Big)\varepsilon + \lim_{u\to\infty}\sum_i \Gamma(\gamma_1+u-h_1+1)\big(\Gamma(\gamma_1+1)\big)^{-1}\times$$
$$\ldots\times\Gamma(\gamma_n+u-h+1)\big(\Gamma(\gamma_n+1)\big)^{-1}\alpha_1^{h_1-u} \ldots \alpha_n^{h_n-u}\Lambda D^h E^u z^{\gamma+\delta_i} w_i(\alpha) = 0.$$

If we show that the limit in (11) is zero then we will have obtained the desired contradiction since $\prod_{i=1}^{n}(1^{\gamma_i}-1)\varepsilon \neq 0$. Now each $w_i(z)$ is bounded



on $X$, so one may obtain an upper bound depending on $u$ for the absolute value of each term in the (finite) sum under the limit sign. Using this estimate, along with the easily proven result that $\prod_{n=1}^{\infty}(1-a_n) = 0$ if each $a_n \geqslant 0$, $\lim_{n\to\infty} a_n = 0$, and $\sum_{n=1}^{\infty} a_n = +\infty$, we see that limit is indeed zero. This contradiction proves part (i).

(ii) Again we may assume that $c(z) \equiv 0$ and again we have

$$\Lambda E_t^u D^h f(z) \equiv \Lambda D^h E_t^u f(z) \equiv \Lambda D^h E^u f(z).$$

Now if $u \geqslant |h|$

$$D^h E^u(\varepsilon z^{\gamma}\log z_1 \ldots \log z_n) = \varepsilon\{\Gamma(\gamma_1+1)\big(\Gamma(\gamma_1+u-h_1+1)\big)^{-1} z^{\gamma_1+u-h_1}\times$$
$$\ldots\times\Gamma(\gamma_n+1)\big(\Gamma(\gamma_n+u-h_n+1)\big)^{-1} z^{\gamma_n+u-h_n}\}\log z_1 \ldots \log z_n + c_1(z)$$

where $c_1(z)$ has property C. Thus if $u \geqslant |h|$,

$$\Lambda D^h E^u f(z) = (2\pi i)^n \varepsilon\Gamma(\gamma_1+1)\big(\Gamma(\gamma_1+u-h_1+1)\big)^{-1} z_1^{\gamma_1+u-h_1}\times$$
$$\ldots\times\Gamma(\gamma_n+1)\big(\Gamma(\gamma_n+u-h_n+1)\big)^{-1} z^{\gamma_n+u-h_n} + \Lambda D^h E^u\Big(\sum_i z^{\gamma+\delta_i} w_i(z)\Big).$$

The same type of argument as before goes through now.

(iii) We wish to show that if $r(z_k) \in R_k^*$ $(1 \leqslant k \leqslant n)$ then $r(z_k) *_k f(z)$ has the same properties as those which were hypothesised for $f(z)$, which will suffice to prove (iii) in light of parts (i) and (ii). Recall $r(z_k) = r_1(z_k) + \eta\delta$ where $r_1(z_k)$ belongs to $R_k'$, $\eta$ belongs to $C$, and $\delta$ is the formal identity. We note that if $m_1(z)$ and $m_2(z)$ both satisfy the hypotheses of (i) (or (ii)), for $\gamma^{(1)}$ and $\gamma^{(2)}$ respectively, and $\gamma^{(1)} > \gamma^{(2)}$ (in the sense that each component of $\gamma^{(1)}$ is larger than or equal to the corresponding component of $\gamma^{(2)}$ with inequality holding at least once) then $m_1(z)+m_2(z)$ satisfies the hypotheses of (i) (or (ii)) for $\gamma^{(2)}$. Thus we need only show that if $r_1(z_k) \in R_k'$ then $r_1(z_k) *_k f(z)$ satisfies the same hypotheses as $f(z)$, but for a larger value of $\gamma$. Obviously we may assume $r_1(z_k) \not\equiv 0$.

Each $r_1(z_k) *_k g_l(z)$ is defined for every $1 \leqslant l \leqslant n$ and has property $A_l$ by Lemma I part (ii). Further, boundedness on each product of finite angular sectors (in the different $Y_k$) follows immediately. In what follows without loss of generality we take $c(z) \equiv 0$. If

$$r_1(z_k) = a_s z^s + z_k^{s+1} r_2(z_k)$$

where $s$ is a positive integer and $a_s \neq 0$ we have

$$r_1(z_k) *_k f(z) = a_s s!\, E_k^{s+1} f(z) + \big(z_k^{s+1} r_2(z_k)\big) *_k f(z).$$

In case (i) above $E_k^{s+1} f(z)$ is of form (i) with $z^{\gamma} z_k^{s+1}$ replacing $z^{\gamma}$ and $c(z) \equiv 0$. In case (ii) we obtain the result that $E_k^{s+1} f(z)$ is of form (ii) with $z^{\gamma} z_k^{s+1}$

replacing $z^\gamma$ and $c(z)$ bounded on each product of finite angular sectors. Finally in each case

$$z^{-\gamma} z_k^{-(s+3/2)}\left[\left(z_k^{s+1} r_2(z_k)\right) *_k f(z)\right]$$

is bounded on every product of finite angular sectors. This proves Theorem III.

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# A general class of sieve generated sequences

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There have been a number of recent investigations (see bibliography) into the density of sequences of integers which are generated by a sieve process. The sieve was always set up to be stochastically similar to the sieve of Eratosthenes, but with the exception of Buschman's [2] recent formulation of the sieve, none of the sieves were stated with enough generality to include the sieve of Eratosthenes. Thus, the theorems which were obtained were only of intrinsic interest, and did not make any real progress toward a new sieve proof of the prime number theorem if such a proof is indeed possible. In this paper, the author describes a sieve process in a very general context so that the prime number sieve as well as the lucky number type sieves can be described. Conditions are then obtained which imply that the sequence generated is prime-like, that is, the sequence $\{a_n\}$ satisfies $a_n \sim n\log n$.

**1. The sieve process.** The sieve process which generates the sequence $A = \{a_n\}$ can be completely described by a nested sequence $A^{(1)} \supset A^{(2)} \supset A^{(3)} \ldots$ where each $A^{(j)}$ is itself a sequence of positive integers which we will denote by $\{a_k^{(j)}\}$. We will take $A^{(1)}$ to be the sequence of all integers greater than 1 so that $a_k^{(1)} = k+1$. $A = \{a_n\}$ is then the set theoretic intersection of the $A^{(j)}$. For each $n \geqslant 1$, we will let the sequence $\{s_n(k)\}$ describe the elements eliminated at the $n$th sieving in the following way: Let

$$a^{(n)}_{s_n(1)} < a^{(n)}_{s_n(2)} < a^{(n)}_{s_n(3)} < \ldots$$

be the elements contained in $A^{(n)}$ but not in $A^{(n+1)}$. Thus the sequences $\{s_n(k)\}$ completely determine the sieve process.

We will furthermore assume the following conditions:

(a) $s_n(1) > n$.

(b) For each $n$, $s_n(k) \sim ka_n$.

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