Finally we prove that $f(q)=0$ for all $q\notin\bigcup_{i=0}^{\infty}\mathscr{S}_i$ $(\overset{\text{def}}{=}\mathscr{T})$ if Q_0 is sufficiently large.

Let $P(y,q)$ denote the number of those $p\leqslant y$ for which $p+1=kq$ and the prime factors of k do not belong to $\mathscr{T}$. We prove that $P(y,q)>0$ if y is large, whence $f(q)=0$ follows.

Indeed,

$$P(y,q)\geqslant\pi(y,q,-1)-\sum_{\substack{q'\in\mathscr{T}\\ q'\leqslant y/q}}\pi(y,q'q,-1). \tag{12}$$

For large y we have

$$\pi(y,q,-1)>\frac{1}{2}\cdot\frac{y}{q\log y}. \tag{13}$$

Furthermore by Lemma 2

$$\Sigma_3\overset{\text{def}}{=}\sum_{\substack{q'\in\mathscr{T}\\ q'\leqslant y/q}}\pi(y,q'q,-1)\leqslant\frac{Cy}{q\log y}\sum_{\substack{q'\in\mathscr{T}\\ q'<y}}\frac{1}{q'}+\frac{y}{q}\sum_{\substack{y^{1/2}\leqslant q'\leqslant y/q\\ q'\in\mathscr{T}}}\frac{1}{q'}.$$

Since $\sum_{q'\in\mathscr{T}}1/q'<\varepsilon$ with large Q_0 and

$$\sum_{y^{1/2}\leqslant q'\leqslant y/q}\frac{1}{q'}\leqslant\log y\max_{Q_l\geqslant y^{1/2}}\sum_{q'\in\mathscr{T}}\frac{1}{q'}\leqslant\frac{1}{\log^2 y},$$

we have

$$\Sigma_3<\frac{1}{4}\cdot\frac{y}{q\log y}.$$

Hence, by (12), (13), $P(y,q)>0$ follows. This completes the proof of the Theorem.

4. The constant K in the Theorem is non-effective since $C(A,B)$ in Lemma 3 is non-effective. It would be very interesting to prove the Theorem with effective K since this would give a possibility to decide with numerical calculation whether $\mathscr{P}_1$ is a set of uniqueness or not.

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A method in diophantine approximation (III)*

by

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Introduction. We begin by giving the hypotheses and statement of a result, called Proposition I below, which was stated and proved in [2] using slightly different notation.

Let D denote differentiation with respect to the complex variable z, let l be an integer greater than one; let each $g_j(z)$ for $1\leqslant j\leqslant l$ be a polynomial of degree less than j with coefficients in the Gaussian field. Suppose that we are in a simply connected region $\overline{D}$ where $a(z)$ is analytic and that $m_1(z),\ldots,m_n(z)$ are $n\geqslant1$ solutions of

$$m(z)=\sum_{j=1}^{l}g_j(z)D^jm(z)+a(z) \tag{1}$$

which are analytic in some open disk $N\subseteq\overline{D}$ about z_0 on which $g_l(z)$ does not vanish. Suppose z_1 belongs to N and z_1 is a Gaussian rational. Let $\mathscr{C}$ be a differentiable path in $\overline{D}$ with endpoints at z_0 which does not pass through any of the zeros of $g_l(z)$. Suppose that $\tilde{m}_1(z)\neq m_1(z),\ldots$ $\ldots,\tilde{m}_n(z)\neq m_n(z)$ are the function elements, analytic on N, obtained by extending $m_1(z),\ldots,m_n(z)$, respectively, along $\mathscr{C}$ and back to z_0 and that $\tilde{m}_1(z)-m_1(z),\ldots,\tilde{m}_n(z)-m_n(z)$ are linearly independent over the complex numbers. Let

$$d=\max_j\left\{\frac{\deg g_j(z)}{j-\deg g_j(z)}\right\}.$$

Let $\|a\|$, for any complex number a, denote the distance from a to the nearest Gaussian integer. Let $(A_1,\ldots,A_n)$ denote any nonzero element of the Cartesian product of the Gaussian integers with themselves n times.

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PROPOSITION I. *For each $\varepsilon > 0$ there exists a $c(\varepsilon) > 0$ such that*

$$\max_{0 \le j < l} \left\{ \left\| \sum_{h=1}^{n} A_h D^j \big(\tilde{m}_h(z_1) - m_h(z_1)\big) \right\| \right\} \ge c(\varepsilon) \min_h \{|A_h|^{-(d+\varepsilon)}\}$$

for all $(A_1, \ldots, A_n)$.

If $n = 1$ and $m_1(z) = m(z)$ the above result may be rephrased as:

COROLLARY. *There exists $c(\varepsilon) > 0$ such that*

$$\max_{0 \le j < l} \left\{ \left\| D^j \big(\tilde{m}(z_1) - m(z_1)\big) - \frac{p}{q} \right\| \right\} \ge c(\varepsilon) \, |q|^{-(d+1+\varepsilon)}$$

for all Gaussian integers p and q with $q \ne 0$.

(Note that if l is zero or one then the requirement that $\deg g_j(z) < j$ for each j forces the $g_j(z)$ to each be constant so all solutions of (1) are analytic and Proposition I and the Corollary are vacuously true.)

What we wish to do in this paper is investigate the structure of the set of functions $m(z)$ which satisfy an equation of type (1) with $l \ge 0$ where $\bar{D}$ is an appropriate sort of simply connected region X. If we assume that X is bounded and starshaped around zero and restrict ourselves to functions $m(z)$ which are "nicely behaved" on such a region we uncover an interesting algebraic structure, which has number-theoretic applications in light of the Corollary above.

Additionally a result drops out of the machinery developed to uncover this structure which is both interesting in its own right and helpful in showing that the restrictions on X assumed throughout the remainer of this paper are not as limiting as might be supposed. This result is:

PROPOSITION II. *If in the Corollary we weaken the hypotheses by letting the $g_j(z)$ have algebraic coefficients and letting z_1 be algebraic, then $m(z+z_1)$ satisfies the hypotheses of the Corollary for a new N containing zero, for $\bar{D}$ shifted by $-z_1$, and for some new l and collection of $g_j(z)$ $(1 \le j \le l)$ belonging to $Q[i, z]$. Hence the conclusion of the Corollary holds but for, in general, new constants d and l.*

(The proof of Proposition II is in the text of the paper.) Proposition II helps to justify the restricted situation which we shall deal with in the balance of this paper, where X is a bounded region starshaped about zero in which the only possible singular point of $m(z)$ is zero. In the situation of the Corollary let w be any singular point of $m(z)$ in $\bar{D}$. Then w is a zero of $g_l(z)$ and, hence, algebraic. Choose r, a rational number, such that $w+r$ belongs to $\bar{D}$ and is not a zero of $g_l(z)$. We see upon applying Proposition II that $m(z+w+r)$ satisfies an equation of type (1) with $a(z)$ analytic on $\bar{D}-(w+r)$. Now translate equation (1) by r and we have placed the singular point at zero. Any point z of $\bar{D}-w$ such that the ray from zero to z lies entirely in $\bar{D}-w$ and does not contain any singularities of $m(z)$ may be included in a bounded region X which is starshaped at zero and such that zero is the only singular point of $m(z)$ in X.

(Further Proposition II allows us to obtain the following result: if in the hypotheses of the Corollary some unique $g_j(z)$ appearing in (1) is of highest degree then $\tilde{m}(z) - m(z) \not\equiv 0$ can not have a power series expansion with only Gaussian rational coefficients about any regular point in X. We see immediately that this holds for all regular algebraic points in X. If $z = \alpha$ is a transcendental number in X, then for some appropriate non-negative integer N the relation

$$D^N \Big(1 - \sum_{j=1} g_j(z) D^j\Big) \big(\tilde{m}(z) - m(z)\big) \equiv 0$$

implies that α is a root of a polynomial of degree $\max_j \{\deg g_j(z)\}$. Since $\tilde{m}(z) - m(z) \not\equiv 0$ we have $\max_j \{\deg g_j(z)\} \ge 1$. Therefore some derivative of $\tilde{m}(z) - m(z)$ is transcendental at $z = \alpha$ and we are through.)

We now make some definitions and give some observations before stating the theorems of this paper.

DEFINITIONS. Let Q denote the rational numbers. Let C denote the complex numbers. Let X denote a bounded starshaped region about zero. Let L be the Riemann surface over C corresponding to $\log z$. Let Y be the Riemann surface over X corresponding to $\log z$. We regard $C - \{0\}$ and Y as embedded in L. Note $0 \notin L$.

If a function is analytic on X we say that it *has property* A. If it is analytic on Y and is bounded on every finite angular sector in Y, i.e. any set of the form $\{z \mid z \in Y \text{ and } \alpha < \arg z < \beta\}$, we say that it *has property* B. We use "regular singular" for "at worst regular singular".

Let R_Y be the set of all functions with property A which satisfy a linear differential equation of the form

$$(Dz)^a r(z) = \sum_{j=0}^{a-1} h_j(D) (Dz)^j r(z) \tag{2}$$

where $a > 0$ and each $h_j(D)$ belongs to $Q[i, D]$.

Let R'_Y be the set of all functions $r(z)$ with property A which satisfy a linear homogeneous differential equation with coefficients in $Q[i, z]$ that has a regular singular point at infinity (possibly a *regular* point).

Let M_Y be the set of all functions with property B which satisfy a linear differential equation of the form

$$m(z) = \sum_{j=1}^{l} g_j(z) D^j m(z) + a(z), \tag{3}$$

where each $g_j(z)$ belongs to $Q[i, z]$, $\deg g_j(z) < j$ for each $j, l \geq 0$, and $a(z)$ has property A.

If $f_1(z)$ has property A and $f_2(z)$ has property B we define $f_1(z)*f_2(z)$ on Y by

$$f_1(z)*f_2(z) = \int_0^z f_1(z-t)f_2(t)\,dt$$

where the path of integration is the ray from zero to z.

Observations. It is clear that $f_1(z)*f_2(z)$ is bounded on any finite angular sector in Y. Suppose z_1 belongs to Y. Then there exists a region

$$\Pi_1 = \{z \in L \mid 0 < |z| < r \text{ and } a_1 < \arg z < a_1+1\}$$

which contains z_1 and is contained in Y. Denote the region in $X-\{0\}$ over which Π_1 lies by Π_1'. Choose $r_2 > 0$ such that $N = \{z \in C \mid |z| < r_2\}$ is contained in X. It is always possible to choose

$$\Pi_2' = \{z \in X \mid 0 < |z| < r \text{ and } \beta_1 < \arg z < \beta_2\} \subset \Pi_1'$$

containing z_1 such that $\Pi_2'-\Pi_2'$, the set of all differences of elements of Π_2', is contained in $\Pi_1' \cup N$. Suppose $\Pi_2 \subset \Pi_1$, lies over Π_2' on Y. Then $f_1(z-t)$ is analytic on $\Pi_2 \times \Pi_2$. Hence on Π_2 we may replace the ray from zero to z_1 by any differentiable path which lies totally in Π_2, except for the initial endpoint zero. It is now easy to set up the difference quotient and calculate the derivative of $f_1(z)*f_2(z)$ in Π_2, which equals

$$\big(Df_1(z)\big)*f_2(z)+f_1(0)f_2(z).$$

Therefore $f_1(z)*f_2(z)$ has property B. If $f_2(z)$ has property A, of course, then so does $f_1(z)*f_2(z)$.

We shall prove the following two theorems about the algebraic properties of R_Y, R_Y', and M_Y.

THEOREM I. (a) *R_Y is a ring under $*$ and $+$.*

(b) *R_Y' is a subring of R_Y.*

(c) *R_Y' is also a ring under multiplication of functional values and $+$.*

(d) *If we replace $Q(i)$ by any finite extension of $Q(i)$ in the definitions of R_Y and R_Y' we obtain the same sets.*

THEOREM II. *M_Y is an R_Y module under $*$.*

EXAMPLES. Given a linear differential equation with coefficients in $Q[i, z]$ which has a regular singular point at z_0 in $Q(i)$ and which has a regular point at $z_0+z_1^{-1}$ in $Q(i)$ we let $z = z_0+(z_1-w)^{-1}$ and see that the transformed equation has a regular point at zero and a regular singular point at infinity. Let X be any region which is starshaped about zero and excludes the singularities of the transformed equation. Set $m(w) = \frac{1}{2}\sin\sqrt{w}$, observing that $m(w) = -2Dm(w)-4wD^2m(w)$. Then $m(w)$ belongs to M_Y. By Theorem II if $f(z)$ was any solution of the original equation then $\frac{1}{2}f\big(z_0+(z_1-w)^{-1}\big)*\sin\sqrt{w}$ belongs to M_Y. Choose w_1 to be a nonzero Gaussian rational in X. Let $\mathscr{C}$ wind once about zero. By the Corollary to Proposition I we conclude that $f\big(z_0+(z_1-w)^{-1}\big)*\sin\sqrt{w}$ has at least one non Gaussian-rational in its power series expansion about w_1. Instead of $\sin\sqrt{w}$ we might use

$$g(w) = \frac{\log w}{2\pi i}(\sqrt{w})J_1(\sqrt{w})$$

where $J_1(w)$ is the Bessel function of order one. Note that $\sqrt{w}J_1(\sqrt{w})$ satisfies the differential equation

$$y = -4wD^2y$$

so

$$g(w) = -4wD^2g(w)+a(w)$$

where

$$a(w) = 4wD\{(2\pi i\sqrt{w})^{-1}J_1(\sqrt{w})\}+4\{(2\pi i)^{-1}D(\sqrt{w}J_1(\sqrt{w}))\}$$

is an entire function. Thus proceeding as above we conclude that

$$f\big(z_0+(z_1-w)^{-1}\big)*\big(\sqrt{w}J_1(\sqrt{w})\big)$$

has a non Gaussian rational in its power series expansion about w_1.

Also we note that $m(z) = f\big(z_0+(z_1-w)^{-1}\big)*\big(\sqrt{w}J_1(\sqrt{w})\big)$ itself satisfies an equation of type (1) with $a(z) \equiv 0$. Thus we may apply the Corollary with $\bar{D} = C$. The different analytic continuations of our new $m(z)$ from X are given by the integral

$$\int_0^z \sqrt{z-t}\,J_1(\sqrt{z-t})f\big(z_0+(z_1-t)^{-1}\big)\,dt$$

where the path must only avoid the singularities of the function from R_Y'. If X is $\{z \mid |z| < \frac{1}{2}\}$ then $(z-1)^i$ and $(z-2)^{-1}$ each belong to R_Y'; hence so does $(z-1)^i(z-2)^{-1}$. Now

$$\int_0^z \sqrt{z-t}\,J_1(\sqrt{z-t})(t-1)^i(t-2)^{-1}\,dt$$

has branch points at $z = 1$ and $z = 2$ and the difference between two branches is of the form

$$\int_z^z \sqrt{z-t}\,J_1(\sqrt{z-t})(t-1)^i(t-2)^{-1}\,dt,$$

where the path of integration winds around the branch points 1 and 2.

Theorems I and II were suggested by study of the effects of Laplace transforms and inverse Laplace transforms upon some of the functions under discussion. The Laplace transform is not used in our proofs because of convergence problems. However, it could be used to formulate a different proof of Theorem I (b).

Section I.

LEMMA I. *R'_Y is the set of all elements of R_Y which satisfy an equation of type* (2) *with* $\deg h_j(D) \leqslant a-j$ *for each* j.

Proof. First we shall show that any equation of type (2) with $\deg h_j(D) \leqslant a-j$ for each j has a regular singular point at infinity. Note that $Dz = zD+1$ so we may rewrite (2), under these conditions, as

$$(zD)^a r(z) = \sum_{j=0}^{a-1} h_j(D)(zD)^j r(z)$$

for some new set of $h_j(D)$ in $Q[i, D]$ with $\deg h_j(D) \leqslant a-j$ for each j. Now set $w = z^{-1}$. We note that zD goes into $-wD$ and D goes into $-w^2D$. This gives

$$(-wD)^a r(w) = \sum_{j=0}^{a-1} h_j(-w^2 D)(-wD)^j r(w),$$

and using $Dw = wD+1$ repeatedly we obtain

$$(4) \qquad (-wD)^a r(w) = \Big(\sum_{j=0}^{a-1} k_j(w)(wD)^j r(w)\Big) + wk_a(w)(wD)^a r(w)$$

for a set of $k_j(w)$ in $Q[i, w]$. It follows from examining (4) that the coefficient of $D^a r(w)$ equals $(-1)^a w^a - k_a(w) w^{a+1}$ for some $k_a(w)$ in $Q[i, w]$ and that each coefficient of each $D^j r(w)$ vanishes to at least the order j at zero. Then zero is a regular singular point, so infinity was a regular singular point of (2).

Now suppose that $r(w)$ satisfies a homogeneous linear differential equation with coefficients in $Q[i, w]$ which has a regular singular point at zero. Setting $z = w^{-1}$ we wish to show that $r(z)$ satisfies an equation of type (2) with $\deg h_j(D) \leqslant a-j$ for each j. Suppose that we write the equation satisfied by $r(w)$ in the form

$$p_\beta(w)(wD)^\beta r(w) = \sum_{j=0}^{\beta-1} p_j(w)(wD)^j r(w)$$

for some $\beta > 0$ where the $p_j(w)$ are in $Q[i, w]$ and $p_\beta(0)$ is nonzero. Change variables to $z = w^{-1}$ obtaining

$$p_\beta(z^{-1})(-zD)^\beta r(z) = \sum_{j=0}^{\beta-1} p_j(z^{-1})(-zD)^j r(z).$$

Let t be the maximum of the degrees of the $p_j(w)$ for $0 \leqslant j \leqslant \beta$. Then

$$(5) \qquad p_\beta(0) D^t z^t(-zD)^\beta = D^t\Big(\sum_{j=0}^{\beta-1} z^t p_j(z^{-1})(-zD)^j r(z)\Big) - D^t z^t [p_\beta(z^{-1}) - p_\beta(0)](-zD)^\beta r(z).$$

Setting $a = \beta + t$ and using $zD = Dz-1$ repeatedly we may put (5) in form (2) with $\deg h_j(D) \leqslant a-j$ for each j.

DEFINITIONS. If $h(D)$ belongs to the n by n matrices over $Q[i, D]$ then $\deg h(D)$ denotes the *maximum of the degrees of its components*.

Let $\bar{R}'_Y$ be the set of all component functions of all n by 1 matrix valued functions (for $n = 1, 2, \ldots$) with property A which satisfy an equation of type (2), where the $h_j(D)$ belong to the n by n matrices over $Q[i, D]$ and for each j, $\deg h_j(D) \leqslant a-j$. Clearly $\bar{R}'_Y \supseteq R'_Y$.

Let

$$(6) \qquad \sum_{i=0}^{k} A_i(z) D^i y = 0$$

be any linear homogeneous differential equation with coefficients in the n by n matrices over $Q[i, z]$. If each $z^{k-i}(A_k(z))^{-1} A_i(z)$ is analytic at zero for $0 \leqslant i \leqslant k$ we say that (6) has a *regular singular point at zero*. If we set $w = z^{-1}$ in (6) and the transformed equation has a regular singular point at zero we say that (6) has a *regular singular point at infinity*.

Let $\bar{\bar{R}}'_Y$ be the set of component functions of all n by 1 matrix valued functions $r(z)$ (for $n = 1, 2, \ldots$) having property A and satisfying a linear homogeneous differential equation with coefficients in the n by n matrices over $Q[i, z]$ that has a regular singular point at infinity. Clearly $\bar{\bar{R}}'_Y \supseteq R'_Y$.

LEMMA II. $\bar{\bar{R}}'_Y = \bar{R}'_Y$.

The proof is the same as the proof of Lemma I, except now the $h_j(D)$, $k_j(D)$, and $p_j(D)$ $(j \neq \beta)$ are matrices. We choose $p_\beta(w)$ as the smallest degree monic polynomial such that each $p_j(w)$ has components in $Q[i, w]$, for $0 \leqslant j \leqslant \beta-1$.

DEFINITIONS. Let $\bar{R}_Y$ be the set of component functions of all n by 1 matrix valued functions $r(z)$ (for $n = 1, 2, \ldots$) with property A which satisfy an equation of type (2) where the $h_j(D)$ belong to the n by n matrices over $Q[i, D]$. Clearly $\bar{R}_Y \supseteq R_Y$.

Let $\bar{M}_Y$ be the set of all component functions of n by 1 matrix valued functions $m(z)$ (for $n = 1, 2, \ldots$) which have property B and satisfy a linear differential equation of form (3) where each $g_j(z)$ belongs to the n by n matrices over $Q[i, z]$, and $a(z)$ has property A. Clearly $\bar{M}_Y \supseteq M_Y$.

LEMMA III. (a) $\bar{R}'_Y = R'_Y$, (b) $\bar{R}_Y = R_Y$, and (c) $\bar{M}_Y = M_Y$.

Proof. (b) We notice that in equation (2) we may apply Dz to the equation and obtain, after simplification, another equation of the same type with $a+1$ replacing a, whether equation (2) is scalar or vector. Applying $Dz, (Dz)^2, \ldots$ to (2) we obtain a set of equations which allows us to express each $(Dz)^N r_h(z)$ $(1 \leqslant h \leqslant n)$ as a sum of terms of the form a polynomial in D times $(Dz)^j r_k(z)$ for $0 \leqslant j \leqslant a-1$ and $1 \leqslant k \leqslant n$. Therefore, the functions $(Dz)^N r_h(z)$, for $N \geqslant 0$, are contained in a finitely generated module over the Noetherian ring $Q[i, D]$. By the ascending chain condition on finitely generated modules over a Noetherian ring we see that for some N, $(Dz)^N r_h(z)$ may be written as a linear combination over $Q[i, D]$ of the $(Dz)^j r_h(z)$ with $0 \leqslant j < N$. This proves (b).

(a) We use $\bar{\bar{R}}' = \bar{R}'$. Recall from the proof of Lemma II that any vector differential equation with coefficients in the n by n matrices over $Q[i, w]$ which has a regular singular point at zero may be written in the form

$$p(w)(wD)^\beta r(w) = \sum_{j=0}^{\beta-1} p_j(w)(wD)^j r(w) \tag{7}$$

where each $p_j(w)$ $(0 \leqslant j \leqslant \beta-1)$ belongs to the n by n matrices over $Q[i, w]$, $p(w) \in Q[i, w]$, and $p(0) \neq 0$. Applying wD repeated to (7) we may express each $p(w)(wD)^{\beta+N} r_h(w)$ $(1 \leqslant h \leqslant n)$ as a linear combination over $Q[i, w]$ of the $(wD)^j r_k(w)$ for $0 \leqslant j < N+\beta$ and $1 \leqslant k \leqslant n$. Using all of these equations we may express each $(p(w))^{N+\beta}(wD)^{N+\beta} r_h(w)$ as a linear combination over $Q[i, w]$ of the $(wD)^j r_k(w)$ (with $0 \leqslant j \leqslant \beta-1$ and $1 \leqslant k \leqslant n$) for each $N \geqslant 0$. By the ascending chain condition we conclude that for some $N \geqslant 0$, $(p(w))^{N+\beta}(wD)^{N+\beta} r_h(w)$ may be written as a linear combination of the $(wD)^j r_h(w)$ (for $0 \leqslant j < N+\beta$) over $Q[i, w]$. This equation has a regular singular point at zero. A change of variables proves the corresponding statement about regular singular points at infinity.

(c) We assume that we are working with the vector version of (3). Differentiating both sides of (3) N times we obtain

$$D^N m(z) - D^N a(z) = \sum_{j=1}^{l} k_j(N, z) D^{N+j} m(z) \tag{8}$$

where each $k_j(N, z)$ belongs to the n by n matrices over the Noetherian ring $Q[i, N, z]$. We use (8) to write, formally, $D^N m_h(z)$—(a function with property A), $D^{N-1} m_h(z)$—(another function with property A), $\ldots$, as a linear combination over $Q[i, N, z]$ of the $D^{N+j} m_k(z)$ $(1 \leqslant j \leqslant l$ and $1 \leqslant k \leqslant n)$. By the ascending chain condition we then have for some positive integer λ, formally

$$D^{N-\lambda} m_h(z) = \sum_{j=1}^{\lambda} p_j(N, z) D^{N-\lambda+j} m_h(z) + b(N, z),$$

where each $p_j(N, z)$ belongs to $Q[i, N, z]$ and $b(N, z)$ has property A. If $N \geqslant \lambda$ then the above relation is valid. Replace $N-\lambda$ by N where now $N \geqslant 0$. Then

$$D^N m_h(z) = \sum_{j=1}^{\lambda} q_j(N, z) D^{N+j} m_h(z) + a(N, z) \tag{9}$$

where each $q_j(N, z)$ belongs to $Q[i, N, z]$ and $a(N, z)$ has property A for each $N \geqslant 0$. We wish to replace $\sum_{j=1}^{\lambda} q_j(N, z) D^{N+j} m_h(z)$ by D^N times a particular N-fold indefinite integral of $\sum_{j=1}^{\lambda} q_j(N, z) D^{N+j} m_h(z)$. Let us first define

$$D^{-r} m_h(z) = \int_0^z \frac{(z-t)^{r-1}}{(r-1)!} m_h(t)\,dt,$$

for each $r > 0$, where the path is the ray from zero to z. Now given any expression of the form $p(z) D^a m_h(z)$, where $p(z)$ is a polynomial in z of degree d and a is any integer, we define T by

$$Tp(z) D^a m_h(z) = \sum_{j=0}^{d} (-1)^j \big(D^j p(z)\big) D^{a-1-j} m_h(z),$$

and extend T by linearity to the space of all sums of terms of the form $p(z) D^a m_h(z)$. If N is any positive integer we have then

$$T^N p(z) D^a m_h(z) = \sum_{j=0}^{d} (-1)^j \frac{N \ldots (N+j-1)}{j!} \big(D^j p(z)\big) D^{a-N-j} m_h(z).$$

Now (9) may be rewritten as

$$\begin{aligned} D^N m_h(z) &= D^N T^N \Big(\sum_{j=1}^{\lambda} q_j(N, z) D^j m_h(z)\Big) + a(N, z) \\ &= D^N \Big(\Big(\sum_{j=-k}^{\lambda} r_j(N, z) D^j m_h(z)\Big) + a(N, z)\Big), \end{aligned} \tag{10}$$

for some $k \geqslant 0$ and a set of $r_j(N, z)$ in $Q[i, N, z]$. We may rewrite (10) in the form

$$D^N m_h(z) = \sum_{s=0}^{\beta} \frac{(N+s) \ldots (N+1)}{s!} D^N \Big(\sum_{j=-k}^{\lambda} r_{s,j}(z) D^j m_h(z)\Big) + a(N, z) \tag{11}$$

where each $r_{s,j}(z)$ belongs to $Q[i, z]$ and β is some non negative integer.

Our immediate objective is to replace the sum on the right side of (11) by a sum of terms similar to those now appearing there, but with polynomials in z replacing the $\frac{(N+s)\ldots(N+1)}{s!}$ as coefficient functions. We notice that

$$(12)\qquad D^{N+s}z^s = \sum_{h=0}^{s} z^h \frac{(N+s)\ldots(N+s+1-h)}{h!} D^N D^{s-h},$$

so

$$(13)\qquad D^N m_h(z) = \sum_{s=0}^{\beta} D^{N+s}z^s\Big(\sum_{j=-k}^{\lambda} r_{s,j}(z)D^j m_h(z)\Big) - \\ - \sum_{s=0}^{\beta}\sum_{h=1}^{s} z^h\Big\{\frac{(N+s)\ldots(N+s+1-h)}{h!}D^N\times \\ \Big(D^{s-h}\sum_{j=-k}^{\lambda} r_{s,j}(z)D^j m_h(z)\Big)\Big\} + a(N,z).$$

Now the term in curly brackets in (13) is of the proper form for us to apply (12) again. After at most β uses of (12) we obtain for a new value of λ

$$(14)\qquad D^N m_h(z) = \sum_{s=0}^{\gamma} z^s D^N\Big\{\sum_{j=-k}^{\lambda} U_{s,j}(z)D^j m_h(z)\Big\} + a(N,z)$$

where $\gamma \geqslant 0$ and the $U_{s,j}(z)$ are in $Q[i,z]$. We now show that $\deg U_{s,j}(z) < j$ for each s and j. Suppose not. Let $a = \max_{s,j}\{\deg U_{s,j}(z) - j\} \geqslant 0$. Let the maximum value of j occurring in $\{(s,j) \mid \deg U_{s,j}(z) - j = a\}$ be b. Let c be the maximum value of s occurring in $\{(s,b) \mid \deg U_{s,b}(z) - b = a\}$. Performing all the indicated differentiations in (14) we see that up to a nonzero coefficient in $Q(i)$ we obtain a term of the form

$$z^c\frac{N\ldots(N-b+1)}{b!}D^{N-a}m_h(z)$$

on the right hand side, and no second term of this type can occur to cancel it out. But, after performing the indicated differentiations, the right hand side of (14) must be identically equal to the right hand side of (9) where no such term appears. This proves $\deg U_{s,j}(z) < j$ for each j. A consequence is that nonpositive values of j may not appear in (14) so

$$(15)\qquad D^N m_h(z) = \sum_{s=0}^{\gamma} z^s D^N\Big\{\sum_{j=1}^{\lambda} U_{s,j}(z)D^j m_h(z)\Big\} + a(N,z).$$

We now show that if $s \geqslant 1$,

$$D^N\Big\{\sum_{j=1}^{\lambda} U_{s,j}(z)D^j m_h(z)\Big\}$$

has property A, for each nonnegative value of N. Differentiating (15) and subtracting from this derivative (15) with $N+1$ substituted for N gives

$$(16)\qquad \sum_{s=1}^{\gamma} sz^{s-1}D^N\Big\{\sum_{j=1}^{\lambda} U_{s,j}(z)D^j m_h(z)\Big\}$$

has property A. Now differentiate (16) and subtract from this derivative (16) with $N+1$ substituted for N. Continuing we obtain the result that each

$$D^N\Big\{\sum_{j=1}^{\lambda} U_{s,j}(z)D^j m_h(z)\Big\}$$

has property A, for each $s \geqslant 1$. Set $N = 0$ in (15) obtaining

$$(17)\qquad m_h(z) = \sum_{j=1}^{\lambda} U_{0,j}(z)D^j m_h(z) + a(0,z) + a(z)$$

where $\deg U_{0,j}(z) < j$ for each j and $a(z)$ has property A. This completes the proof of Lemma III.

We suppose that $f_1(z)$ has property A and $f_2(z)$ has property B below. Then:

Lemma IV.

(a) $z(f_1(z)*f_2(z)) = (zf_1(z))*f_2(z) + f_1(z)*(zf_2(z))$;

(b) $D(f_1(z)*f_2(z)) = (Df_1(z))*f_2(z) + f_1(0)f_2(z)$, *which equals* $f_1(z)*(Df_2(z)) + f_1(z)f_2(0)$ *if* $Df_2(z)$ *has property* B; *and*

(c) $Dz(f_1(z)*f_2(z)) = (Dzf_1(z))*f_2(z) + f_1(z)*(Dzf_2(z))$, *if* $Df_2(z)$ *has property* B.

Proof. (a) This is trivial. (b) The second part of this follows upon using integration by parts on $\int_0^z (Df_1(z-t))f_2(t)\,dt$. (The first part was covered under Observations in the Introduction.) (c) Use (a) and (b). This proves Lemma IV.

We assume below that $r(z)$ satisfies an αth order scalar differential equation of type (2) (alternately, with the extra condition $\deg h_j(D) \leqslant \alpha - j$ for each j). Let $\bar r(z)$ be the $(\alpha+1)(\alpha)/2$ by one matrix consisting of the component functions $D^i(Dz)^k r(z)$, for $0 \leqslant i+k \leqslant \alpha-1$.

Lemma V. *Then $\bar r(z)$ satisfies an α-th order vector differential equation of type* (2) (*alternately, with* $\deg h_j(D) \leqslant \alpha - j$ *for each* j).

Proof. Let us write (2) as

$$(18)\qquad (Dz)^a r(z) = \sum_{q\geqslant 0}\sum_{s=0}^{a-1}\beta_{q,s}D^q(Dz)^s r(z),$$

(alternately, with $q+s\leqslant a$) for a set of Gaussian rational $\beta_{q,s}$. Operate on (18) with $D^i(Dz)^k$ where $0\leqslant i+k\leqslant a-1$. We notice that

$$D^i(Dz)^k(Dz)^a r(z) = (Dz)^a D^i(Dz)^k r(z),$$

plus terms involving $k+a-\theta$ factors of z and $i+k+a-\theta$ factors of D where $1\leqslant\theta<a$. The terms on the right hand side of (18), after operating with $D^i(Dz)^k$, each have $k+s$ factors of z and $(q+i)+k+s$ factors of D occurring. Writing $s=a-\theta$ where $1\leqslant\theta\leqslant a$, they have $k+a-\theta$ factors of z and $q+i+k+a-\theta$ factors of D for $0\leqslant\theta\leqslant a$ (alternately, $0\leqslant q\leqslant\theta\leqslant a$). Consolidating terms above we may write $(Dz)^a\big(D^i(Dz)^k r(z)\big)$, as a sum of terms involving $k+a-\theta$ factors of z and $(q+i)+k+a-\theta$ factors of D where $0\leqslant\theta\leqslant a$ (alternately, $0\leqslant q\leqslant\theta\leqslant a$). Using $zD=Dz-1$ repeatedly we attempt to write $(Dz)^a\big(D^i(Dz)^k r(z)\big)$ as a sum of terms each having a factor of $(Dz)^{a-1}$ directly in front of $r(z)$. If this fails in a particular term we have run out of factors of z. Thus, more generally, we attempt to place $D^{a-1-t}(Dz)^t$ directly in front of $r(z)$ for $0\leqslant t\leqslant a-1$. This latter procedure can only fail when we run out of powers of D. In this case only, we obtain terms of the form $D^{a-1-t-\varphi}(Dz)^t r(z)$ for $0\leqslant\varphi\leqslant a-1-t$. Otherwise our procedure does not fail and we continue, using $zD=Dz-1$, to obtain a sum of terms of the form $D^u(Dz)^v\big(D^{a-1-t}(Dz)^t r(z)\big)$ for $u\geqslant 0$ and $v\geqslant 0$ where if $t\leqslant a-1$, $v=0$. (Note that the total number of factors of D occurring above can not be greater than $q+i+k+a-\theta$ which, alternately, is less than or equal to $i+k+a\leqslant 2a-1$. Thus in this case $u+v\leqslant a$.) Now writing the above equations (for each i and k in the desired range) in matrix form gives the desired equation.

Section II.

Proof of Theorem I. We note (see Observations in the Introduction) that property A is preserved under $+$ and $*$. Also, by Lemma I, $R_Y\supseteq R'_{Y'}$. Thus we need only show closure of R_Y and R'_Y, respectively, under $+$ and $*$. Because of Lemmas II and III we need only show that the sums and products lie in $\bar R_Y$ or $\bar R'_Y=\bar{\bar R}'_Y$, respectively.

Suppose that $r_1(z)$ and $r_2(z)$ are in R_Y (alternately, R'_Y). We shall show that using matrix block notation

$$r(z) = \begin{pmatrix} r_1(z)+r_2(z)\\ r_1(z)\\ r_2(z)\end{pmatrix}$$

satisfies an equation of type (2) (alternately, where $\deg h_j(D)\leqslant a-j$ for each j). Applying powers of Dz to the equation satisfied by $r_1(z)$ or $r_2(z)$ we see that we may take $a_1=a_2=a$. Then

$$(Dz)^a r(z) = \begin{pmatrix} (Dz)^a r_1(z)+(Dz)^a r_2(z)\\ (Dz)^a r_1(z)\\ (Dz)^a r_2(z)\end{pmatrix}.$$

Each of the rows on the right above may be expressed as a sum of terms of the appropriate sort, using the equations of type (2) for $r_1(z)$ and $r_2(z)$. This proves closure under $+$.

Keeping $r_1(z), r_2(z)$, and a as above, form $\bar r_1(z)$ and $\bar r_2(z)$. (See the paragraph before the statement of Lemma V.) Then by Lemma V, $\bar r_1(z)$ and $\bar r_2(z)$ each satisfies an ath order equation of type (2) (alternately, with $\deg h_j(D)\leqslant a-j$ for each j). Let $\bar r_1(z)*\bar r_2(z)$ denote the column vector of all possible convolution products of the form a component of $\bar r_1(z)$ times a component of $\bar r_2(z)$. Let $\big(D^N\bar r_1(0)\big)\,\bar r_2(z)$ denote the column vector of all possible ordinary products of the form a component of $D^j\bar r_1(0)$, for any $0\leqslant j\leqslant N$ (where N will be determined later), times a component of $\bar r_2(z)$. We define $\bar r_1(z)\big(D^N\bar r_2(0)\big)$ analogously. Using matrix block notation define the column vector $\tilde r(z)$ by

$$\tilde r(z) = \begin{pmatrix} \bar r_1(z)*\bar r_2(z)\\ \big(D^N\bar r_1(0)\big)\,\bar r_2(z)\\ \bar r_1(z)\,\big(D^N\bar r_2(0)\big)\end{pmatrix}.$$

Operating on $\tilde r(z)$ with $(Dz)^a$ and applying Lemma V, we see that the bottom two blocks of rows of $(Dz)^a\tilde r(z)$ may be expressed as a sum of terms appropriate for an equation of type (2). A typical element of $\bar r_1(z)*\bar r_2(z)$ is $\big(D^a(Dz)^b r_1(z)\big)*\big(D^c(Dz)^d r_2(z)\big)$ where a, b, c, and d are non-negative integers satisfying $a+b\leqslant a-1$ and $c+d\leqslant a-1$. Operating on a term of this form with $(Dz)^a$ yields a sum of terms of the form (see Lemma IV (c))

$$\big((Dz)^{a-e}D^a(Dz)^b r_1(z)\big)*\big((Dz)^e D^c(Dz)^d r_2(z)\big)$$

where $0\leqslant e\leqslant a$. We may write each term of this sort as a sum of terms of the form

$$\big(D^a(Dz)^f r_1(z)\big)*\big(D^c(Dz)^g r_2(z)\big)$$

where $a+f\leqslant a+(a-e+b)\leqslant 2a-1-e$ and $c+g\leqslant c+(d+e)\leqslant a-1+e$. We show that in general we may take $f\leqslant a-1$ and $g\leqslant a-1$. To see this rewrite $(Dz)^f r_1(z)$ (or $(Dz)^g r_2(z)$) using the equation for $r_1(z)$ (or $r_2(z)$) after it has been operated on by $(Dz)^{f-a}$ (or $(Dz)^{g-a}$). Continuing we obtain a sum of terms of the form

$$\big(D^a(Dz)^f r_1(z)\big) * \big(D^c(Dz)^g r_2(z)\big)$$

with $f \leqslant a-1$ and $g \leqslant a-1$. If $r_1(z)$ and $r_2(z)$ belong to R'_Y then we still have $a+f \leqslant 2a-1-e$ and $c+g \leqslant a-1+e$. Otherwise we merely know that we may pick $N \geqslant 3a-2$, depending on the degrees of the polynomials $h_j(D)$ appearing in the equations for $r_1(z)$ and $r_2(z)$, such that $a+f+c+g \leqslant N$. This specifies our N in the definition of $\tilde{r}(z)$. Applying Lemma IV (b) repeatedly (use both formulas) we obtain terms which are derivatives of components of $\tilde{r}(z)$ appearing in the bottom two blocks of rows and a term of the form

$$D^j\big\{\big(D^h(Dz)^f r_1(z)\big) * \big(D^k(Dz)^g r_2(z)\big)\big\}$$

where $h+f \leqslant a-1$, $k+g \leqslant a-1$, and $j \leqslant \max\{0, a+f+c+g-2(a-1)\}$. (Alternately, $j \leqslant \max\{0, a+f+c+g-2(a-1)\} \leqslant \max\{0, a\} = a$.) This proves Theorem I parts (a) and (b).

Part (c). Suppose $r_1(z)$ and $r_2(z)$ belong to R'_Y. We may assume without loss of generality that for some $a \geqslant 0$

$$p^{(1)}(z)(zD)^a r_1\left(\frac{1}{z}\right) = \sum_{j=1}^{a} p_j^{(1)}(z)(zD)^{a-j} r_1\left(\frac{1}{z}\right),$$

$$p^{(2)}(z)(zD)^a r_2\left(\frac{1}{z}\right) = \sum_{j=1}^{a} p_j^{(2)}(z)(zD)^{a-j} r_2\left(\frac{1}{z}\right),$$

where $p^1(z), p^{(2)}(z)$, each $p_j^{(1)}(z)$, and each $p_j^{(2)}(z)$ belong to $Q[i, z]$ and $p^{(1)}(z)p^{(2)}(z)$ does not vanish at zero. Then the module over $Q[i, z]$ generated by the elements of

$$\left\{[p^{(1)}(z)p^{(2)}(z)]^N (zD)^N r_1\left(\frac{1}{z}\right) r_2\left(\frac{1}{z}\right), \text{ for each } N \geqslant 0\right\}$$

is finitely generated. By the ascending chain condition we have for some $N \geqslant 0$

$$\big(p^{(1)}(z)p^{(2)}(z)\big)^N (zD)^N r_1\left(\frac{1}{z}\right) r_2\left(\frac{1}{z}\right) = \sum_{j=1}^{N} Q_j(z)\big[\big(p^{(1)}(z)p^{(2)}(z)\big)\big]^{N-j}(zD)^{N-j} r_1\left(\frac{1}{z}\right) r_2\left(\frac{1}{z}\right)$$

for a set of $Q_j(z)$ in $Q[i, z]$. Thus $r_1(z)r_2(z)$ belongs to R'_Y.

(d) We have shown that $R_Y = \bar{R}_Y$ and $R'_Y = \bar{R}'_Y$. Suppose that the $h_j(D)$ in (2) are in $K[D]$ for some K with $[K: Q(i)] < \infty$. Choose a basis $1 = w_0, \ldots, w_n$ for K over $Q(i)$ and write an equation of type (2) for the column vector $(w_j y)$, $0 \leqslant j \leqslant n$, with coefficients which are matrices over $Q[i, D]$. Then $w_0 y = y$ belongs to $\bar{R}_Y = R_Y$ or $\bar{R}'_Y = R'_Y$.

Proof of Theorem II. From Observations in the Introduction we recall that if $r(z)$ has property A and $m_1(z)$ has property B then $r(z) * m_1(z)$ has property B. Also, by Lemma III, $\bar{M}_Y = M_Y$. Therefore, all we must show is that if $m_1(z)$ and $m_2(z)$ belong to M_Y and $r(z)$ belongs to R_Y then $m_1(z) + m_2(z)$ and $r(z) * m_1(z)$ belong to $\bar{M}_Y$.

Set

$$m(z) = \begin{pmatrix} m_1(z) + m_2(z) \\ m_1(z) \\ m_2(z) \end{pmatrix}.$$

Using the equations of type (3) satisfied by $m_1(z)$ and $m_2(z)$ we may express $m(z)$ as a sum of terms $g_j(z)D^j m(z)$ (where the $g_j(z)$ belong to the 3 by 3 matrices over $Q[i, z]$, $\deg g_j(z) < j$, and $1 \leqslant j \leqslant \max\{l_1, l_2\}$) plus a term $a(z)$ with property A. This proves closure under $+$.

Let $m_1(z) = m(z)$ and $l_1 = l$. If $l = 0$ or 1 then by a remark in the Introduction $m(z)$ has property A. In this case $r(z) * m(z)$ has property A, so it is in M_Y. Therefore we may assume $l \geqslant 2$ in what follows. Further, as we shall now show, we may assume that $D^l m(z)$ has property B. Using indefinite integration l times on equation (3) we see that

$$(19) \quad \int_0^z \frac{(z-t)^{l-1}}{(l-1)!} m(t)\,dt - \sum_{j=1}^{l} \sum_{0 \leqslant i < j} (-1)^i \frac{l \ldots (l-1+i)}{i!} \times \big(D^i g_j(z)\big) D^{j-i}\left(\int_0^z \frac{(z-t)^{l-1}}{(l-1)!} m(t)\,dt\right) - \int_0^z \frac{(z-t)^{l-1}}{(l-1)!} a(t)\,dt$$

is a polynomial. Thus

$$\int_0^z \frac{(z-t)^{l-1}}{(l-1)!} m(t)\,dt$$

belongs to M_Y, and

$$D^l \int_0^z \frac{(z-t)^{l-1}}{(l-1)!} m(t)\,dt = m(z)$$

has property B. If we know that

$$r(z) * \int_0^z \frac{(z-t)^{l-1}}{(l-1)!} m(t)\,dt$$

belongs to M_Y it follows that

$$D^l\left(r(z)*\left(\int_0^z \frac{(z-t)^{l-1}}{(l-1)!}\,m(t)\,dt\right)\right)$$

satisfies an equation of the appropriate sort for membership in M_Y. Using Lemma IV (b) l times we see that the above iterated derivative equals $r(z)*m(z)$. Further $r(z)*m(z)$ has property B since $r(z)$ has property A and $m(z)$ has property B. Thus $r(z)*m(z)$ belongs to M_Y.

In what follows we take $l\geqslant 2$ and assume that $D^l m(z)$ has property B. Let $\big((Dz)^{a-1}r(z)\big)*m(z)$ denote the column vector of all

$$\big((Dz)^j r(z)\big)*m(z) \quad \text{for} \quad 0\leqslant j\leqslant a-1$$

and let $\big(D^N r(0)\big)m(z)$ denote the column vector of all $\big(D^k r(0)\big)m(z)$ for $0\leqslant k\leqslant N = 2a\max_j\{\deg h_j(D)\mid h_j(D) \text{ appears in } (2)\}$.

Set

$$\overline{m}(z) = \begin{pmatrix}\big((Dz)^{a-1}r(z)\big)*m(z)\\ \big(D^N r(0)\big)m(z)\end{pmatrix},$$

using matrix block notation. We may write the bottom block of rows as equal to a sum of terms of the appropriate sort for an equation of type (3) in $\overline{m}(z)$, by using equation (3) $N+1$ times. Consider any entry from the top block of rows, say $\big((Dz)^j r(z)\big)*m(z)$ where $0\leqslant j\leqslant a-1$. Using equation (3) we may write this as a sum of terms of the form

$$\big((Dz)^j r(z)\big)*\big((Dz)^p D^q m(z)\big),$$

where $p+q\leqslant l$ and $0<q$, plus $\big((Dz)^j r(z)\big)*a(z)$ which has property A. By Lemma IV (c) we may write each $\big((Dz)^j r(z)\big)*\big((Dz)^p D^q m(z)\big)$ as a sum of terms of the form

$$(Dz)^{p-u}\big[\big((Dz)^{j+u}r(z)\big)*\big(D^q m(z)\big)\big]$$

where $0\leqslant u\leqslant p$. Now we may use the equation of type (2) satisfied by $r(z)$ to write each term above as a sum of terms of the form

$$(Dz)^{p-u}\big[\big(D^s(Dz)^t r(z)\big)*\big(D^q m(z)\big)\big]$$

with $t\leqslant a-1$ and $s\leqslant (i+u)\big(\max_j\{\deg h_j(D)\}\big)\leqslant 2a\big(\max_j\{\deg h_j(D)\}\big) = N$.

Using Lemma IV (b) s times we obtain

$$(Dz)^{p-u}D^s\big[\big((Dz)^t r(z)\big)*\big(D^q m(z)\big)\big],$$

minus terms of the form

$$(Dz)^{p-u}D^{s-j}\big[\big(D^{j-1}r(0)\big)\big(D^q m(z)\big)\big] = (Dz)^{p-u}D^{q+s-j}\big(D^{j-1}r(0)\big)m(z)$$

where $1\leqslant j\leqslant s\leqslant N$. Since $0<q$ these latter terms are of an appropriate sort for an equation of type (2) in $\overline{m}(z)$. Finally using Lemma IV (b) q times we obtain

$$(Dz)^{p-u}D^{s+q}\big[\big((Dz)^t r(z)\big)*m(z)\big],$$

where $t\leqslant a-1$ and $0<q$, plus terms with property A. This proves Theorem II.

Proof of Proposition II. We must show that under the hypotheses of the Corollary $m(z+z_1)$ satisfies an equation of type (1) with coefficients in $Q(i)$ where $z=0$ is a regular point. Let K be the extension of the Gaussian rationals generated by the coefficients of the $g_j(z)$ and by z_1. If $w_0=1, w_1,\ldots,w_n$ is a basis for K over $Q(i)$ then we can write a matrix equation of type (1) for the column vector with components $w_j m(z+z_1)$, $0\leqslant j\leqslant n$, where the coefficients have entries in $Q[i,z]$. We now wish to use the proof that $\overline{M}_Y = M_Y$ in Lemma III. If we let X in the proof of Lemma III be an open disk about zero in $N-z_1 = \{z-z_1\mid z\in N\}$ and replace property A by the property of being analytic on $\overline{D}-z_1 = \{z-z_1\mid z\in\overline{D}\}\supseteq X$, the proof goes through line by line to prove that $m(z+z_1)$ satisfies an equation of type (1) with coefficients in $Q(i)$. (The point is that the class of functions satisfying our "new property A" is closed under addition, multiplication of functional values, indefinite integration, and differentiation just as was the class of functions satisfying our "old property A".) In what follows assume $m(z+z_1)$ satisfies (1).

If $m(z+z_1)$ satisfies any linear differential equation with coefficients in $Q[i,z]$ of the form $Lm(z+z_1)=a_1(z)$ where $a_1(z)$ is analytic on $\overline{D}-z_1$ then for some $N\geqslant 1$

$$(20)\quad m(z+z_1) = \sum_{j=1}^{l} g_j(z)D^j m(z+z_1) + a(z) + D^N\big(Lm(z+z_1)-a_1(z)\big)$$

is an equation of type (1) which has singular points precisely at the singularities of L or of $a(z)-D^N a(z)$.

Now go back to the scalar equation for $m(z+z_1)$ which was regular at zero. We may easily obtain an equation of the form $p(z)D^n m(z+z_1)$ equals a linear combination of the $D^j m(z+z_1)$, $0<j\leqslant n-1$, over $K[z]$ plus some $a(z)$ analytic on $\overline{D}-z_1$, where $p(z)$ belongs to $Q[i,z]$ and $p(0)\neq 0$. Applying the ascending chain condition to the module over $Q[i,z]$ generated by the elements of

$$\{(p(z))^N D^N m(z+z_1)\mid N\geqslant 0\}$$

we obtain an equation of the desired sort $Lm(z+z_1)=a_1(z)$ with $a_1(z)$ analytic on $\overline{D}-z_1$ and $z=0$ a regular point of L.

Now we can apply the Corollary to (20) and the objects $\overline{D}'$, N', z_0', z_1', and $\mathscr{C}'$ where

$$\overline{D}' = \overline{D} - z_1 = \{z - z_1 \mid z \in \overline{D}\},$$

N' is a sufficiently small neighborhood of zero, $z_0' = z_1' = 0$, and $\mathscr{C}'$ is the translation of $\mathscr{C}$. Thus the conclusion of Proposition I holds here but for possibly new values of l and d.

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A method in diophantine approximation (IV)*

by

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Introduction. In this paper we shall extend to functions of $n > 1$ complex variables some of the results of Part III of this series of papers (referred to below as Part III).

DEFINITION. By $Z, Q, Q(i)$, and C we shall mean, respectively, the integers, the rational numbers, the Gaussian rational numbers, and the complex numbers. Throughout this paper t and z will stand for n-tuples of complex numbers.

By D^h we shall mean $\frac{\partial^{h_1}}{\partial z_1^{h_1}} \cdots \frac{\partial^{h_n}}{\partial z_n^{h_n}}$, where each h_k $(1 \leqslant k \leqslant n)$ is a non-negative integer. Analogously we define D^j and D^∞. We define D^{h+j} to be $D^h \cdot D^j$. We use D_k to denote $\frac{\partial}{\partial z_k}$ and θ to denote $\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_n}$. By $|h|$ or $|j|$ we shall mean $\max\{h_k \mid 1 \leqslant k \leqslant n\}$ or $\max\{j_k \mid 1 \leqslant k \leqslant n\}$, respectively.

If for some positive integer N, $g(z)$ belongs to the N by N matrices over $Q[i, z]$, then by $\deg_k g(z)$ we mean $\min_i \{i \geqslant 0 \mid D_k^{i+1} g(z) \equiv 0\}$.

We define a norm, $\| \ \|$, on matrices over the complex numbers by letting $\|\text{matrix}\|$ denote the maximum of the absolute values of the entries of the matrix.

For each $1 \leqslant k \leqslant n$ choose X_k to be a bounded starshaped region about zero in C which shall remain fixed throughout this paper. Let L_k and Y_k denote the Riemann surfaces generated by $\log z_k$ over C and X_k respectively. Note zero is not in either L_k or Y_k. We shall regard $C - \{0\}$ and $X_k - \{0\}$ as being embedded in L_k and Y_k respectively. Set $L = \prod_{k=1}^{n} L_k$, $Y = \prod_{k=1}^{n} Y_k$ and $X = \prod_{k=1}^{n} (X_k - \{0\})$. Then $X \subset Y \subset L$. From now on any

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