Finally we prove that \( f(q) = 0 \) for all \( q \in \bigcup_{i=0}^{\infty} \mathcal{F}_i \) if \( Q_0 \) is sufficiently large.

Let \( P(y, q) \) denote the number of those \( p \leq y \) for which \( p + 1 = kq \) and the prime factors of \( k \) do not belong to \( \mathcal{F} \). We prove that \( P(y, q) > 0 \) if \( y \) is large, whence \( f(q) = 0 \) follows.

Indeed,
\[
P(y, q) \geq \pi(y, q, -1) - \sum_{\sigma \leq q} \pi(y, q', -1).
\]

For large \( y \) we have
\[
\pi(y, q, -1) > \frac{1}{2} \frac{y}{\log y}.
\]

Furthermore by Lemma 2
\[
\sum_{\sigma \leq q} \pi(y, q', -1) \leq \frac{C_1}{\log y} \sum_{\sigma \leq q} \frac{1}{q} + \frac{y}{q} \sum_{\sigma \leq q} \sum_{q' \leq y/q} \frac{1}{q'}.
\]

Since \( \sum 1/q' \leq C_1 \) with large \( Q_0 \) and
\[
\sum_{q' \leq y/q} \frac{1}{q'} \leq \log y \max_{q' \leq y/q} \sum_{q' \leq y/q} \frac{1}{q'} \leq \frac{1}{\log^2 y},
\]

we have
\[
\sum_{\sigma \leq q} \frac{1}{q} \leq \frac{1}{4} \frac{y}{\log y}.
\]

Hence, by (12), (13), \( P(y, q) > 0 \) follows. This completes the proof of the Theorem.

4. The constant \( K \) in the Theorem is non-effective since \( C(A, B) \) in Lemma 3 is non-effective. It would be very interesting to prove the Theorem with effective \( K \) since this would give a possibility to decide with numerical calculation whether \( \mathcal{F}_i \) is a set of uniqueness or not.

References


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A method in diophantine approximation (III)*

by

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Introduction. We begin by giving the hypotheses and statement of a result, called Proposition I below, which was stated and proved in [2] using slightly different notation.

Let \( D \) denote differentiation with respect to the complex variable \( z \), let \( l \) be an integer greater than one; let each \( g_l(z) \) for \( 1 \leq j \leq l \) be a polynomial of degree less than \( j \) with coefficients in the Gaussian field. Suppose that we are in a simply connected region \( D \) where \( a(z) \) is analytic and that \( m_1(z), \ldots, m_n(z) \) are \( n \geq 1 \) solutions of
\[
m(z) = \sum_{j=1}^{l} g_j(z) D^j m(z) + a(z)
\]

which are analytic in some open disk \( N \subset D \) about \( z_0 \) on which \( g_l(z) \) does not vanish. Suppose \( z_1 \) belongs to \( N \) and \( z_2 \) is a Gaussian rational.

Let \( \gamma \) be a differentiable path in \( D \) with endpoints at \( z_0 \) which does not pass through any of the zeros of \( g_l(z) \). Suppose that \( \tilde{m}_1(z) \neq m_1(z), \ldots, \tilde{m}_n(z) \neq m_n(z) \) are the function elements, analytic on \( N \), obtained by extending \( m_1(z), \ldots, m_n(z) \), respectively, along \( \gamma \) and back to \( z_0 \) and that \( \tilde{m}_1(z) - m_1(z), \ldots, \tilde{m}_n(z) - m_n(z) \) are linearly independent over the complex numbers. Let
\[
d = \max \left\{ \frac{\deg g_j(z)}{j - \deg g_1(z)} \right\}_j.
\]

Let \( ||a||_\gamma \), for any complex number \( a \), denote the distance from \( a \) to the nearest Gaussian integer. Let \( (A_1, \ldots, A_n) \) denote any nonzero element of the Cartesian product of the Gaussian integers with themselves \( n \) times.

* This paper was written while the author was on a postdoctoral Research Associateship at the National Bureau of Standards, Washington, D. C. (This award is given by the National Bureau of Standards in association with the National Academy of Sciences and the National Research Council.)
Proposition I. For each \(\varepsilon > 0\) there exists a \( c(\varepsilon) > 0 \) such that
\[
\max_{s \leq l < t} \left\{ \left\| \sum_{j=1}^{n} A_j D'(\tilde{m}_s(z_j) - m_s(z_j)) \right\| \right\} \geq c(\varepsilon) \min_{h} \{ |A_h|^{-1/(2l+\varepsilon)} \}
\]
for all \((A_1, \ldots, A_n)\).

If \( n = 1 \) and \( m_1(z) = m(z) \) the above result may be rephrased as:

Corollary. There exists \( c(\varepsilon) > 0 \) such that
\[
\max_{s \leq l < t} \left\{ \left\| D'(\tilde{m}_s(z_1) - m(z_1)) \right\| \right\} \geq c(\varepsilon) \left\| q^{-1/(2l+\varepsilon)} \right\|
\]
for all Gaussian integers \( p \) and \( q \) with \( q \neq 0 \).

(Note that if \( l \) is zero or one then the requirement that \( \deg g_j(z) < j \)
for each \( j \) forces the \( g_j(z) \) to each be constant so all solutions of (1) are
analytic and Proposition I and the Corollary are vacuously true.)

What we wish to do in this paper is investigate the structure of the
set of functions \( m(z) \) which satisfy an equation of type (1) with \( l \geq 0 \)
where \( D \) is an appropriate sort of simply connected region \( X \). If we assume
that \( X \) is bounded and starshaped around zero and restrict ourselves to
functions \( m(z) \) which are “nicely behaved” on such a region we uncover
an interesting algebraic structure, which has number-theoretic applications
in light of the Corollary above.

Additionally a result drops out of the machinery developed to uncover
this structure which is both interesting in its own right and helpful in
showing that the restrictions on \( X \) assumed throughout the remainder
of this paper are not as limiting as might be supposed. This result is:

Proposition II. If in the Corollary we weaken the hypotheses by letting
the \( g_j(z) \) have algebraic coefficients and letting \( z_j \) be algebraic, then \( m(z + z_j) \)
satisfies the hypotheses of the Corollary for a new \( N \) containing zero, for
\( D \) shifted by \(-z_j\), and for some new \( l \) and collection of \( g_j(z) \) \((1 \leq j \leq l)\)
belonging to \( Q[i, z] \). Hence the conclusion of the Corollary holds but for,
in general, new constants \( \tilde{d} \) and \( \tilde{l} \).

(The proof of Proposition II is in the text of the paper.) Proposition II
helps to justify the restricted situation which we shall deal with in the
balance of this paper, where \( X \) is a bounded region starshaped about
zero in which the only possible singular point of \( m(z) \) is zero. In the situa-
tion of the Corollary let \( w \) be any singular point of \( m(z) \) in \( D \). Then \( w \)
is a zero of \( g(z) \) and, hence, algebraic. Choose \( r \), a rational number, such
that \( w + w \) belongs to \( D \) and is not a zero of \( g(z) \). We see upon applying
Proposition II that \( m(z + w + r) \) satisfies an equation of type (1) with
\( a(z) \) analytic on \( D - (w + r) \). Now translate equation (1) by \( r \) and we
have placed the singular point at zero. Any point \( z \) of \( D - w \) such that the
ray from zero to \( z \) lies entirely in \( D - w \) and does not contain any singulari-
ties of \( m(z) \) may be included in a bounded region \( X \) which is starshaped
at zero and such that zero is the only singular point of \( m(z) \) in \( X \).

(Further Proposition II allows us to obtain the following result:
if in the hypotheses of the Corollary some unique \( g(z) \) appearing in (1)
is of highest degree then \( \tilde{m}(z) - m(z) \equiv 0 \) can not have a power series
expansion with only Gaussian rational coefficients about any regular
point in \( X \). We see immediately that this holds for all regular algebraic
points in \( X \). If \( z = \alpha \) is a transcendental number in \( X \), then for some
appropriate non-negative integer \( N \) the relation
\[
D_N \left[ 1 - \sum_{j=1}^{l} g_j(z) \right] (\tilde{m}(z) - m(z)) = 0
\]
implies that \( \alpha \) is a root of a polynomial of degree \( \max \{ \deg g_j(z) \} \). Since
\( \tilde{m}(z) - m(z) \equiv 0 \) we have \( \max \{ \deg g_j(z) \} \geq 1 \). Therefore some derivative
of \( \tilde{m}(z) - m(z) \) is transcendental at \( z = \alpha \) and we are through.)

We now make some definitions and give some observations before
stating the theorems of this paper.

Definitions. Let \( Q \) denote the rational numbers. Let \( C \) denote the
complex numbers. Let \( X \) denote a bounded starshaped region about
zero. Let \( L \) be the Riemann surface over \( C \) corresponding to \( L_g \). Let \( Y \)
be the Riemann surface over \( X \) corresponding to \( L_g \). We regard \( C \{ 0 \} \)
and \( Y \) as embedded in \( L \). Note \( 0 \in Y \).

If a function is analytic on \( X \) we say that it has property \( A \). If it is
analytic on \( X \) and is bounded on every finite angular sector in \( Y \), i.e.
any set of the form \( \{ z \in Y : \alpha < \arg z < \beta \} \), we say that it has property \( B \). We use “regular singular” for “at worst regular singular”.

Let \( R_Y \) be the set of all functions with property \( A \) which satisfy
a linear differential equation of the form
\[
(\Delta z)^{a} (z) = \sum_{j=0}^{a} h_j(D) (\Delta z)^{a} (z)
\]
where \( a > 0 \) and each \( h_j(D) \) belongs to \( Q[i, z] \).

Let \( E_Y \) be the set of all functions \( r(z) \) with property \( A \) which satisfy
a linear homogeneous differential equation with coefficients in \( Q[i, z] \)
that has a regular singular point at infinity (possibly a regular point).

Let \( M_Y \) be the set of all functions with property \( B \) which satisfy
a linear differential equation of the form
\[
m(z) = \sum_{j=1}^{l} g_j(z) D^j m(z) + a(z),
\]
where each \( g_j(x) \) belongs to \( Q[i, z_i] \) deg\( g_j(x) < j \) for each \( j, l > 0 \), and \( a(z) \) has property A.

If \( f_1(x) \) has property A and \( f_2(x) \) has property B we define \( f_1(x)*f_2(x) \) on \( Y \) by

\[
f_1(x)*f_2(x) = \int_0^x f_1(x-t)f_2(t)\,dt
\]

where the path of integration is the ray from zero to \( z \).

Observations. It is clear that \( f_1(x)*f_2(x) \) is bounded on any finite angular sector in \( Y \). Suppose \( z_1 \) belongs to \( Y \). Then there exists a region

\[
\Pi_1 = \{z \in U \mid 0 < |z| < r \text{ and } \alpha_1 < \arg z < \alpha_1 + 1\}
\]

which contains \( z_1 \) and is contained in \( Y \). Denote the region in \( X - \{0\} \) over which \( \Pi_1 \) lies by \( \Pi'_1 \). Choose \( t_1 > 0 \) such that \( N = \{z \in U \mid |z| < t_1\} \) is contained in \( X \). It is always possible to choose

\[
\Pi'_1 = \{z \in X \mid 0 < |z| < r \text{ and } \beta_1 < \arg z < \beta_2 \subset \Pi'_1
\]

containing \( z_1 \) such that \( \Pi'_1 - \Pi_1 \), the set of all differences of elements of \( \Pi'_1 \), is contained in \( N \cup N' \). Suppose \( \Pi_1 \neq \Pi'_1 \), lies over \( \Pi'_1 \) on \( Y \). Then \( f_1(x-t) \) is analytic on \( \Pi_1 \times \Pi'_1 \). Hence on \( \Pi_1 \) we may replace the ray from zero to \( z_1 \) by any differentiable path which lies totally in \( \Pi_1 \), except for the initial endpoint zero. It is now easy to set up the difference quotient and calculate the derivative of \( f_1(x)*f_2(x) \) in \( \Pi'_1 \), which equals

\[
(Df_1(x))*f_2(x) + f_1(0)f_2(x).
\]

Therefore \( f_1(x)*f_2(x) \) has property B. If \( f_2(x) \) has property A, of course, then so does \( f_1(x)*f_2(x) \).

We shall prove the following two theorems about the algebraic properties of \( R_X, R_Y, \) and \( M_X \).

**Theorem I.** (a) \( R_X \) is a ring under \( * \) and \( + \).
(b) \( E_X \) is a subring of \( R_X \).
(c) \( E_X \) is also a ring under multiplication of functional values and \( + \).
(d) If we replace \( Q(i) \) by any finite extension of \( Q(i) \) in the definitions of \( R_X \) and \( E_X \) we obtain the same sets.

**Theorem II.** \( M_X \).h is an \( R_X \) module under \( * \).

**Examples.** Given a linear differential equation with coefficients in \( Q[i, z] \) which has a regular singular point at \( z_1 \) in \( Q(i) \) and which has a regular point at \( z_1 + \tilde{z}_1 \) in \( Q(i) \) we let \( x = x_1 + (z_1 - w)^{-1} \) and see that the transformed equation has a regular point at zero and a regular singular point at infinity. Let \( X \) be any region which is starshaped about zero and excludes the singularities of the transformed equation. Set \( m(w) = \frac{1}{2} \sin \sqrt{w} \), observing that \( m(w) = -2m(\sqrt{w}) - 4wD^2m(\sqrt{w}) \). Then \( m(w) \) belongs to \( M_X \). By Theorem II if \( f(x) \) was any solution of the original equation then \( \frac{1}{2} f(x) = f(x_1 + (z_1 - w)^{-1}) \sin \sqrt{w} \) belongs to \( M_X \). Choose \( w_1 \) to be a nonzero Gaussian rational in \( X \). Let \( g \) wind once about zero. By the Corollary to Proposition I we conclude that \( f(x_1 + (z_1 - w)^{-1}) \sin \sqrt{w} \) has at least one non-Gaussian rational in its power series expansion about \( w_1 \). Instead of \( \sin \sqrt{w} \) we might use

\[
g(w) = \log \sqrt{w}/2\pi i J_1(\sqrt{w})/J_4(\sqrt{w})
\]

where \( J_4(w) \) is the Bessel function of order one. Note that \( \sqrt{w} J_1(\sqrt{w}) \) satisfies the differential equation

\[
y = -4wD^2y
\]

so

\[
g(w) = -4wD^2g(w) + a(w)
\]

where

\[
a(w) = 4w D((2\pi i \sqrt{w})^{-1} J_1(\sqrt{w})) - 4((2\pi i)^{-1} D[\sqrt{w}J_1(\sqrt{w})])
\]

is an entire function. Thus proceeding as above we conclude that

\[
f(x_1 + (z_1 - w)^{-1}) \sin \sqrt{w} \sin J_1(\sqrt{w})\]

has a non Gaussian rational in its power series expansion about \( w_1 \).

Also we note that \( m(w) = f(x_1 + (z_1 - w)^{-1}) \sin \sqrt{w} J_1(\sqrt{w}) \) itself satisfies an equation of type (1) with \( a(z) = 0 \). Thus we may apply the Corollary with \( D = C \). The different analytic continuations of our new \( m(z) \) from \( X \) are given by the integral

\[
\int_0^x f(z-t) J_1(\sqrt{z-t}) \left((z_1 - t)^{-1} \right) dt
\]

where the path must only avoid the singularities of the function from \( E_X \).

If \( X \) is \( \{z \mid |z| < \tilde{z}_1\} \) then \( (z-1)^{-1} \) and \( (z-2)^{-1} \) each belong to \( E_X \); hence so does \( (z-1)^{-1}(z-2)^{-1} \). Now

\[
\int_0^x f(z-t) J_1(\sqrt{z-t}) (t-1)^{-1} (t-2)^{-1} dt
\]

has branch points at \( z = 1 \) and \( z = 2 \) and the difference between two branches is of the form

\[
\int_0^x f(z-t) J_1(\sqrt{z-t}) (t-1)^{k} (t-2)^{-1} dt,
\]

where the path of integration winds around the branch points 1 and 2.
Theorems I and II were suggested by study of the effects of Laplace transforms and inverse Laplace transforms upon some of the functions under discussion. The Laplace transform is not used in our proofs because of convergence problems. However, it could be used to formulate a different proof of Theorem I (b).

**Section I.**

**Lemma I.** \( R_F \) is the set of all elements of \( E_F \) which satisfy an equation of type (2) with \( \deg h_j(D) \leq a - j \) for each \( j \).

**Proof.** First we shall show that any equation of type (2) with \( \deg h_j(D) \leq a - j \) for each \( j \) has a regular singular point at infinity. Note that \( Dz = zD + 1 \) so we may rewrite (2), under these conditions, as

\[
(zD)^j r(z) = \sum_{j=0}^{\alpha-1} h_j(D)(zD)^j r(z)
\]

for some new set of \( h_j(D) \) in \( Q[i,D] \) with \( \deg h_j(D) \leq a - j \) for each \( j \). Now set \( w = x^{-1} \). We note that \( zD \) goes into \( -wD \) and \( D \) goes into \( -wD \). This gives

\[
(-wD)^j r(w) = \sum_{j=0}^{a-1} h_j(-wD)(-wD)^j r(w),
\]

and using \( Dw = wD + 1 \) repeatedly we obtain

\[
(-wD)^j r(w) = \sum_{j=0}^{a-1} h_j(w)(wD)^j r(w) + \sum_{j=0}^{a-1} k_j(w)(wD)^j r(w)
\]

for a set of \( h_j(w) \) in \( Q[i,w] \). It follows from examining (4) that the coefficient of \( D^j r(w) \) equals \( (-1)^n w^n h_n(w) w^{n+1} \) for some \( h_n(w) \) in \( Q[i,w] \) and that each coefficient of each \( D^j r(w) \) vanishes to at least the order \( j \) at zero. Then zero is a regular singular point, so infinity was a regular singular point of (2).

Now suppose that \( r(w) \) satisfies a homogeneous linear differential equation with coefficients in \( Q[i,w] \) which has a regular singular point at zero. Setting \( z = w^{-1} \) we wish to show that \( r(z) \) satisfies an equation of type (2) with \( \deg h_j(D) \leq a - j \) for each \( j \). Suppose that we write the equation satisfied by \( r(w) \) in the form

\[
p_j(w)(wD)^\beta r(w) = \sum_{j=0}^{\beta-1} p_j(w)(wD)^j r(w)
\]

for some \( \beta > 0 \) where the \( p_j(w) \) are in \( Q[i,w] \) and \( p_j(0) \) is nonzero. Change variables to \( z = w^{-1} \) obtaining

\[
p_j(z^{-1})(-zD)^\beta r(z) = \sum_{j=0}^{\beta-1} p_j(z^{-1})(-zD)^j r(z).
\]

Let \( t \) be the maximum of the degrees of the \( p_j(w) \) for \( 0 \leq j \leq \beta \). Then

\[
p_j(w)(wD)^\beta r(w) = D^{\beta}(\sum_{j=0}^{\beta-1} p_j(z^{-1})(-zD)^j r(z) - D^\beta \sum_{j=0}^{\beta-1} p_j(z^{-1})(-zD)^j r(z)).
\]

Setting \( a = \beta + 1 \) and using \( wD = zD - 1 \) repeatedly we may put (5) in form (2) with \( \deg h_j(D) \leq a - j \) for each \( j \).

**Definitions.** If \( h(D) \) belongs to the \( n \) by \( n \) matrices over \( Q[i,D] \) then \( \deg h(D) \) denotes the maximum of the degrees of its components.

Let \( \overline{E}_F \) be the set of all component functions of all \( n \) by \( 1 \) matrix valued functions \( 0 = 1, 2, \ldots \) with property \( \Lambda \) which satisfy an equation of type (2), where the \( h_j(D) \) belong to the \( n \) by \( n \) matrices over \( Q[i,D] \) and for each \( j \), \( \deg h_j(D) \leq a - j \). Clearly \( \overline{E}_F \supseteq E_F \).

Let

\[
A_A(x) = \sum_{i=0}^{k} A_A(x) D^i y = 0
\]

be any linear homogeneous differential equation with coefficients in the \( n \) by \( n \) matrices over \( Q[i,x] \). If each \( D^k A_A(x) A_A(x) \) is analytic at zero for \( 0 \leq i \leq k \) we say that (6) has a regular singular point at zero. If we set \( w = z^{-1} \) in (6) and the transformed equation has a regular singular point at zero we say that (6) has a regular singular point at infinity.

Let \( \overline{E}_F \) be the set of component functions of all \( n \) by \( 1 \) matrix valued functions \( r(x) \) (for \( n = 1, 2, \ldots \) ) having property \( \Lambda \) and satisfying a linear homogeneous differential equation with coefficients in the \( n \) by \( n \) matrices over \( Q[i,x] \) that has a regular singular point at infinity. Clearly \( \overline{E}_F \supseteq E_F \).

**Lemma II.** \( \overline{E}_F \supseteq E_F \).

The proof is the same as the proof of Lemma I, except now the \( h_j(D), h_j(D), \) and \( p_j(D) \) of \( j \neq \beta \) are matrices. We choose \( p_j(w) \) as the smallest degree monic polynomial such that each \( p_j(w) \) has components in \( Q[i,w] \), for \( 0 \leq j \leq \beta - 1 \).

**Definitions.** Let \( \overline{E}_F \) be the set of component functions of all \( n \) by \( 1 \) matrix valued functions \( m(x) \) (for \( n = 1, 2, \ldots \) ) which have property \( B \) and satisfy a linear differential equation of form (3) where each \( g_j(x) \) belongs to the \( n \) by \( n \) matrices over \( Q[i,x] \), and \( a(x) \) has property \( \Lambda \). Clearly \( \overline{E}_F \supseteq E_F \).
**Lemma III.** (a) \( R_T = R_Y \), (b) \( R_T = R_Y \), and (c) \( M_T = M_Y \).

**Proof.** (b) We notice that in equation (2) we may apply \( D_a \) to the equation and obtain, after simplification, another equation of the same type with \( a + 1 \) replacing \( a \), whether equation (2) is scalar or vector. Applying \( D_{a+1} \) to (2) we obtain a set of equations which allows us to express each \( (D_{a+1})^r_a \) as a sum of terms of the form a polynomial in \( D \) times \( (D_{a+1})^r_a \) for \( 0 \leq j \leq a - 1 \) and \( 1 \leq k \leq n \). Therefore, the functions \( (D_{a+1})^r_a \), for \( N \geq 0 \), are contained in a finitely generated module over the Noetherian ring \( Q[i, D] \). By the ascending chain condition on finitely generated modules over a Noetherian ring we see that for some \( N \), \( (D_{a+1})^r_a \) may be written as a linear combination over \( Q[i, D] \) of \( (D_{a+1})^r_a \) with \( 0 \leq j < N \). This proves (b).

(a) We use \( R_T = R_Y \). Recall from the proof of Lemma II that any vector differential equation with coefficients in \( A \) by \( n \) matrices over \( Q[i, w] \) which has a regular singular point at zero may be written in the form

\[
p(w)(wD)^p r(w) = \sum_{j=0}^{p-1} p_j(w)(wD)^j r(w)
\]

where each \( p_j(w) \) \( (0 \leq j < p - 1) \) belongs to the \( n \) by \( n \) matrices over \( Q[i, w] \), \( p(0) \) is not zero. Applying \( wD \) repeated to (7) we may express each \( (wD)^N p_j(w)(wD)^j r(w) \) \( (1 \leq k \leq n) \) as a linear combination over \( Q[i, w] \) of \( (wD)^r_a \) \( (0 \leq j < N + 1) \) and \( 1 \leq k \leq n \). Using all of these equations we may express each \( (wD)^N p_j(w)(wD)^j r(w) \) \( (0 \leq j < N + 1) \) and \( 1 \leq k \leq n \) as a linear combination over \( Q[i, w] \) of \( (wD)^r_a \) \( (0 \leq j < N + 1) \) and \( 1 \leq k \leq n \) for each \( N \geq 0 \). By the ascending chain condition we conclude that for some \( N \geq 0 \), \( (wD)^N p_j(w)(wD)^j r(w) \) may be written as a linear combination of \( (wD)^r_a \) \( (0 \leq j < N + 1) \) and \( 1 \leq k \leq n \) over \( Q[i, w] \). This equation has a regular singular point at zero. A change of variables proves the corresponding statement about regular singular points at infinity.

Ci) We assume that we are working with the vector version of (3). Differentiating both sides of (3) \( N \) times we obtain

\[
D^N m(z) - D^N a(z) = \sum_{j=1}^{N} k_j(N, z) D^N m(z)
\]

where each \( k_j(N, z) \) belongs to the \( n \) by \( n \) matrices over the Noetherian ring \( Q[i, N, z] \). We use (8) to write, formally, \( D^N m(z) \) \( = (a \text{ function with property } \Lambda) \), \( D^{N-1} m(z) \) \( = (a \text{ function with property } \Lambda) \), . . . , as a linear combination over \( Q[i, N, z] \) of the \( D^{N-1} m(z) \) \( (1 \leq j \leq t \) and \( 1 \leq k \leq n) \). By the ascending chain condition we then have for some positive integer \( \lambda \), formally

\[
D^N m(z) = \sum_{j=1}^{N} g_j(N, z) D^{N-j} m(z) + b(N, z),
\]

where each \( g_j(N, z) \) belongs to \( Q[i, N, z] \) and \( b(N, z) \) has property \( \Lambda \). If \( N \geq \lambda \) then the above relation is valid. Replace \( N \) by \( N \) where now \( N \geq N \). Then

\[
D^{N} m(z) = \sum_{j=1}^{N} g_j(N, z) D^{N-j} m(z) + a(N, z)
\]

where each \( g_j(N, z) \) belongs to \( Q[i, N, z] \) and \( a(N, z) \) has property \( \Lambda \) for each \( N \geq 0 \). We wish to replace \( \sum g_j(N, z) D^{N-j} m(z) \) by \( D^{N} m(z) \) a particular \( N \)-fold indefinite integral of \( \sum g_j(N, z) D^{N-j} m(z) \). Let us first define

\[
D^{-r} m(z) = \frac{z}{r} \frac{(z-1)^{r-1}}{(r-1)!} m_0(t) dt,
\]

for each \( r > 0 \), where the path is the ray from zero to \( z \). Now given any expression of the form \( p(z) D^{m}(z) \), where \( p(z) \) is a polynomial in \( z \) of degree \( d \) and \( a \) is any integer, we define \( T \) by

\[
T p(z) D^{m}(z) = \sum_{j=0}^{d} (-1)^j D^{j} p(z) D^{m-j}(z),
\]

and extend \( T \) by linearity to the space of all sums of terms of the form \( p(z) D^{m}(z) \). If \( N \) is any positive integer we have then

\[
T^N p(z) D^m(z) = \sum_{j=0}^{d} (-1)^j \frac{(N+j-1)!}{j!} D^{j} p(z) D^{N-j}(z).
\]

Now (9) may be rewritten as

\[
D^{N} m(z) = D^{N} T^N \left( \sum_{j=1}^{N} g_j(N, z) D^{N-j} m(z) + b(N, z) \right)
\]

\[
= D^{N} \left( \sum_{j=1}^{N} r_j(N, z) D^{N-j} m(z) + a(N, z) \right),
\]

for some \( k \geq 0 \) and a set of \( r_j(N, z) \) in \( Q[i, N, z] \). We may rewrite (10) in the form

\[
D^{N} m(z) = \sum_{j=0}^{N} \frac{(N+j)!}{j!} D^{j} \left( \sum_{j=1}^{N} r_j(N, z) D^{N-j} m(z) + a(N, z) \right)
\]

where each \( r_{j,N}(z) \) belongs to \( Q[i, z] \) and \( \beta \) is some non-negative integer.
We now show that if \( s \geq 1 \),
\[
D^N \left\{ \sum_{j=1}^{s} U_{s,j}(x) D^j m_s(z) \right\}
\]
has property A, for each nonnegative value of \( N \). Differentiating (15) and subtracting from this derivative (15) with \( N+1 \) substituted for \( N \) gives
\[
\sum_{j=1}^{s} s^j D^N \left\{ \sum_{j=1}^{s} U_{s,j}(x) D^j m_s(z) \right\}
\]
has property A. Now differentiate (16) and subtract from this derivative (16) with \( N+1 \) substituted for \( N \). Continuing we obtain the result that each
\[
D^N \left\{ \sum_{j=1}^{s} U_{s,j}(x) D^j m_s(z) \right\}
\]
has property A, for each \( s \geq 1 \). Set \( N = 0 \) in (15) obtaining
\[
m_h(z) = \sum_{j=1}^{s} U_{0,j}(x) D^j m_h(z) + a(0, z) + a(z)
\]
where \( U_{0,j}(z) < j \) for each \( j \) and \( a(z) \) have property A. This completes the proof of Lemma III.

We suppose that \( f_1(z) \) has property A and \( f_2(z) \) has property B below. Then:

**Lemma IV.**

- (a) \( \varphi(1) \) \( f_1(z) \) \( f_2(z) \) = \( \varphi(1) \) \( f_1(z) \) \( f_2(z) \),
- (b) \( D^j f_1(z) \) \( f_2(z) \) = \( D^j f_1(z) \) \( f_2(z) \) \( f_1(0) \) \( f_2(0) \), which equals \( f_1(z) \) \( D^j f_1(z) \) \( f_2(z) \), \( f_1(0) \) \( f_2(0) \), if \( D^j f_1(z) \) has property B.
- (c) \( D^j f_1(z) \) \( f_2(z) \) = \( D^j f_1(z) \) \( f_2(z) \) \( f_1(0) \) \( f_2(0) \), if \( D^j f_1(z) \) has property B.

Proof. (a) This is trivial. (b) The second part of this follows upon using integration by parts on \( f_1(z) \) \( f_2(z) \) \( f_1(0) \) \( f_2(0) \) \( D^j f_1(z) \) \( f_1(0) \) \( f_2(0) \), (The first part was covered under Observations in the Introduction.) (c) Use (a) and (b). This proves Lemma IV.

We assume below that \( r(z) \) satisfies an \( \alpha \)th order scalar differential equation of type (2) (alternately, with the extra condition \( \deg h_i(D) \leq a-j \) for each \( j \)). Let \( r(z) \) be the \( (a+1)\alpha /2 \) by one matrix consisting of the component functions \( D^1(D^2)^y(z) \), for \( 0 \leq i+h \leq a-1 \).

**Lemma V.** Then \( r(z) \) satisfies an \( \alpha \)th order vector differential equation of type (2) (alternately, with \( \deg h_i(D) \leq a-j \) for each \( j \)).
Proof. Let us write (2) as

\[(D_2)^s r(z) = \sum_{0 \leq i + k \leq a - 1} \beta_{i+k} (D_1)^i (D_2)^k r(z),\]

(alternatively, with \(q + s \leq a\)) for a set of Gaussian rational \(\beta_{i+k}\). Operate on (18) with \((D_1)^s (D_2)^s\) where \(0 \leq i + k \leq a - 1\). We notice that

\[D_1(D_2)^s (D_2)^s r(z) = (D_2)^s (D_1)^s (D_2)^s r(z),\]

plus terms involving \(k + a = 0\) factors of \(z\) and \(i + k + a = 0\) factors of \(D_1\) where \(1 \leq \theta \leq a\). The terms on the right hand side of (18), after operating with \((D_1)^s (D_2)^s\), each have \(k + a\) factors of \(z\) and \((q + s) + k + a\) factors of \(D_1\) occurring. Writing \(z = a - \theta\) where \(1 \leq \theta \leq a\), we have \(k + a = \theta\) factors of \(z\) and \(q + s + \theta + a\) factors of \(D_1\) for \(0 \leq \theta \leq a\) (alternatively, \(0 < q \leq \theta \leq a\)). Consolidating terms above we may write \((D_2)^s (D_1)^s (D_2)^s r(z)\)

as a sum of terms involving \(k + a = 0\) factors of \(z\) and \((q + s) + k + a = 0\) factors of \(D_1\) where \(0 \leq \theta \leq a\) (alternatively, \(0 < q \leq \theta \leq a\)). Using \(aD = D_2 - 1\) repeatedly we attempt to write \((D_2)^s (D_1)^s (D_2)^s r(z)\)

as a sum of terms each having a factor of \((D_2)^s\) directly in front of \(r(z)\). If this fails in a particular term we have run out of factors of \(z\). Thus, more generally, we attempt to place \((D_2)^{s-a} (D_2)^{a} r(z)\)

directly in front of \(r(z)\) for \(0 \leq s \leq a - 1 - t\). Otherwise our procedure does not fail and we continue, using \(aD = D_2 - 1\), to obtain a sum of terms of the form \((D_2)^s (D_1)^s (D_2)^s r(z)\)

for \(u \geq 0\) and \(v \geq 0\) where \(i \leq s - 1, v = 0\). (Note the total number of factors of \(D_1\) occurring above can not be greater than \(q + s + \theta + a\) which, alternately, is less than or equal to \(i + k + a = 2a - 1\). Thus in this case \(a + s \leq a\).) Now writing the above equations (for each \(i \) and \(k\) in the desired range) in matrix form gives the desired equation.

Section II.

Proof of Theorem I. We note (see Observations in the Introduction) that property A is preserved under + and *. Also, by Lemma I, \(R_T \supseteq R'_T\).

Thus we only need show closure of \(R_T\) and \(R_T'\), respectively, under + and *.

Because of Lemmas II and III we need only show that the sums and products lie in \(R_T\) or \(R'_T\), respectively.

Suppose that \(r_1(z)\) and \(r_2(z)\) are in \(R_T\) (alternately, \(R'_T\)). We shall show that using matrix block notation

\[r(z) = \begin{pmatrix} r_1(z) + r_2(z) \\ r_1(z) \\ r_2(z) \end{pmatrix},\]

satisfies an equation of type (2) (alternatively, where \(\text{deg} h_1(D) \leq a - j\) for each \(j\)). Applying powers of \(D_2\) to the equation satisfied by \(r_1(z)\) or \(r_2(z)\) we see that we may take \(a_1 = a_2 = a\). Then

\[(D_2)^s r_1(z) = \begin{pmatrix} (D_2)^s r_1(z) \\ (D_2)^s r_2(z) \end{pmatrix}.\]

Each of the rows on the right above may be expressed as a sum of terms of the appropriate sort, using the equations of type (2) for \(r_1(z)\) and \(r_2(z)\). This proves closure under +.

Keeping \(r_1(z)\), \(r_2(z)\), and \(a\) as above, form \(\tilde{r}_1(z)\) and \(\tilde{r}_2(z)\). (See the paragraph before the statement of Lemma V.) Then by Lemma V, \(\tilde{r}_1(z)\) and \(\tilde{r}_2(z)\) each satisfy an \(\alpha\)th order equation of type (2) (alternatively, with \(\text{deg} h_1(D) \leq a - j\) for each \(j\)). Let \(\tilde{r}_1(z) \ast \tilde{r}_2(z)\) denote the column vector of all possible convolution products of the form a component of \(r_1(z)\) times a component of \(r_2(z)\). Let \(D_2^{N} \tilde{r}_1(z)\) \(\tilde{r}_2(z)\) denote the column vector of all possible ordinary products of the form a component of \(D_2^{N} \tilde{r}_1(z)\) for any \(0 \leq j \leq N\) (where \(N\) will be determined later), times a component of \(r_2(z)\). We define \(\tilde{r}_1(z) \ast D_2^{N} \tilde{r}_2(0)\) analogously. Using matrix block notation define the column vector \(\tilde{r}(z)\) by

\[\tilde{r}(z) = \begin{pmatrix} \tilde{r}_1(z) \\ \tilde{r}_2(z) \end{pmatrix}.\]

Operating on \(\tilde{r}(z)\) with \((D_2)^s\) and applying Lemma V, we see that the bottom two blocks of rows of \((D_2)^s \tilde{r}(z)\) may be expressed as a sum of terms appropriate for an equation of type (2). A typical element of \(\tilde{r}_1(z) \ast \tilde{r}_2(z)\) is \((D_2)^s (D_1)^s (D_2)^s r(z)\) where \(a, b, c, \) and \(d\) are non-negative integers satisfying \(a + b \leq a - 1\) and \(c + d \leq a - 1\). Operating on a term of this form with \((D_2)^s\) yields a sum of terms of the form (see Lemma IV (c))

\[(D_2)^{s - d} (D_2)^{d} r(z) \ast \begin{pmatrix} (D_2)^s (D_1)^s (D_2)^s r(z) \end{pmatrix},\]

where \(0 \leq e \leq a\). We may write each term of this sort as a sum of terms of the form

\[(D_2)^s (D_1)^s r(z) \ast \begin{pmatrix} (D_2)^s (D_1)^s (D_2)^s r(z) \end{pmatrix}.\]

where \(a + f \leq a + (a - e + b) \leq 2a - 1 - e\) and \(c + e \leq c + (d - e) \leq a - 1 + e\). We show that in general we may take \(f \leq a - 1\) and \(g \leq a - 1\). To see this rewrite \((D_2)^s r_1(z) \ast \begin{pmatrix} (D_2)^s r_2(z) \end{pmatrix}\) using the equation for \(r_1(z)\) or \(r_2(z)\) after it has been operated on by \((D_2)^{s - a}\). Continuing we obtain

\[A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
a sum of terms of the form

\[(D^{f}(D^{g})r_{1}(z)) \ast (D^{f}(D^{g})r_{2}(z))\]

with \(f \leq a - 1\) and \(g \leq a - 1\). If \(r_{1}(z)\) and \(r_{2}(z)\) belong to \(K[D]\) then we still have \(a + f \leq 2a - 1 - \varepsilon\) and \(a + g \leq a - 1 + \varepsilon\). Otherwise we merely know that we may pick \(N \geq 3a - 2,\) depending on the degrees of the polynomials \(h_{0}(D)\) appearing in the equations for \(r_{1}(z)\) and \(r_{2}(z)\), such that \(a + f + c + g \leq N\). This specifies our \(N\) in the definition of \(\tilde{r}(z)\). Applying Lemma IV (b) repeatedly (use both formulas) we obtain terms which are derivatives of components of \(\tilde{r}(z)\) appearing in the bottom two blocks of rows and a term of the form

\[D^{j}(D^{k}D^{l}r_{1}(z)) \ast (D^{j}D^{k}r_{2}(z))\]

where \(k + f \leq a - 1, k + g \leq a - 1,\) and \(j \leq \max\{0, a + f + c + g - 2(a - 1)\}\). (Alternatively, \(j \leq \max\{0, a + f + c + g - 2(a - 1)\} \leq \max\{0, a\} = a\). This proves Theorem I parts (a) and (b).

Part (c). Suppose \(r_{1}(z)\) and \(r_{2}(z)\) belong to \(R_{F}\). We may assume without loss of generality that for some \(a \geq 0\)

\[p^{i}(z)(zD)^{a}r_{1}(z) = \sum_{i=0}^{a} p^{i}(z)(zD)^{-a} r_{1}(z),\]

\[p^{i}(z)(zD)^{a}r_{2}(z) = \sum_{i=0}^{a} p^{i}(z)(zD)^{-a} r_{2}(z),\]

where \(p^{i}(z), p^{0}(z), \) each \(p^{i}(z),\) and each \(p^{0}(z)\) belong to \(Q[i, z]\) and \(p^{0}(z)p^{0}(z)\) does not vanish at zero. Then the module over \(Q[i, z]\) generated by the elements of

\[\left\{(p^{i}(z)p^{0}(z))(zD)^{N}r_{1}(z), (zD)^{N}r_{2}(z)\right\}, \text{ for each } N \geq 0\]

is finitely generated. By the ascending chain condition we have for some \(N \geq 0\)

\[\left\{(p^{i}(z)p^{0}(z))(zD)^{N}r_{1}(z), (zD)^{N}r_{2}(z)\right\}\]

\[= \sum_{j=0}^{N} Q_{0}(z) \left\{(p^{i}(z)p^{0}(z))(zD)^{N-j}(zD)^{N-j}r_{1}(z) r_{2}(z)\right\}\]

for a set of \(Q_{0}(z)\) in \(Q[i, z]\). Thus \(r_{1}(z)r_{2}(z)\) belongs to \(R_{F}\).

(d) We have shown that \(R_{F} = R_{F} \ast R_{F} = R_{F}\). Suppose that the \(h_{0}(D)\) in (2) are in \(K[D]\) for some \(K\) with \([K: Q(i)] < \infty\). Choose

a basis \(\omega_{0}, \ldots, \omega_{n}\), for \(K\) over \(Q(i)\) and write an equation of type (2) for the column vector \((\omega_{j})\), \(0 \leq j \leq n\), with coefficients which are matrices over \(Q[i, D]\). Then \(\omega_{0} \cdot y\) belongs to \(R_{F} = R_{F} \ast R_{F} = R_{F}\).

Proof of Theorem II. From Observations in the Introduction we recall that if \(r(z)\) has property A and \(m(z)\) has property B then \(r(z) \ast m(z)\) has property B. Also, by Lemma III, \(R_{F} = M_{F}\). Therefore, all we must show is that \(m_{2}(z)\) and \(m_{3}(z)\) belong to \(M_{F}\) and \(r(z)\) belongs to \(R_{F}\) then \(m_{1}(z) + m_{2}(z)\) and \(r(z) \ast m_{1}(z)\) belong to \(M_{F}\).

Set

\[m(z) = \begin{pmatrix} m_{1}(z) \\ m_{2}(z) \end{pmatrix}.\]

Using the equations of type (3) satisfied by \(m_{1}(z)\) and \(m_{2}(z)\) we may express \(m(z)\) as a sum of terms \(g_{l}(z)(zD)^{m}(z)\) (where the \(g_{l}(z)\) belong to the 3 by 3 matrices over \(Q[i, z]\), \(\deg g_{l}(z) < j,\) and \(1 \leq j \leq \max\{l_{1}, l_{2}\}\)) plus a term \(a(z)\) with property A. This proves closure under \(\ast\).

Let \(m_{3}(z) = m(z)\) and \(l_{1} = l\). If \(l = 0\) or \(1\) then by a remark in the Introduction \(m(z)\) has property A. In this case \(r(z) \ast m(z)\) has property A, so it is in \(M_{F}\). Therefore we may assume \(l \geq 2\) in what follows. Further, as we shall now show, we may assume that \(D^{l}(m(z))\) has property B. Using indefinite integration \(l\) times on equation (3) we see that

\[
\int_{0}^{\pi} \frac{(z-t)^{-1}}{(l-1)!} \cdot m(t) dt - \sum_{j=1}^{l} \sum_{j=1}^{l} \frac{(-1)^{j} \ldots (l-1-j+1)}{(l-1)!} \cdot (z-t)^{j-1} \cdot m(t) dt \cdot \frac{(z-t)^{-1}}{(l-1)!} \cdot a(t) dt
\]

is a polynomial. Thus

\[
\int_{0}^{\pi} \frac{(z-t)^{-1}}{(l-1)!} \cdot m(t) dt
\]

belongs to \(M_{F}\), and

\[
D^{l} \int_{0}^{\pi} \frac{(z-t)^{-1}}{(l-1)!} \cdot m(t) dt = m(z)
\]

has property B. If we know that

\[
r(z) \ast \int_{0}^{\pi} \frac{(z-t)^{-1}}{(l-1)!} \cdot m(t) dt
\]
belongs to \( M_T \) it follows that
\[
D^l \left( r(z) \ast \left( \sum_{i=1}^{s} \frac{(z-t)^{i-1}}{(l-1)!} \frac{d^n}{dt^n} m(t) dt \right) \right)
\]
satisfies an equation of the appropriate sort for membership in \( M_T \). Using Lemma IV (b) \( l \) times we see that the above iterated derivative equals \( r(z) \ast m(z) \). Further \( r(z) \ast m(z) \) has property \( B \) since \( r(z) \) has property \( A \) and \( m(z) \) has property \( B \). Thus \( r(z) \ast m(z) \) belongs to \( M_T \).

In what follows we take \( k \geq 2 \) and assume that \( D^m m(z) \) has property \( B \).

Let \((Dz)^{a-1} r(z) \ast m(z)\) denote the column vector of all
\[
[(Dz)^j r(z) \ast m(z)] \quad \text{for} \quad 0 \leq j \leq a-1
\]
and let \((Dz)^{a-1} r(0) \ast m(0)\) denote the column vector of all
\[
(D^k r(0) \ast m(z)) \quad \text{for} \quad 0 \leq k \leq N = 2a \max_j \{ \deg h_j(D) \} \quad \text{appears in (9)}.\]

Set
\[
\overline{m}(z) = \left( (Dz)^{a-1} r(z) \ast m(z) \right) - (Dz)^{a-1} r(0) \ast m(z),
\]
using matrix block notation. We may write the bottom block of rows as equal to a sum of terms of the appropriate sort for an equation of type (3) in \( \overline{m}(z) \), by using equation (3) \( N+1 \) times. Consider any entry from the top block of rows, say \((Dz)^j r(z) \ast m(z)\) where \( 0 \leq j \leq a-1 \). Using equation (3) we may write this as a sum of terms of the form
\[
(Dz)^{a-1} r(z) \ast ((Dz)^p D^q m(z)),
\]
where \( p + q = l \) and \( 0 \leq q \leq l \), plus \((Dz)^a r(z) \ast a(z)\) which has property \( A \).

By Lemma IV (c) we may write each \((Dz)^a r(z) \ast ((Dz)^p D^q m(z))\) as a sum of terms of the form
\[
(Dz)^{a-u} \left( (Dz)^{u} r(z) \ast (D^q m(z)) \right),
\]
where \( 0 \leq l \leq p \). Now we may use the equation of type (2) satisfied by \( r(z) \) to write each term above as a sum of terms of the form
\[
(Dz)^{a-l} \left( D^l (Dz)^{u} r(z) \ast (D^q m(z)) \right),
\]
with \( t \leq a-1 \) and \( s \leq (i-1) \{ \max_j \{ \deg h_j(D) \} \leq 3a \max \{ \deg h_j(D) \} \} \leq N \).

Using Lemma IV (b) \( s \) times we obtain
\[
(Dz)^{a-s} D^s \left( (Dz)^{u} r(z) \ast (D^q m(z)) \right),
\]
minus terms of the form
\[
(Dz)^{a-s} D^{s-t} \left( (D^{s-t} r(z) \ast (D^q m(z)) \right) = (Dz)^{a-s} D^{s-t} \left( (D^{s-t} r(z) \ast (D^q m(z)) \right),
\]
where \( 1 \leq j \leq s \leq N. \) Since \( 0 < q \) these latter terms are of an appropriate sort for an equation of type (2) in \( \overline{m}(z) \). Finally using Lemma IV (b) \( q \) times we obtain
\[
(Dz)^{a-u} D^q \left( (Dz)^{a-1} r(z) \ast m(z) \right),
\]
where \( t \leq a-1 \) and \( 0 \leq q \), plus terms with property \( A \). This proves Theorem II.

Proof of Proposition II. We must show that under the hypotheses of the Corollary \( m(z+\alpha) \) satisfies an equation of type (1) with coefficients in \( Q(t) \) where \( z = 0 \) is a regular point. Let \( K \) be the extension of the Gaussian rationals generated by the coefficients of the \( g_i(z) \) and by \( \alpha \).

If \( w_0, w_1, \ldots, w_n \) is a basis for \( K \) over \( Q(t) \) then we can write a matrix equation of type (1) for the column vector with components \( w_0 m(z+\alpha), \ldots, w_n m(z+\alpha) \), \( 0 \leq j \leq n \), where the coefficients have entries in \( Q[1, z] \). We now wish to use the proof that \( M_T = M_T \) in Lemma III. If we let \( X \) in the proof of Lemma III be an open disk about zero in \( N-\alpha = \{ z \in C \mid z \leq N \} \) and replace property \( A \) by the property of being analytic on \( D-\alpha \), \( \alpha \) in place of \( \alpha \), the proof goes through line by line to prove that \( m(z+\alpha) \) satisfies an equation of type (1) with coefficients in \( Q(t) \). (The point is that the class of functions satisfying our “new property \( A' \)” is closed under addition, multiplication of functional values, indefinite integration, and differentiation just as was the class of functions satisfying our “old property \( A' \)”.) In what follows assume \( m(z+\alpha) \) satisfies (1).

If \( m(z+\alpha) \) satisfies any linear differential equation with coefficients in \( Q[1, z] \) of the form \( Lm(z+\alpha) = a(z) \) where \( a(z) \) is analytic on \( D-\alpha \), then for some \( N \geq 1 \)
\[(20) \quad m(z+\alpha) = \sum_{j=1}^{l} g_j(z) D^j m(z) + a(z) + D^N (Lm(z+\alpha) - a(z)) \]
is an equation of type (1) which has singular points precisely at the singularities of \( L \) or of \( a(z) - D^N a(z) \).

Now go back to the scalar equation for \( m(z+\alpha) \) which was regular at zero. We may easily obtain an equation of the form \( p(z) D^m m(z+\alpha) \) equals a linear combination of the \( D^m m(z+\alpha), 0 \leq s \leq a-1 \), over \( K[1, z] \) plus some \( a(z) \) analytic on \( D-\alpha \), where \( p(z) \) belongs to \( Q[1, z] \) and \( p(0) \neq 0 \). Applying the ascending chain condition to the module over \( Q[1, z] \) generated by the elements of
\[
\{ (p(z)^N D^N m(z+\alpha) \} \quad N \geq 0 \}
\]
we obtain an equation of the desired sort \( Lm(z+\alpha) = a(z) \) with \( a(z) \) analytic on \( D-\alpha \) and \( z = 0 \) a regular point of \( L \).
Now we can apply the Corollary to (20) and the objects $\bar{D}', N', z_i', z_i$, and $\varphi'$ where

$$\bar{D}' = \bar{D} - z_i = \{z - z_i \mid z \in \bar{D}\},$$

$N'$ is a sufficiently small neighborhood of zero, $z_i' = z_i = 0$, and $\varphi'$ is the translation of $\varphi$. Thus the conclusion of Proposition I holds here but for possibly new values of $t$ and $d$.

References


A method in diophantine approximation (IV)*

by

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Introduction. In this paper we shall extend to functions of $n \geq 1$ complex variables some of the results of Part III of this series of papers (referred to below as Part III).

Definition. By $Z$, $Q$, $Q(i)$, and $C$ we shall mean, respectively, the integers, the rational numbers, the Gaussian rational numbers, and the complex numbers. Throughout this paper $t$ and $z$ will stand for $n$-tuples of complex numbers.

By $D^k$ we shall mean $\frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_k}}{\partial x_k^{n_k}}$, where each $h_k$ ($1 \leq k \leq n$) is a non-negative integer. Analogously we define $D'$ and $D''$. We define $D^{k+1}$ to be $D^k D'$. We use $D_k$ to denote $\frac{\partial}{\partial x_k}$ and $\theta$ to denote $\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n}$.

By $|h|$ or $|j|$ we shall mean $\max\{h_k \mid 1 \leq k \leq n\}$ or $\max\{j_k \mid 1 \leq k \leq n\}$, respectively.

If for some positive integer $N$, $g(x)$ belongs to the $N$ by $N$ matrices over $Q[i, z]$, then by $\deg g(x)$ we mean $\min_{i} \{i \geq 0 \mid D_g^{(i)} g(x) = 0\}$.

We define a norm, $\| \|$, on matrices over the complex numbers by letting $\| \text{ matrix } \|$ denote the maximum of the absolute values of the entries of the matrix.

For each $1 \leq k \leq n$ choose $X_k$ to be a bounded starshaped region about zero in $C$ which shall remain fixed throughout this paper. Let $L_k$ and $Y_k$ denote the Riemann surfaces generated by $\log z_k$ over $C$ and $X_k$ respectively. Note zero is not in either $L_k$ or $Y_k$. We shall regard $C - \{0\}$ and $X_k - \{0\}$ as being embedded in $L_k$ and $Y_k$ respectively. Set $L = \prod_{k=1}^{n} L_k$, $Y = \prod_{k=1}^{n} Y_k$ and $X = \prod_{k=1}^{n} (X_k - \{0\})$. Then $X \subset Y \subset L$. From now on any

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