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ACTA ARITHMETICA
XVI (1969)

On sets characterizing number-theoretical functions (II)

(The set of "prime plus one" s is a set of quasi-uniqueness)

by

I. Kátai (Budapest)

1. In [2] it was proved that the set $\mathscr{P}_1 = \{p+1\}$, p runs over the primes, is a set of quasi-uniqueness, under the assumption of the validity of the Riemann–Piltz conjecture. Here we prove this assertion without any unproved hypothesis.

THEOREM. *There is a numerical constant K with the following property: If $f(n)$ is a completely additive number-theoretic function such that $f(p) = 0$ for $p \leqslant K$ and $f(p+1) = 0$ for all primes p, then $f(n) = 0$ identically.*

The proof is based on Bombieri's result in the theory of the large sieve.

2. Notation and lemmas. The letters $p, p_1, p_2, \ldots; q, q_1, \ldots; q'$ stand for prime numbers. Let $c, c_1, \ldots$ denote numerical positive constants, $\varepsilon, \varepsilon_1, \ldots, \varepsilon', \delta, \delta'$ sufficiently small positive constants, not necessarily the same at every occurrence. $C(\ldots)$ denote constants which depend only on the values stated in the bracket.

Let

$$\pi(x, k, l) = \sum_{\substack{p\leqslant x\\ p\equiv l \pmod k}} 1.$$

For the proof we need some lemmas.

LEMMA 1. *Let $N(x, k)$ denote the number of the couples of primes satisfying the conditions $p+1 = kq$, $p \leqslant x$. Then*

$$N(x, k) < c\frac{x}{\varphi(k)\log^2(x/k)} \quad (2 \leqslant k < x). \tag{1}$$

For the proof see [3], p. 51, Theorem 4.6.

LEMMA 2 [3]. *Let $\delta > 0$. Then for $k \leqslant x^{1-\delta}$, $(l, k) = 1$*

$$\pi(x, k, l) < C(\delta)\frac{x}{\varphi(k)\log x}. \tag{2}$$

LEMMA 3. *Let* $x = Q^2(\log Q)^B$, $B \geqslant 4A+40$, A, B *being arbitrary positive constants. Then*

$$(3)\qquad \sum_{Q \leqslant q \leqslant 2Q} \left| \pi(x, q, -1) - \frac{\operatorname{li} x}{q-1} \right| \leqslant C(A, B) \frac{x}{(\log Q)^A} \qquad (Q \to \infty).$$

This is an immediate consequence of Bombieri's theorem ([1], Theorem 2).

3. The proof of the Theorem. Let Q_0 be a large constant, $Q_l = Q_0 \cdot 2^l$, $I_0 = [1, Q_0]$, $I_l = [Q_{l-1}, Q_l]$ $(l \geqslant 1)$. Let $\mathscr{S}_l$ be a set of prime numbers in the interval I_l defined by induction as follows. $\mathscr{S}_0$ is the empty set. Assume that $\mathscr{S}_0, \ldots, \mathscr{S}_{m-1}$ are defined. Then $\mathscr{S}_m$ is the set of those primes q in I_m for which there exist no k and p satisfying the following condition:

(A) $p+1 = kq$, *the prime factors of* k *are all smaller than* Q_{m-1} *and do not belong to the set* $\bigcup_{i=0}^{m-1} \mathscr{S}_i$.

Let R_l be the number of elements of $\mathscr{S}_l$.

First we prove the following

LEMMA 4. *For sufficiently large* Q_0 *we have*

$$(4)\qquad R_l < \frac{Q_l}{(\log Q_l)^3} \qquad (l = 0, 1, \ldots).$$

Proof. Since $\mathscr{S}_0$ is an empty set, (4) holds for $l = 0$. Suppose that (4) holds for $l = 0, \ldots, m-1$. Applying Lemma 3 by choosing $Q = Q_{m-1}$, $A = 4$, $B = 100$, $x = Q_{m-1}^2(\log Q_{m-1})^{100}$ we have

$$(5)\qquad \sum_{Q_{m-1} \leqslant q \leqslant Q_m} \left| \pi(x, q, -1) - \frac{\operatorname{li} x}{q-1} \right| \leqslant C \frac{x}{(\log Q_m)^4}.$$

Hence it follows that

$$(6)\qquad \pi(x, q, -1) > \frac{3}{4} \cdot \frac{x}{q \log x}$$

for all $q \in I_m$ except at most $Q_m/(\log Q_m)^3$.

Now we prove that the validity of (6) implies that $q \notin \mathscr{S}_m$. Indeed, let $\Pi(x, q)$ denote the number of those $p \leqslant x$ for which (A) holds. Then

$$(7)\qquad \Pi(x, q) \geqslant \pi(x, q, -1) - \Sigma_1 - \Sigma_2,$$

where Σ_1 denote the number of those primes $p \leqslant x$ for which $p+1 = kq$ and k has at least one prime factor not exceeding $Q_{m-1}^{1-\delta}$ which belongs to $\bigcup_{j \geqslant 0} \mathscr{S}_j$. Σ_2 denotes the number of those $p \leqslant x$ for which, in $p+1 = kq$, k has a prime factor greater than $Q_{m-1}^{1-\delta}$.

If p occurs in Σ_2, then $p+1$ has the form $p+1 = jq_1q$, where $j \leqslant x/(qq_1) \leqslant xQ_{m-1}^{-2+\delta} \leqslant x^{2\delta}$, say. Consequently by (1) we have

$$\Sigma_2 \leqslant \sum_{j \leqslant x^{2\delta}} N(x, jq) \leqslant c_1 \frac{x}{q \log^2 x} \sum_{j \leqslant x^{2\delta}} \frac{1}{\varphi(j)} \leqslant c_2 \delta \frac{x}{q \log x}.$$

Choosing $\delta < \varepsilon / c_2$ we obtain

$$(8)\qquad \Sigma_2 < \varepsilon \frac{x}{q \log x}.$$

Furthermore we have

$$\Sigma_1 \leqslant \sum_{\substack{q' \in \bigcup_l \mathscr{S}_l \\ q' \leqslant Q_{m-1}^{1-\delta}}} \pi(x, qq', -1).$$

By Lemma 2 we deduce

$$\Sigma_1 \leqslant C(\delta) \frac{\operatorname{li} x}{q-1} \sum_{Q_\nu \leqslant Q_{m-1}^{1-\delta}} \sum_{q' \in \mathscr{S}_\nu} \frac{1}{q'-1} \leqslant 2C(\delta) \frac{x}{q \log x} \sum_{\nu \leqslant m-2} \frac{R_\nu}{Q_\nu},$$

if Q_0 is so large that $Q_{m-2} > Q_{m-1}^{1-\delta}$. Assuming (4) for $l \leqslant m-2$ we have

$$(9)\qquad \Sigma_1 \leqslant 2C(\delta) \frac{x}{q \log x} \sum_{\nu=0}^{\infty} \frac{1}{(\log Q_0 + \nu)^3} < \varepsilon \frac{x}{q \log x}.$$

Consequently choosing a small δ and after this a large Q_0, we have

$$(10)\qquad \Sigma_1 + \Sigma_2 < \frac{1}{4} \cdot \frac{x}{q \log x}.$$

Hence we infer that (6) implies $\Pi(x, q, -1) > 0$, i.e. that $q \notin \mathscr{S}_m$. Thus (4) holds for $l = m$. This completes the proof of Lemma 4.

Now we begin the proof of the Theorem. Let $f(n)$ be a completely additive function satisfying the conditions stated in the Theorem with $K \geqslant Q_0$, where Q_0 is such a large constant as is implied by Lemma 4.

First we prove that $f(q) = 0$ for all $q \notin \bigcup_{m=0}^{\infty} \mathscr{S}_m$. Indeed, this holds for $q \in I_0$. Assume that

$$(11)\qquad f(q) = 0 \quad \text{for all } q \in I_j,\ q \notin \mathscr{S}_j \quad \text{for } j \leqslant m-1.$$

Let $q \in I_m$, $q \notin \mathscr{S}_m$. Then there exists a p such that $p+1 = kq$ for which $k = p_1^{a_1} \ldots p_r^{a_r}$, $p_i < Q_{m-1}$, $p_i \notin \bigcup_{j=0}^{m-1} \mathscr{S}_j$ $(i = 1, \ldots, r)$. Hence $f(k) = 0$ and thus $0 = f(p+1) = f(k) + f(q) = f(q)$ follows. This proves (11) for $j = m$.

Finally we prove that $f(q) = 0$ for all $q \notin \bigcup_{i=0}^{\infty} \mathscr{S}_i$ ($\overset{\text{def}}{=} \mathscr{T}$) if Q_0 is sufficiently large.

Let $P(y, q)$ denote the number of those $p \leqslant y$ for which $p+1 = kq$ and the prime factors of k do not belong to $\mathscr{T}$. We prove that $P(y, q) > 0$ if y is large, whence $f(q) = 0$ follows.

Indeed,

$$P(y, q) \geqslant \pi(y, q, -1) - \sum_{\substack{q' \in \mathscr{T} \\ q' \leqslant y/q}} \pi(y, q'q, -1). \tag{12}$$

For large y we have

$$\pi(y, q, -1) > \frac{1}{2} \cdot \frac{y}{q \log y}. \tag{13}$$

Furthermore by Lemma 2

$$\Sigma_3 \overset{\text{def}}{=} \sum_{\substack{q' \in \mathscr{T} \\ q' \leqslant y/q}} \pi(y, q'q, -1) \leqslant \frac{Cy}{q \log y} \sum_{\substack{q' \in \mathscr{T} \\ q' < y}} \frac{1}{q'} + \frac{y}{q} \sum_{\substack{y^{1/2} \leqslant q' \leqslant y/q \\ q' \in \mathscr{T}}} \frac{1}{q'}.$$

Since $\sum_{q' \in \mathscr{T}} 1/q' < \varepsilon$ with large Q_0 and

$$\sum_{y^{1/2} \leqslant q' \leqslant y/q} \frac{1}{q'} \leqslant \log y \max_{Q_l \geqslant y^{1/2}} \sum_{q' \in \mathscr{T}} \frac{1}{q'} \leqslant \frac{1}{\log^2 y},$$

we have

$$\Sigma_3 < \frac{1}{4} \cdot \frac{y}{q \log y}.$$

Hence, by (12), (13), $P(y, q) > 0$ follows. This completes the proof of the Theorem.

4. The constant K in the Theorem is non-effective since $C(A, B)$ in Lemma 3 is non-effective. It would be very interesting to prove the Theorem with effective K since this would give a possibility to decide with numerical calculation whether $\mathscr{P}_1$ is a set of uniqueness or not.

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A method in diophantine approximation (III)*

by

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Introduction. We begin by giving the hypotheses and statement of a result, called Proposition I below, which was stated and proved in [2] using slightly different notation.

Let D denote differentiation with respect to the complex variable z, let l be an integer greater than one; let each $g_j(z)$ for $1 \leqslant j \leqslant l$ be a polynomial of degree less than j with coefficients in the Gaussian field. Suppose that we are in a simply connected region $\bar{D}$ where $a(z)$ is analytic and that $m_1(z), \ldots, m_n(z)$ are $n \geqslant 1$ solutions of

$$m(z) = \sum_{j=1}^{l} g_j(z) D^j m(z) + a(z) \tag{1}$$

which are analytic in some open disk $N \subseteq \bar{D}$ about z_0 on which $g_l(z)$ does not vanish. Suppose z_1 belongs to N and z_1 is a Gaussian rational. Let $\mathscr{C}$ be a differentiable path in $\bar{D}$ with endpoints at z_0 which does not pass through any of the zeros of $g_l(z)$. Suppose that $\tilde{m}_1(z) \neq m_1(z), \ldots$ $\ldots, \tilde{m}_n(z) \neq m_n(z)$ are the function elements, analytic on N, obtained by extending $m_1(z), \ldots, m_n(z)$, respectively, along $\mathscr{C}$ and back to z_0 and that $\tilde{m}_1(z) - m_1(z), \ldots, \tilde{m}_n(z) - m_n(z)$ are linearly independent over the complex numbers. Let

$$d = \max_j \left\{ \frac{\deg g_j(z)}{j - \deg g_j(z)} \right\}.$$

Let $\|a\|$, for any complex number a, denote the distance from a to the nearest Gaussian integer. Let $(A_1, \ldots, A_n)$ denote any nonzero element of the Cartesian product of the Gaussian integers with themselves n times.

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