On sets characterizing number-theoretical functions (II)

(The set of "prime plus one" is a set of quasi-uniqueness)

by

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1. In [2] it was proved that the set \( P = \{ p+1 \} \), \( p \) runs over the primes, is a set of quasi-uniqueness, under the assumption of the validity of the Riemann–Piltz conjecture. Here we prove this assertion without any improved hypothesis.

\textbf{Theorem.} There is a numerical constant \( K \) with the following property:

If \( f(n) \) is a completely additive number-theoretical function such that \( f(p) = 0 \) for \( p \leq K \) and \( f(p+1) = 0 \) for all primes \( p \) then \( f(n) = 0 \) identically.

The proof is based on Bombieri's result in the theory of the large sieve.

2. Notation and lemmas. The letters \( p, p_1, p_2, \ldots; q, q_1, q_2, \ldots; q' \) stand for prime numbers. Let \( c, c_1, \ldots \) denote numerical positive constants, \( e, e_1, \ldots, e', \delta, \delta' \) sufficiently small positive constants, not necessarily the same at every occurrence. \( C(...) \) denote constants which depend only on the values stated in the bracket.

Let

\[ \pi(x, k, l) = \sum_{p \leq x, k \mid p} 1. \]

For the proof we need some lemmas.

\textbf{Lemma 1.} Let \( N(x, k) \) denote the number of the couples of primes satisfying the conditions \( p+1 = kq, p \leq x \). Then

\[ (1) \quad N(x, k) < c \frac{x}{\varphi(h) \log^2 x} \quad (2 \leq k < x). \]

For the proof see [3], p. 51, Theorem 4.6.

\textbf{Lemma 2 [3].} Let \( \delta > 0 \). Then for \( k \leq n^{-4} \), \( l, k = 1 \)

\[ (2) \quad \pi(x, k, l) < C(\delta) \frac{x}{\varphi(k) \log^2 x}. \]
Lemma 3. Let \( x = Q^2 (\log Q)^2, B \geq 4A + 40, A, B \) be arbitrary positive constants. Then

\[
\sum_{q \leq Q} \pi(x, q, -1) \frac{\ln x}{q - 1} \leq C(A, B) \frac{x}{(\log Q)^2} \quad (Q \to \infty).
\]

This is an immediate consequence of Bombieri's theorem ([1], Theorem 2).

3. The proof of the Theorem. Let \( Q_0 \) be a large constant, \( Q_1 = Q_0, Q_i, I_i = [Q_{i-1}, Q_i] \) \((i \geq 1)\). Let \( J_i \) be a set of prime numbers in the interval \( I_i \) defined by induction as follows. \( J_0 \) is the empty set. Assume that \( J_0, \ldots, J_{m-1} \) are defined. Then \( J_m \) is the set of those primes \( q \in I_m \) for which there exist no \( k \) and \( p \) satisfying the following condition:

(A) \( p + 1 = kq \), the prime factors of \( k \) are all smaller than \( Q_{m-1} \) and do not belong to the set \( \bigcup_{i=0}^{m-1} J_i \).

Let \( B_l \) be the number of elements of \( J_l \).

First we prove the following

Lemma 4. For sufficiently large \( Q_0 \) we have

\[
B_l < \frac{Q_l}{(\log Q_l)^3} \quad (l = 0, 1, \ldots).
\]

Proof. Since \( J_0 \) is an empty set, (4) holds for \( l = 0 \). Suppose that (4) holds for \( l = 0, \ldots, m-1 \). Applying Lemma 3 by choosing \( Q = Q_{m-1}, A = 4, B = 100, x = Q_{m-1}^2 (\log Q_{m-1})^{150} \) we have

\[
\sum_{q \leq Q_{m-1}} \pi(x, q, -1) \frac{\ln x}{q - 1} \leq C \frac{x}{(\log Q_m)^2}.
\]

Hence it follows that

\[
\pi(x, q, -1) > \frac{3x}{4 Q_m}\frac{1}{\log x}
\]

for all \( q \in I_m \) except at most \( Q_m^3 (\log Q_m)^2 \).

Now we prove that the validity of (6) implies that \( q \notin J_m \). Indeed, let \( \Pi(x, q) \) denote the number of those \( p \leq x \) for which (A) holds. Then

\[
\Pi(x, q) \geq \pi(x, q, -1) - \Sigma_1 - \Sigma_2,
\]

where \( \Sigma_1 \) denote the number of those primes \( p \leq x \) for which \( p + 1 = kq \) and \( k \) has at least one prime factor not exceeding \( Q_{m-1}^2 \), which belongs to \( \bigcup_{i=0}^{m-1} J_i \). \( \Sigma_2 \) denotes the number of those \( p \leq x \) for which, in \( p + 1 = kq \), \( k \) has a prime factor greater than \( Q_{m-1}^2 \).

If \( p \) occurs in \( \Sigma_2 \), then \( p + 1 \) has the form \( p + 1 = jq \), where \( j \leq \varphi(q) \leq \varphi(Q_{m-1}^2) \leq A^2 \), say. Consequently by (1) we have

\[
\sum_{q \leq Q_{m-1}} \frac{1}{q \log q} \leq C(A, B) \frac{x}{(\log Q_{m-1})^2} \sum_{q \leq Q_{m-1}} \frac{1}{q \log q} \leq C(A, B) \frac{x}{(\log Q_{m-1})^2} < \frac{\varphi(Q_{m-1}^2)}{\varphi(Q_{m-1}^2) - 1} < \frac{1}{\log Q_{m-1}}.
\]

Choosing \( \delta < s/\varphi(Q_{m-1}^2) \) we obtain

\[
\Sigma_2 < e^{-\frac{x}{\log x}}.
\]

Furthermore we have

\[
\Sigma_1 \leq \sum_{q \leq Q_{m-1}} \pi(x, q, -1).
\]

By Lemma 2 we deduce

\[
\Sigma_1 \leq 2C(B) \frac{x}{\log x} \sum_{q \leq Q_{m-1}} \frac{1}{(\log Q_{m-1} + 1)^2} < e^{-\frac{x}{\log x}}.
\]

Consequently choosing a small \( \delta \) and after this a large \( Q_0 \), we have

\[
\Sigma_1 + \Sigma_2 < \frac{1}{4} \frac{x}{\log x}.
\]

Hence we infer that (6) implies \( \Pi(x, q, -1) > 0 \), i.e. that \( q \notin J_m \).

Thus (4) holds for \( l = m \). This completes the proof of Lemma 4.

Now we begin the proof of the Theorem. Let \( f(x) \) be a completely additive function satisfying the conditions stated in the Theorem with \( K > Q_0 \), where \( Q_0 \) is such a large constant as is implied by Lemma 4.

First we prove that \( f(q) = 0 \) for all \( q \notin \bigcup_{m=0}^{\infty} J_m \). Indeed, this holds for \( q \notin J_0 \). Assume that

\[
f(q) = 0 \quad \text{for all } q \notin J_l, q \notin J_i \quad \text{for } j \leq m-1.
\]

Let \( q \in I_m, q \notin J_m \). Then there exists a \( p \) such that \( p + 1 = kq \) for which \( k = p_1^{a_1} \cdots p_r^{a_r}, p_1 \leq Q_{m-1}, p_i \in J_i \) \((i = 1, \ldots, r)\). Hence \( f(k) = 0 \) and thus \( 0 = f(p + 1) = f(k) + f(q) = f(q) \) follows. This proves (11) for \( j = m \).
Finally we prove that \( f(q) = 0 \) for all \( q \in \mathcal{S}_1 \) if \( Q_q \) is sufficiently large.

Let \( P(y, q) \) denote the number of those \( p \leq y \) for which \( p + 1 = kq \) and the prime factors of \( k \) do not belong to \( \mathcal{S} \). We prove that \( P(y, q) > 0 \) if \( y \) is large, whence \( f(q) = 0 \) follows.

Indeed,

\[
P(y, q) \gg \pi(y, q, -1) - \sum_{q' < q} \pi(y, q', -1).
\]

For large \( y \) we have

\[
\pi(y, q, -1) > \frac{1}{2} \frac{y}{q \log y}.
\]

Furthermore by Lemma 2

\[
\sum_{q \in \mathcal{S}} \pi(y, q, -1) \ll \frac{C y}{q \log y} \sum_{q < y} \frac{1}{q} + \frac{y}{q} \sum_{q' < q} \frac{1}{q'}.
\]

Since \( \sum 1/q' < \varepsilon \) with large \( Q_q \) and

\[
\sum_{q' \in \mathcal{S}} \frac{1}{q'} \ll \log y \max_{q \in \mathcal{S}} \sum_{q' < q} \frac{1}{q'} \ll \frac{1}{\log^2 y},
\]

we have

\[
\sum_{q' \in \mathcal{S}} \frac{1}{q' y} \ll \frac{y}{q \log y},
\]

Hence, by (12), (13), \( P(y, q) > 0 \) follows. This completes the proof of the Theorem.

4. The constant \( K \) in the Theorem is non-effective since \( C(A, B) \) in Lemma 3 is non-effective. It would be very interesting to prove the Theorem with effective \( K \) since this would give a possibility to decide with numerical calculation whether \( \mathcal{S}_1 \) is a set of uniqueness or not.

References


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**A method in diophantine approximation (III)**

by

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**Introduction.** We begin by giving the hypotheses and statement of a result, called Proposition I below, which was stated and proved in [2] using slightly different notation.

Let \( D \) denote differentiation with respect to the complex variable \( z \); let \( l \) be an integer greater than one; let each \( g_l(z) \) for \( 1 \leq j \leq l \) be a polynomial of degree less than \( j \) with coefficients in the Gaussian field. Suppose that we are in a simply connected region \( \tilde{D} \) where \( a(z) \) is analytic and that \( m_1(z), \ldots, m_n(z) \) are \( n \geq 1 \) solutions of

\[
m(z) = \sum_{j=1}^{l} g_l(z) D^{j} m(z) + a(z)
\]

which are analytic in some open disk \( N \subset \tilde{D} \) about \( z_0 \) on which \( g_l(z) \) does not vanish. Suppose \( z_1 \) belongs to \( N \) and \( z_1 \) is a Gaussian rational. Let \( \mathscr{G} \) be a differentiable path in \( \tilde{D} \) with endpoints at \( z_0 \) which does not pass through any of the zeros of \( g_l(z) \). Suppose that \( \tilde{m}_1(z) \neq m_1(z), \ldots, \tilde{m}_n(z) \neq m_n(z) \) are the function elements, analytic on \( N \), obtained by extending \( m_1(z), \ldots, m_n(z) \), respectively, along \( \mathscr{G} \) and back to \( z_0 \) and that \( \tilde{m}_1(z) - m_1(z), \ldots, \tilde{m}_n(z) - m_n(z) \) are linearly independent over the complex numbers. Let

\[
d = \max_{j} \left\{ \frac{\deg g_j(z)}{\deg g_l(z)} \right\}.
\]

Let \( \|a\| \), for any complex number \( a \), denote the distance from \( a \) to the nearest Gaussian integer. Let \( (A_1, \ldots, A_n) \) denote any nonzero element of the Cartesian product of the Gaussian integers with themselves \( n \) times.

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