

of squares of two linear forms. The number of representations of f as a sum of two squares is $r_2(2\eta)$.

Proof. If $\eta = 0$, $r_2(2\eta) = r_2(0)$ is infinite as is the number of representations of f as a sum of two squares. Suppose $\eta \neq 0$. From Theorem 3 we see that η must be even, and with every factorization $\eta = 2\alpha_1\beta_1$ there is associated a representation of f as a sum of squares of two linear forms. We have only to count the number of factors α_1, β_1 . We may write $2\eta = i^r(1+i)^s\pi_1^{k_1}\pi_2^{k_2}\dots\pi_n^{k_n}$ where the π_i are odd primary primes, $r = 0, 1, 2$, or 3 , and $s \geq 4$. The number of factors of $2\eta/4$ is then $4(s-3)(k_1+1)\dots(k_n+1)$ which is just $r_2(2\eta)$.

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On a problem of P. Erdős and S. Stein

by

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The system of congruences

$$(1) \quad a_i \pmod{n_i}, \quad n_1 < \dots < n_k$$

is called a *covering system* if every integer satisfies at least one of the congruences (1). An old conjecture of P. Erdős states that for every integer c there is a covering system with $n_1 = c$. Selfridge and others settled this question for $c \leq 8$. The general case is still unsettled and seems difficult.

A system (1) is called *disjoint* if every integer satisfies at most one of the congruences (1). It is trivial that in a disjoint system we must have

$$(n_i, n_j) > 1 \quad \text{and} \quad \sum_{i=1}^k 1/n_i \leq 1.$$

It is known that a disjoint system can never be covering [2] and that for a disjoint system we have [3]

$$(2) \quad \sum_{i=1}^k \frac{1}{n_i} \leq 1 - \frac{1}{2^k}.$$

(2) is easily seen to be best possible.

Denote by $f(x)$ the largest value of k for which there exists a disjoint system (1) satisfying $n_k \leq x$. P. Erdős and S. Stein conjectured that $f(x) = o(x)$.

The main purpose of this paper will be to prove this conjecture. In fact, we prove the following

THEOREM 1. *For every $\varepsilon > 0$ if $x > x_0(\varepsilon)$ we have (c_1, c_2, \dots) denote suitable positive constants)*

$$(3) \quad \frac{x}{\exp((\log x)^{1/2+\varepsilon})} < f(x) < \frac{x}{(\log x)^{c_1}}.$$

The proof of the lower bound we obtained with the help of S. Stein [3]. First we outline the proof of the lower bound in (3) leaving some details to the reader.

Let p_r be the least prime greater than $\exp((\log x)^{1/2})$, $n_1 < \dots < n_k$ are the squarefree integers not exceeding x the greatest prime factor of which is p_r . Put

$$n_j = p_{i_1} \dots p_{i_l} p_r, \quad p_{i_1} < \dots < p_{i_l} < p_r.$$

Let

$$(4) \quad \begin{aligned} a_j &\equiv 0 \pmod{p_{i_1}}, & a_j &\equiv p_{i_{s-1}} \pmod{p_{i_s}}, & 1 < s \leq l, \\ & & a_j &\equiv p_{i_l} \pmod{p_r}. \end{aligned}$$

The congruences (4) determine a_j uniquely $\pmod{n_j}$. It is easy to see that the system $a_j \pmod{n_j}$, $1 \leq j \leq k$, is disjoint. Clearly k equals $\psi_1(x/p_r, p_r)$ where $\psi_1(u, v)$ denotes the number of squarefree integers not exceeding u all whose prime factors do not exceed v . It easily follows from the results of de Bruijn and others [1] that for $x > x_0(\varepsilon)$

$$\psi_1(x/p_r, p_r) > \frac{x}{\exp((\log x)^{1/2+\varepsilon})},$$

which proves the lower bound in (3).

The proof of the upper bound will be considerably more difficult. Let $N = \{n_1 < \dots < n_k \leq x\}$ be an arbitrary sequence of integers. Denote by $g_N(d)$ the largest j for which there are j n 's the greatest common divisor of any two of which is d . ($g_N(1)$ is thus the largest integer for which there are $g_N(1)$ n 's which are pairwise relatively prime.)

Now we prove the following

LEMMA 1. Assume that the system (1) is disjoint. Then we have for every $d \geq 1$

$$(5) \quad g_N(d) \leq d.$$

Assume that (5) is not satisfied for a certain d and assume that the greatest common divisor of any two of the integers $n_{i_1}, \dots, n_{i_{d+1}}$ is d . We show that the congruences

$$(6) \quad a_{i_j} \pmod{n_{i_j}}, \quad 1 \leq j \leq d+1,$$

cannot be disjoint. To see this put $n_{i_j} = dm_{i_j}$, $1 \leq j \leq d+1$, where any two of the m 's are relatively prime. By the box principle, there are two integers $1 \leq j_1 < j_2 \leq d+1$ satisfying $a_{i_{j_1}} \equiv a_{i_{j_2}} \pmod{d}$, but then the congruences $a_{i_{j_1}} \pmod{d}$ and $a_{i_{j_2}} \pmod{d}$ have a common solution, or the system (6) is not disjoint, which proves (5) and the lemma.

Denote $A_N(x) = \sum_{n_j \leq x} 1$. Put $F(x) = \max A_N(x)$ where the maximum is taken over all the sequences N which satisfy (5) for every $d \geq 1$. By Lemma 1 we have

$$(6) \quad F(x) \geq f(x).$$

Now we prove

THEOREM 2. Let $c_3 > 0$ be sufficiently small and c_2 sufficiently large. Then

$$(7) \quad \frac{x}{(\log x)^{c_2}} < F(x) < \frac{x}{(\log x)^{c_3}}.$$

Theorem 2 and Lemma 1 prove the upper bound in (3) and this completes the proof of Theorem 1.

It is quite possible that $f(x) < x/\exp(\log x)^{c_4}$ for some $c_4 > 0$, but the lower bound in (7) shows that the method used in this paper cannot give $f(x) < x/(\log x)^{c_2}$.

To prove Theorem 2 we need some lemmas.

LEMMA 2. The number of integers $n \leq x$ divisible by the square of a prime $p > \log x$ is $o(x/\log x)$.

The number of these integers is clearly less than

$$\sum_{p > \log x} \frac{x}{p^2} = o\left(\frac{x}{\log x}\right)$$

which proves the lemma.

LEMMA 3. Put $n = \prod_{i=1}^k p_i^{a_i}$, $p_1 < \dots < p_k$. Let $c_3 > 0$ be sufficiently small. All but $o(x/(\log x)^{c_3})$ integers $n \leq x$ have a prime factor p_i satisfying

$$(8) \quad p_i > (\log x)^{10} \prod_{i=1}^{j-1} p_i^{a_i} \quad ((\log x)^{10} = T_1).$$

A well known theorem of Hardy and Ramanujan [4] states that for a sufficiently small $c_3 > 0$ for all but $o(x/(\log x)^{c_3})$ integers $n \leq x$ we have

$$(9) \quad \sum_{i=1}^k a_i < (1 + \frac{1}{10}) \log \log x.$$

Hence we clearly can assume that n satisfies (9) and

$$(10) \quad x/\log x < n \leq x.$$

Denote by p_r the greatest prime factor of n which is less than $\log x$. By Lemma 2 we can assume that $a_{r+i} = 1$ for all $1 \leq i \leq k-r$. Further

since n satisfies (9) we evidently have

$$(11) \quad \prod_{i=1}^r p_i^{c_i} < (\log x)^{2 \log \log x} = T_2.$$

If (8) fails to hold for every $r < j \leq k$ we have from (11)

$$(12) \quad p_{r+1} < T_1 T_2, \quad p_{r+2} < T_1^2 T_2^2$$

and by induction with respect to i (using (11) and (12))

$$(13) \quad p_{r+i} < (T_1 T_2)^{2^{i-1}}.$$

Hence finally from (13) and (9) by a simple calculation ($\exp x = e^x$)

$$(14) \quad p_k < (T_1 T_2)^{2^{k-1}} < \exp(2^{(1+1/10) \log \log x} \log 2 \cdot \log T_1 T_2) < x^{1/(\log \log x)^2}.$$

From (14), (11) and (9) we obtain

$$n < T_2 p_k^{2 \log \log x} < x^{1/2}$$

which contradicts (10) and hence Lemma 3 is proved.

Now we are ready to prove the upper bound in (7). Let $n_1 < \dots < n_r \leq x$ be a sequence of integers which satisfies (5) for all $d \geq 1$. Assume that

$$(15) \quad r \geq x/(\log x)^{c_3}.$$

We shall show that (15) leads to a contradiction. First of all if (15) holds then by Lemma 3 we can assume that for at least $r/2$ n_i 's there is a d_i so that $d_i | n_i$ and all prime factors of n_i/d_i are greater than $d_i(\log x)^{10}$. If d_i has these properties we say that d_i corresponds to n_i . Now we prove the simple but crucial

LEMMA 4. *There is at least one d which corresponds to at least $x/d(\log x)^5$ values of n_i .*

From (15) and what we just stated it follows that at least one d_i ($1 \leq d_i \leq x$) corresponds to more than $r/2 > x/2(\log x)^{c_3}$ n_i 's. Thus if our lemma would be false we would have

$$\frac{x}{2(\log x)^{c_3}} < \frac{r}{2} \leq \frac{x}{(\log x)^5} \sum_{d=1}^x \frac{1}{d} = o\left(\frac{x}{\log x}\right),$$

an evident contradiction for $c_3 < 1$, which proves Lemma 4.

Let now d be an integer which satisfies Lemma 4 and let $n_1 < \dots < n_s \leq x$, $s > x/d(\log x)^5$ be the n 's to which d corresponds. Put

$$(16) \quad n_i = d v_i, \quad 1 \leq i \leq s, \quad v_i \leq \frac{x}{d}, \quad s > \frac{x}{d(\log x)^5},$$

where all prime factors of v_i are greater than $d(\log x)^{10}$. Let v_{i_1}, \dots, v_{i_t} be a maximal set of v 's which are pairwise relatively prime. We evidently have by (5)

$$(17) \quad d \geq g_N(d) \geq t$$

since $(n_{i_1}, n_{i_2}) = d$, $1 \leq j_1 < j_2 \leq t$. Now we show that (16) and (17) contradict each other and this will complete the proof of the upper bound in (7).

Let $q_1 < \dots < q_s$ be the set of prime factors of $\prod_{r=1}^t v_{i_r}$. Clearly

$$(18) \quad z < t \log x$$

since every $m \leq x$ has fewer than $\log x$ distinct prime factors. The maximality property of v_{i_1}, \dots, v_{i_t} implies that every v is divisible by at least one of the q 's. Thus by (16), (18) and $q_1 > d(\log x)^{10}$ we evidently have

$$\frac{x}{d(\log x)^5} < s < \frac{x}{d} \sum_{i=1}^s \frac{1}{q_i} < \frac{x}{d} \cdot \frac{t \log x}{q_1} < \frac{x}{d} \cdot \frac{t}{d(\log x)^9},$$

or $t > d(\log x)^4$ which contradicts (17) and completes our proof. Thus as stated previously Theorem 1 is also proved.

To complete the proof of Theorem 2 we outline the proof of the lower bound in (7), leaving many of the details to the reader. Let n be squarefree, put $n = p_1 \dots p_k$, $p_1 < \dots < p_k$. Denote by N the set of all integers n for which

$$(19) \quad p_i < \prod_{j=1}^{i-1} p_j, \quad p_1 = 3, \quad p_2 = 5,$$

holds for every prime factor p_i , $i \geq 3$.

Now we show that the sequence N satisfies (5) for every $d \geq 1$.

To see this let $n_{i_1} < \dots < n_{i_s}$, $s = g_N(d)$ be a maximal set of n 's the greatest common divisor of any two of which is d . Write now $n_{i_j} = d v_j$. By (19) each v_j must have a prime factor less than d and since we must have $(v_{j_1}, v_{j_2}) = 1$, $1 \leq j_1 < j_2 \leq s$, we clearly have

$$s = g_N(d) \leq \pi(d) < d$$

which proves that the sequence N satisfies (5) for every $d \geq 1$. To complete the proof of Theorem 2 we only have to show that for sufficiently large c_2 ($n_i \in N$ satisfies (19))

$$(20) \quad N(x) = \sum_{n_i \leq x} 1 > \frac{x}{(\log x)^{c_2}}.$$

We only outline the proof of (20). Let $x^{1/2} < a_1 < \dots < a_k < x^{3/4}$ be the sequence of squarefree integers $\equiv 0 \pmod{3, 5, 7, 11}$ so that if

p_i and p_{i+1} are two consecutive prime factors of a_j , $p_{i+1} > 11$, then $p_{i+1} < p_i^{5/4}$. It is immediate that the a_j 's satisfy (19) and it is not hard to prove that

$$(21) \quad \sum_{j=1}^k \frac{1}{a_j} > \frac{1}{(\log x)^{64}}.$$

It is immediate that the integers of the form

$$(22) \quad a_j p, \quad p < x/a_j, \quad (p, a_j) = 1,$$

also satisfy (19). From (21) we obtain that the number of integers of the form (22) is less than $\nu(a_j)$ ($\nu(a_j)$ denotes the number of prime factors of a_j)

$$(23) \quad \frac{1}{\log x} \sum_{j=1}^k \left(\pi\left(\frac{x}{a_j}\right) - \nu(a_j) \right) > \frac{x}{(\log x)^{63}}.$$

The factor $1/\log x$ in (23) comes from the fact that an integer $n \leq x$ can be represented in the form $a_j p$ at most $\nu(n) < \log x$ times. (23) clearly implies (20), and thus the proof of Theorem 2 is complete.

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