In the footnote on p. 254, replace $A \equiv B+1 \pmod 4$ by $A \equiv 1 \pmod 4$.

The formula for $Q_n^{(i)}(\alpha, \beta)$ on p. 255 must be replaced by

$$Q_n^{(i)}(\alpha, \beta) = \psi(\chi_{n/\eta_e}; \alpha^{1/e}, \zeta_e^i \beta^{1/e})$$

and the corresponding change must be made in formula (15), p. 256, which as it stands applies to $Q_n^{(0)}(\alpha, \beta)$.

In foımula (16) replace $\psi(\chi^i_{n/\eta_e}; x, y)$ by $\psi(\chi_{n/\eta_e}; x, \zeta_e^i y)$.

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A note on solid partitions

by

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1. The purpose of this paper is to extend some of the results of [2] to solid partitions.

2. We define a solid partition of a positive integer n to be a representation of n as the solution to a Diophantine equation of the form

$$n = \sum_{i,j,k=1}^{\infty} n_{i,j,k}$$

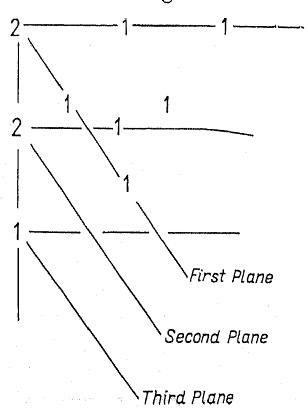
where the $n_{i,j,k}$ are non-negative integers satisfying $n_{i,j,k} \geq n_{i+1,j,k}$, $n_{i,j,k} \geq n_{i,j+1,k}$, and $n_{i,j,k} \geq n_{i,j,k+1}$. We may represent such a partition as a three dimensional array obtained by placing the non-zero part $n_{i,j,k}$ in a position $i-1$ units to the right of, j units below and k units underneath some origin. For example, the solid partitions of 4 are:

First Plane	4	31	3 1	3	22	2 2	2	211	21 1	2 1 1
Second Plane				1			2			
Third Plane										
Fourth Plane										

First Plane	21	2	2	1111	111	11	11	1	111
		1			1	11	1	1	
							1	1	
								1	
Second Plane	1	1	1						1
Third Plane			1						
Fourth Plane									

First Plane	11	1	11	11	1	1	1
	1	1			1	1	
		1					
Second Plane	1	1	11	1	1	1	1
					1		
Third Plane				1		1	1
Fourth Plane							1

where one imagines the ith plane placed under the $(i-1)$st plane, e.g. the solid partition of 11 with 211 on the first plane, 21 on the second plane
11
1
and 1 on the third plane can be thought of as



We define $a(n)$ to be the number of solid partitions. We see from the previously enumerated partitions that $a(4) = 26$. We define

$$A(x) = \sum_{n=0}^{\infty} a(n)x^n$$

to be the generating function for solid partitions. MacMahon [3] conjectured that

$$A(x) = \prod_{\nu=1}^{\infty} (1-x^\nu)^{-\nu(\nu+1)/2}$$

but recent computations, [1], have shown this to be incorrect.

We will also consider solid partitions with the restriction that $n_{i,j,k} > n_{i+1,j,k}$, i.e. partitions with distinct parts along rows. We define $b(n)$ to be the number of such partitions and the corresponding generating function

$$B(x) = \sum_{n=0}^{\infty} b(n)x^n.$$

For example $b(4) = 16$, the relevant partitions being

First Plane	4	31	3	3	2	2	21	2	21
			1		2		1	1	
								1	
Second Plane				1		2			1
Third Plane									
Fourth Plane									

First Plane	2	2	1	1	1	1	1
	1		1	1	1	1	
			1	1			
			1				
Second Plane	1	1		1	1	1	1
					1		
Third Plane		1				1	1
Fourth Plane							1

3. THEOREM. *Let $f_{i,j}$ be integers such that*

$$f_{i,j}\geqslant f_{i+1,j}\geqslant 0 \quad \textit{and} \quad f_{i,j}\geqslant f_{i,j+1}\geqslant 0 \quad (i=1,\dots,\varrho,\ j=1,\dots,\sigma).$$

Let $b(n;f_{1,1},\dots,f_{\varrho,\sigma})$ be the number of solid partitions of n strictly decreasing on rows with the property that there are precisely $f_{i,j}$ parts on the i-th row of the j-th plane. We call such a partition a partition of type $(f_{1,1},\dots,f_{\varrho,\sigma})$. Let $B(x;f_{1,1},\dots,f_{\varrho,\sigma})$ be the corresponding generating function. Then $B(x;f_{1,1},\dots,f_{\varrho,\sigma})$ satisfies the recursion formula

$$B(x;f_{1,1},\dots,f_{\varrho,\sigma}) = x^f\sum_{\varepsilon_{i,j}=0}^{1} B(x;f_{1,1}-\varepsilon_{1,1},\dots,f_{\varrho,\sigma}-\varepsilon_{\varrho,\sigma})$$

where $f=\sum_{i,j} f_{i,j}$; and is uniquely defined by this recursion subject to initial conditions that

$$B(x;f_{1,1},\dots,f_{\varrho,\sigma}) = \begin{cases} 0 & \textit{unless} \quad f_{i,j}\geqslant f_{i+1,j}\geqslant 0,\ f_{i,j}\geqslant f_{i,j+1}\geqslant 0,\\ 1 & \textit{if} \quad f_{1,1}=\dots=f_{\varrho,\sigma}=0.\end{cases}$$

In addition, $b(n;f_{1,1},\dots,f_{\varrho,\sigma})$ satisfies the partial difference equation

$$b(n;f_{1,1},\dots,f_{\varrho,\sigma}) = \sum_{\varepsilon_{i,j}=0}^{1} b(n-f;f_{1,1}-\varepsilon_{1,1},\dots,f_{\varrho,\sigma}-\varepsilon_{\varrho,\sigma})$$

and are uniquely defined by the initial conditions

$$b(n;f_{1,1},\dots,f_{\varrho,\sigma}) = \begin{cases} 0 & \textit{unless} \quad n\geqslant 0,\ f_{i,j}\geqslant f_{i+1,j}\geqslant 0,\ f_{i,j}\geqslant f_{i,j+1}\geqslant 0,\\ 1 & \textit{if} \quad n=f_{1,1}=\dots=f_{\varrho,\sigma}=0.\end{cases}$$

Proof. The proof is analogous to that of [2]. We can classify the set of partitions of type $(f_{1,1},\dots,f_{\varrho,\sigma})$ according as to which rows end in 1. Thus for each matrix

$$\varepsilon = \begin{pmatrix}\varepsilon_{1,1} & \dots & \varepsilon_{1,\sigma}\\ \cdot & \cdot\ \cdot\ \cdot & \cdot\\ \varepsilon_{\varrho,1} & \dots & \varepsilon_{\varrho,\sigma}\end{pmatrix}$$

where $\varepsilon_{i,j}=1$ or 0, we can associate those partitions of type $(f_{1,1},\dots,f_{\varrho,\sigma})$ whose i,jth row ends in 1 or not according as $\varepsilon_{i,j}=1$ or 0. It we subtract 1 from each part of a partition related to a matrix ε we obtain a partition of type $n-f$ of type $(f_{1,1}-\varepsilon_{1,1},\dots,f_{\varrho,\sigma}-\varepsilon_{\varrho,\sigma})$. Since this correspondence is bijective, the number of partitions associated with ε is equal to $b(n-f;f_{1,1}-\varepsilon_{1,1},\dots,f_{\varrho,\sigma}-\varepsilon_{\varrho,\sigma})$. Summation over all matrices ε yields

$$b(n;f_{1,1},\dots,f_{\varrho,\sigma}) = \sum_{\varepsilon_{i,j}=0}^{1} b(n-f;f_{1,1}-\varepsilon_{1,1},\dots,f_{\varrho,\sigma}-\varepsilon_{\varrho,\sigma}).$$

We can multiply both sides by x^n and sum over all $n\geqslant 0$ to obtain

$$\sum_{n=0}^{\infty} b(n;f_{1,1},\dots,f_{\varrho,\sigma})x^n = \sum_{n=0}^{\infty}\sum_{\varepsilon_{i,j}=0}^{1} x^n b(n-f;f_{1,1}-\varepsilon_{1,1},\dots,f_{\varrho,\sigma}-\varepsilon_{\varrho,\sigma}).$$

But the left side is $B(x;f_{1,1},\dots,f_{\varrho,\sigma})$ hence

$$\begin{aligned} B(x;f_{1,1},\dots,f_{\varrho,\sigma}) &= \sum_{\varepsilon_{i,j}=0}^{1} x^f \sum_{n=0}^{\infty} b(n-f;f_{1,1}-\varepsilon_{1,1},\dots,f_{\varrho,\sigma}-\varepsilon_{\varrho,\sigma})x^{n-f}\\ &= x^f\sum_{\varepsilon_{i,j}=0}^{1} B(x;f_{1,1}-\varepsilon_{1,1},\dots,f_{\varrho,\sigma}-\varepsilon_{\varrho,\sigma})\end{aligned}$$

since

$$b(n-f;f_{1,1}-\varepsilon_{1,1},\dots,f_{\varrho,\sigma}-\varepsilon_{\varrho,\sigma})=0 \quad \text{for} \quad n<f.$$

The uniqueness is obvious. This completes the proof.

Remark. If we define $B_{\varrho,\sigma}(x)$ to be the generating function for the number of ϱ row, σ plane partitions of n, i.e., solid partitions with $n_{i,j,k}=0$ for $i>\varrho$ and $k>\sigma$, then

$$B_{\varrho,\sigma}(x) = \sum_{\substack{f_{i,j}\geqslant f_{i,j+1}\geqslant 0\\ f_{i,j}\geqslant f_{i+1,j}\geqslant 0}} B(x;f_{1,1},\dots,f_{\varrho,\sigma}).$$

Hence

$$B(x) = \lim_{\varrho,\sigma\to\infty} B_{\varrho,\sigma}(x).$$

4. Let $f_{1,1}=\dots=f_{\varrho,\sigma}=g$. The partitions enumerated by $b(n;g,\dots,g)$ are precisely those obtained by superimposing any unrestricted ϱ row, σ plane partitions with at most g parts on any row upon the array

First Plane	g	$g-1$	$g-2$	$\dots$	1	ϱ rows
	g	$g-1$	$g-2$	$\dots$	1	
	$\cdot$					
	$\cdot$					
	g	$g-1$	$g-2$	$\dots$	1	
	$\cdot$					$\cdot$
	$\cdot$					$\cdot$
	$\cdot$					$\cdot$
σth Plane	g	$g-1$	$g-2$	$\dots$	1	ϱ rows
	g	$g-1$	$g-2$	$\dots$	1	
	$\cdot$					
	$\cdot$					
	$\cdot$					
	g	$g-1$	$g-2$	$\dots$	1	

The sum of the numbers in this array is $\varrho\sigma\binom{g+1}{2}$. Hence if $a(n; f_{1,1}, \ldots, f_{\varrho,\sigma})$ denotes the number of unrestricted ϱ row, σ plane partitions of n with at most $f_{i,j}$ non-zero parts on the i, jth row. Hence

$$b(n; g, \ldots, g) = \sum_{f_{i,j} \leqslant g} a\left(n - \varrho\sigma\binom{g+1}{2}; f_{1,1}, \ldots, f_{\varrho,\sigma}\right).$$

Multiply both sides by $x^{n-\varrho\sigma\binom{g+1}{2}}$ and sum over n. We obtain

$$B(x; g, \ldots, g) = x^{\varrho\sigma\binom{g+1}{2}} \sum_{f_{i,j} \leqslant g} \sum_{n=0}^{\infty} a(n; f_{1,1}, \ldots, f_{\varrho,\sigma}) x^n$$
$$= x^{\varrho\sigma\binom{g+1}{2}} \sum_{f_{i,j} \leqslant g} A(x; f_{1,1}, \ldots, f_{\varrho,\sigma})$$

where we may replace $n - \varrho\sigma\binom{g+1}{2}$ by n as the summation index in the right since the terms vanish for $n < \varrho\sigma\binom{g+1}{2}$. If we let $g \to \infty$ and note that

$$A_{\varrho,\sigma}(x) = \sum_{f_{i,j}} A(x; f_{1,1}, \ldots, f_{\varrho,\sigma})$$

we can obtain an expression for $A_{\varrho,\sigma}(x)$. Further we can see that since

$$A(x) = \lim_{\substack{\varrho\to\infty \\ \sigma\to\infty}} A_{\varrho,\sigma}(x),$$

a solution to the recursion of the theorem will enable us also to obtain a solution to the unrestricted case. At present only a numerical solution is available.

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A note on the representability of binary quadratic forms with Gaussian integer coefficients as sums of squares of two linear forms

by

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1. Notations. Let G denote the ring of Gaussian integers. Small Greek letters will denote elements of G, except for the unit i, and small Latin letters will denote ordinary integers in Z. If α is in G, the norm of α will be denoted by $N(\alpha)$.

DEFINITION. α in G is called *odd* if $N(\alpha)$ is odd. α in G is called *even* if $N(\alpha)$ is even.

With each integer $a+bi$ in G, there are associated three other integers, namely $-a-bi$, $-b+ai$, $b-ai$.

DEFINITION. The number $x+yi$ of the four associated odd integers $a+bi$, $-a-bi$, $-b+ai$, $b-ai$ is called *primary* if

$$x \equiv 1\ (4), \quad y \equiv 0\ (4)$$

or

$$x \equiv 3\ (4), \quad y \equiv 2\ (4).$$

In any group of four associated odd integers, exactly one is primary.

DEFINITION. If α in G is even, we distinguish between the associates of α by taking as *primary* that one which can be written as $(1+i)^k\beta$ where β is an odd, primary integer.

DEFINITION. Let α, β, δ be Gaussian integers. δ will be called the *greatest common divisor* of α and β if

1) δ is a common divisor of α and β,
2) if γ in G is a common divisor of α and β, then $\gamma \mid \delta$,
3) δ is primary.

We shall write $\delta = (\alpha, \beta)$.

2. The following result may be found in [2].

THEOREM. *If α is an odd Gaussian integer of the form $a+2bi$, then α can be expressed as a sum of two squares of integers in G. If α is even,*