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Approximation to real numbers by algebraic integers

by

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1. Introduction. The problem of approximating to a real numbet ξ by algebraic numbers was investigated by Wirsing [2]. He proved thae if $n>1$ and ξ is not an algebraic number of degree at most n, there arr infinitely many algebraic numbers α of degree at most n which satisfy(¹)

$$|\xi-\alpha| \ll H(\alpha)^{-(n+3)/2}, \tag{1}$$

where $H(\alpha)$ denotes the height of α. The constant implied by the notation $\ll$ depends on n and ξ, but its dependence on ξ is of a simple nature. In the case $n=2$ we showed [1] that the result holds with $H(\alpha)^{-3}$ on the right, and this is best possible.

In the present paper we investigate approximation to a real number ξ by real algebraic *integers* α.

It is instructive to consider first the case $n=2$, even though this is every simple. In the first place, if ξ is rational there are infinitely many quadratic integers α satisfying

$$0<|\xi-\alpha| \ll H(\alpha)^{-1}. \tag{2}$$

For there are infinitely many integer pairs x, y satisfying

$$0<|\xi^2+\xi x+y| \leqslant 1,$$

and if we put $t^2+tx+y = (t-\alpha)(t-\alpha')$ we have

$$(\alpha-\alpha')^2 = x^2-4y = (2\xi+x)^2+O(1).$$

Without loss of generality we can suppose that

$$|\xi-\alpha'| \gg |2\xi+x| \gg |x|,$$

and as $0<|(\xi-\alpha)(\xi-\alpha')| \leqslant 1$, this implies that

$$0<|\xi-\alpha| \ll |x|^{-1},$$

(¹) Wirsing also gave an exponent slightly better than that in (1).

whence (2). It is also plain that the right hand side of (2) cannot be improved to $o(H(\alpha)^{-1})$, since $|\xi^2+\xi x+y|$ has a positive lower bound.

If ξ is irrational we can do better than (2). By theorems of Minkowski and Hurwitz, there are infinitely many integer pairs x, y satisfying

$$0<|\xi^2+\xi x+y| \ll |x|^{-1}.$$

The same argument as above gives infinitely many algebra[illegible] egers α, of degree at most 2, satisfying

$$(3) \qquad 0<|\xi-\alpha| \ll H(\alpha)^{-2}.$$

This is best possible if ξ is any quadratic irrational, for then if $\xi^2+\xi x+y$ is not 0 we have

$$|\xi^2+\xi x+y| \gg |\xi'^2+\xi' x+y|^{-1} \gg |x|^{-1},$$

where ξ' denotes the algebraic conjugate of ξ.

Similar considerations for general n show that in order to assert a result of the form

$$0<|\xi-\alpha| = o\big(H(\alpha)^{-k}\big)$$

for an infinity of algebraic integers α of degree at most n, we must exclude the possibility that ξ is an algebraic number of degree at most k.

In the case $n=3$ we prove:

THEOREM 1. *Suppose that ξ is neither rational nor a quadratic irrational. Then there are infinitely many algebraic integers α of degree at most 3 which satisfy*

$$(4) \qquad 0<|\xi-\alpha| \ll H(\alpha)^{-\theta},$$

where

$$(5) \qquad \theta = \tfrac{1}{2}(3+\sqrt{5}) = 2.618\ldots$$

For general n we prove:

THEOREM 2. *Suppose that $n \geq 3$ and that ξ is not an algebraic number of degree at most $\frac{1}{2}(n-1)$. Then there are infinitely many algebraic integers α of degree at most n which satisfy*

$$(6) \qquad 0<|\xi-\alpha| \ll H(\alpha)^{-[\frac{1}{2}(n+1)]}.$$

It will be seen that if $n=3$ the exponent is -2, and so the result is then inferior to that of Theorem 1. The exponent is still -2 if $n=4$. We prove in Theorem 4 (§ 4) that in the case $n=4$ it is possible to get the exponent -3.

We have no reason to think that the exponents in these theorems are best possible.

It will be clear from the earlier discussion of the case $n=2$ that the problem is essentially that of finding a small value of the non-homogeneous linear form

$$(7) \qquad \xi^n + x_0\xi^{n-1} + x_1\xi^{n-2} + \ldots + x_{n-1}$$

in terms of $\max(|x_0|, \ldots, |x_{n-2}|)$. We need also that the derived linear form

$$n\xi^{n-1} + (n-1)x_0\xi^{n-2} + \ldots + x_{n-2}$$

shall be large, but (rather unexpectedly) this condition, which presented the main difficulty in our paper [1], gives no serious trouble here.

The natural way of attempting to find small values of the non-homogeneous form (7) is as follows. We consider the convex body defined by

$$(8) \qquad |x_0\xi^{n-1} + \ldots + x_{n-1}| < R^{-n+1}, \qquad \max(|x_0|, \ldots, |x_{n-2}|) < R,$$

and we denote by $\tau_1, \ldots, \tau_n$ the successive minima of this body, in the sense of Minkowski. If we can find large values of R for which τ_n has a good upper bound, we can deduce a small value of the form (7).

By the principle of duality (or 'transference'), a good upper bound for τ_n corresponds to a good lower bound for τ_1', the first minimum of the body which is polar to (8). This body is defined by

$$(9) \qquad |y_0| < R^{n-1}, \qquad |y_0\xi^m - y_m| < R^{-1} \qquad (m=1, \ldots, n-1),$$

and we have to seek values of R for which these inequalities do not have a particularly small solution.

The preceding considerations serve to explain why Theorems 1 and 2 above are derived from the following results.

THEOREM 1a. *Suppose ξ is neither rational nor a quadratic irrational. Then there are arbitrarily large values of X such that the inequalities*

$$(10) \qquad |x_0| < X, \qquad |x_0\xi - x_1| < cX^{-\theta+2}, \qquad |x_0\xi^2 - x_2| < cX^{-\theta+2},$$

where c is a suitable positive number depending on ξ, have no solution in integers x_0, x_1, x_2, not all 0.

THEOREM 2a. *Suppose that $n \geq 3$ and that ξ is not an algebraic number of degree at most $\frac{1}{2}(n-1)$. Then there are arbitrarily large values of X such that the inequalities*

$$(11) \qquad |x_0| < X, \qquad |x_0\xi^m - x_m| < cX^{-\gamma} \qquad (m=1, \ldots, n-1),$$

where

$$\gamma = [\tfrac{1}{2}(n-1)]^{-1}$$

and c is a suitable positive number depending on n and ξ, have no solution in integers $x_0, \ldots, x_{n-1}$, not all 0.

For the sake of symmetry, we prove an analogue of the last result for the polar reciprocal body, though we shall make no use of it. This is as follows.

THEOREM 2b. *Suppose that $n \geqslant 2$ and that ξ is not an algebraic number of degree at most $n-1$. Then there are arbitrarily large values of X such that the inequalities*

$$(12)\qquad |x_0\xi^{n-1}+\ldots+x_{n-1}| < cX^{-2n+3}, \quad |x_m| < X \quad (m = 0, \ldots, n-2),$$

where c is a suitable positive number depending on n and ξ, have no solution in integers $x_0, \ldots, x_{n-1}$, not all 0.

In the proof of Theorem 2a, we shall make use of the following result, which may be of independent interest.

THEOREM 3. *Suppose that $1 \leqslant h \leqslant m$, and let $a_0, a_1, \ldots, a_h$ be integers with no common factor throughout. Let $\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ be integer points in h dimensional space which span the whole space and which satisfy the recurrence relation*

$$(13)\qquad a_0\boldsymbol{y}_i+a_1\boldsymbol{y}_{i+1}+\ldots+a_h\boldsymbol{y}_{i+h} = 0 \quad (0 \leqslant i \leqslant m-h).$$

Suppose that all $h\times h$ determinants formed from any h of the vectors $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_m$ have absolute values at most Z. Then

$$(14)\qquad \max(|a_0|, |a_1|, \ldots, |a_h|) \ll Z^{1/(m-h+1)}.$$

2. Deduction of Theorems 1, 2 from Theorems 1a, 2a.

LEMMA 1. *Suppose that $n \geqslant 2$, that ξ is real, and that $\lambda > 0$. Suppose that for some $c > 0$ there are arbitrarily large values of X such that the inequalities*

$$(15)\qquad |x_0| \leqslant X, \quad |x_0\xi^m - x_m| \leqslant cX^{-\lambda} \quad (m = 1, \ldots, n-1)$$

have no solution in integers $x_0, \ldots, x_{n-1}$, not all 0. Then there are infinitely many algebraic integers α of degree at most n which satisfy

$$(16)\qquad 0 < |\xi-\alpha| \ll H(\alpha)^{-1-1/\lambda}.$$

When $\lambda = \theta-2 = \frac{1}{2}(-1+\sqrt{5})$ we get $1+\lambda^{-1} = \theta$. Thus Theorem 1a implies Theorem 1. Similarly when $\lambda = [\frac{1}{2}(n-1)]^{-1}$ we get $1+\lambda^{-1} = [\frac{1}{2}(n+1)]$, and Theorem 2a implies Theorem 2.

Proof of Lemma 1. We may assume that $c < 1$. Let X be one of the large numbers specified in the enunciation, and let

$$(17)\qquad Y = X^{(\lambda+1)/n}.$$

Let $K(Y)$ be the parallelepiped defined by

$$(18)\qquad |x_0| \leqslant Y^{n-1}, \quad |x_0\xi^m - x_m| \leqslant Y^{-1} \quad (m = 1, \ldots, n-1).$$

Then the first minimum of $K(Y)$, say $\tau_1(Y)$, satisfies

$$(19)\qquad \tau_1(Y) \geqslant cY^{-\varrho},$$

where

$$(20)\qquad \varrho = \frac{\lambda n}{\lambda+1} - 1.$$

For in the contrary case there would be a non-trivial solution of

$$|x_0| \leqslant \tau_1(Y)Y^{n-1} < Y^{n-1-\varrho} = X,$$
$$|x_0\xi^m - x_m| \leqslant \tau_1(Y)Y^{-1} < cY^{-1-\varrho} = cX^{-\lambda} \quad (1 \leqslant m \leqslant n-1),$$

which is contrary to the hypothesis.

The polar body $K^*(Y)$ of $K(Y)$ is defined by

$$(21)\qquad \begin{aligned} &|x_{n-1}\xi^{n-1}+\ldots+x_1\xi+x_0| \leqslant Y^{-n+1},\\ &|x_m| \leqslant Y \quad (m = 1, \ldots, n-1).\end{aligned}$$

It follows from (19), by Mahler's theorem on polar reciprocal bodies, that the nth successive minimum $\tau_n^*(Y)$ of $K^*(Y)$ satisfies

$$(22)\qquad \tau_n^*(Y) \leqslant c_1 Y^{\varrho}.$$

By the definition of successive minima, there exist n linearly independent integers points

$$\boldsymbol{x}^{(i)} = (x_0^{(i)}, x_1^{(i)}, \ldots, x_{n-1}^{(i)}) \quad (i = 1, \ldots, n),$$

satisfying

$$(23)\qquad \begin{aligned} &|x_{n-1}^{(i)}\xi^{n-1}+\ldots+x_0^{(i)}| \leqslant c_1 Y^{\varrho-n+1},\\ &|x_m^{(i)}| \leqslant c_1 Y^{\varrho+1} \quad (1 \leqslant m \leqslant n-1),\end{aligned}$$

for $i = 1, \ldots, n$. Put

$$L^{(i)} = x_{n-1}^{(i)}\xi^{n-1}+\ldots+x_0^{(i)},$$
$$P^{(i)} = (n-1)x_{n-1}^{(i)}\xi^{n-2}+\ldots+x_1^{(i)}.$$

There exist real numbers $\theta_1, \ldots, \theta_n$ satisfying

$$(24)\qquad \theta_1 x_m^{(1)}+\ldots+\theta_n x_m^{(n)} = 0 \quad (m = 2, 3, \ldots, n-1),$$

$$(25)\qquad n\xi^{n-1}+\theta_1 P^{(1)}+\ldots+\theta_n P^{(n)} = Y^{1+\varrho}+|P^{(1)}|+\ldots+|P^{(n)}|,$$

$$(26)\qquad \xi^n+\theta_1 L^{(1)}+\ldots+\theta_n L^{(n)} = (n+1)c_1 Y^{\varrho-n+1},$$

since these are n non-homogeneous linear equations in n unknowns with non-zero determinant. Let $t_1, \ldots, t_n$ be integers, not all 0, with

$$|t_i - \theta_i| \leqslant 1 \quad (i = 1, \ldots, n)$$

and put

$$\boldsymbol{x} = t_1\boldsymbol{x}^{(1)}+\ldots+t_n\boldsymbol{x}^{(n)}.$$

Then $\boldsymbol{x} \neq 0$ and by (23), (24) we have

$$|x_m| \leqslant nc_1 Y^{1+\varrho} \quad (m = 2, 3, \ldots, n-1).$$

By (23), (25) we have

$$\begin{aligned} Y^{1+\varrho} &\leqslant |n\xi^{n-1}+(n-1)x_{n-1}\xi^{n-2}+\ldots+x_1| \\ &\leqslant Y^{1+\varrho}+2(|P^{(1)}|+\ldots+|P^{(n)}|) \ll Y^{1+\varrho}. \end{aligned}$$

Finally, (23) and (26) imply that

$$0 < |\xi^n+x_{n-1}\xi^{n-1}+\ldots+x_0| \leqslant (2n+1)c_1 Y^{\varrho-n+1}.$$

The polynomial

$$Q(t) = t^n+x_{n-1}t^{n-1}+\ldots+x_0$$

has height

$$\|Q\| \ll Y^{1+\varrho},$$

and by the preceding inequalities it satisfies

$$0 < |Q(\xi)| \ll Y^{\varrho-n+1}, \quad |Q'(\xi)| \gg Y^{1+\varrho}.$$

Since $\|Q\| \ll Y^{1+\varrho}$, the derivative of Q remains $\gg Y^{1+\varrho}$ in any interval containing ξ whose length is small compared with 1. Hence Q has a real zero α satisfying

$$0 < |\xi-\alpha| \ll Y^{\varrho-n+1-(1+\varrho)} = Y^{-n}.$$

Since $H(\alpha) \ll \|Q\| \ll Y^{1+\varrho}$, the algebraic integer α satisfies

$$0 < |\xi-\alpha| \ll H(\alpha)^{-n/(1+\varrho)}.$$

The exponent here is $n/(1+\varrho) = 1+\lambda^{-1}$ by (20), and the result of the lemma is established.

3. Proof of Theorem 1a. Suppose that ξ is neither rational nor a quadratic irrational, and suppose that the inequalities

$$(27) \qquad |x_0| \leqslant X, \quad |x_0\xi-x_1| \leqslant cX^{-\theta+2}, \quad |x_0\xi^2-x_2| \leqslant cX^{-\theta+2}$$

can be solved in integers x_0, x_1, x_2, not all 0, for every large X. We shall reach a contradiction if c is sufficiently small, and this will prove the Theorem.

For each real $X > 1$ we consider the set of integer points $\boldsymbol{x} = (x_0, x_1, x_2)$ with

$$1 \leqslant x_0 \leqslant X, \quad |x_0\xi-x_1| \leqslant 1, \quad |x_0\xi^2-x_2| \leqslant 1.$$

Among them we choose the unique point for which

$$\max(|x_0\xi-x_1|, |x_0\xi^2-x_2|)$$

has its least value, and we call this the *minimal point corresponding to X.*

It is obvious that if $\boldsymbol{x}$ is the minimal point corresponding both to X' and to X'' it is also the minimal point corresponding to any X between X' and X''. Hence there is a sequence of integers

$$X_1 < X_2 < \ldots$$

such that the same minimal point corresponds to all X in the range $X_i \leqslant X < X_{i+1}$ but to no X outside this range. If we denote this point by

$$\boldsymbol{x}_i = (x_{i0}, x_{i1}, x_{i2}),$$

we obviously have $x_{i0} = X_i$ and

$$(28) \qquad |\boldsymbol{x}_i| \ll X_i.$$

If we write for brevity

$$L_i = \max(|x_{i0}\xi-x_{i1}|, |x_{i0}\xi^2-x_{i2}|),$$

then obviously

$$(29) \qquad L_1 > L_2 > \ldots$$

The minimal point corresponding to $X_{i+1}-\varepsilon$, where ε is a small positive number, is $\boldsymbol{x}_i$, and by our hypothesis we must have

$$L_i \leqslant c(X_{i+1}-\varepsilon)^{-\theta+2}.$$

Since ε is arbitrarily small, this implies that

$$(30) \qquad L_i \leqslant cX_{i+1}^{-\theta+2}.$$

Lemma 2. *For all sufficiently large i we have*

$$(31) \qquad \begin{vmatrix} x_{i0} & x_{i1} \\ x_{i1} & x_{i2} \end{vmatrix} \neq 0.$$

Proof. If the result were false, we should have

$$x_{i0}x_{i2} = x_{i1}^2.$$

Since x_{i0}, x_{i1}, x_{i2} have no common factor, it follows that

$$x_{i0} = m^2, \quad x_{i1} = mn, \quad x_{i2} = n^2,$$

where m, n are integers. We have

$$(32) \qquad |m| = X_i^{1/2}.$$

Since $|m(m\xi-n)| = |x_{i0}\xi - x_{i1}| \ll X_{i+1}^{-\theta+2}$, we obtain

$$|m\xi-n| \ll X_i^{-1/2} X_{i+1}^{-\theta+2}. \tag{33}$$

The vectors $\boldsymbol{x}_{i-1}, \boldsymbol{x}_i$ are linearly independent, and therefore the matrix

$$\begin{pmatrix} x_{i-1,0} & x_{i-1,1} & x_{i-1,2} \\ x_{i0} & x_{i1} & x_{i2} \end{pmatrix}$$

has rank 2. Since $x_{i-1,1}$ and x_{i1} are distinct from zero, at least one of the two determinants

$$\begin{vmatrix} x_{i-1,0} & x_{i-1,1} \\ x_{i0} & x_{i1} \end{vmatrix}, \quad \begin{vmatrix} x_{i-1,1} & x_{i-1,2} \\ x_{i1} & x_{i2} \end{vmatrix} \tag{34}$$

is not zero.

Suppose the first determinant in (34) is not zero. This implies that

$$\begin{vmatrix} x_{i-1,0} & x_{i-1,1} \\ m & n \end{vmatrix} \neq 0. \tag{35}$$

By (30) we have

$$|x_{i-1,0}\xi - x_{i-1,1}| \ll X_i^{-\theta+2}. \tag{36}$$

Using (33) and (36) we obtain

$$\begin{vmatrix} x_{i-1,0} & x_{i-1,0}\xi - x_{i-1,1} \\ m & m\xi-n \end{vmatrix} \ll X_i^{-1/2} X_{i+1}^{-\theta+2} X_{i-1} + X_i^{-\theta+2} X_i^{1/2} \ll X_i^{-\theta+5/2}.$$

Since $\theta > \frac{5}{2}$, this contradicts (35) if i is sufficiently large.

The argument is similar if the second of the determinants (34) is not zero. It is a consequence of (30) that

$$|x_{i-1,1}\xi - x_{i-1,2}| \ll X_i^{-\theta+2},$$

and one uses this in place of (36).

LEMMA 3. *For all large i we have*

$$X_{i+1}^{\theta-2} \leqslant 2c(1+|\xi|)X_i. \tag{37}$$

Proof. We have, by (30),

$$|x_{i0}\xi - x_{i1}| \leqslant cX_{i+1}^{-\theta+2},$$

$$|x_{i1}\xi - x_{i2}| \leqslant c(1+|\xi|)X_{i+1}^{-\theta+2}.$$

These imply that

$$\begin{vmatrix} x_{i0} & x_{i1} \\ x_{i1} & x_{i2} \end{vmatrix} \leqslant cX_{i+1}^{-\theta+2}\left(|x_{i0}|(1+|\xi|)+|x_{i1}|\right) \leqslant 2c(1+|\xi|)X_i X_{i+1}^{-\theta+2}.$$

The determinant on the left is a non-zero integer by Lemma 2, and the result follows.

LEMMA 4. *Suppose that $\boldsymbol{x}_{i-1}, \boldsymbol{x}_i, \boldsymbol{x}_{i+1}$ are linearly independent. Then*

$$6c^2 X_{i+1}^{3-\theta} \geqslant X_i^{\theta-2}. \tag{38}$$

Proof. We have

$$\begin{vmatrix} x_{i-1,0} & x_{i-1,1} & x_{i-1,2} \\ x_{i0} & x_{i1} & x_{i2} \\ x_{i+1,0} & x_{i+1,1} & x_{i+1,2} \end{vmatrix} = \begin{vmatrix} x_{i-1,0} & x_{i-1,0}\xi - x_{i-1,1} & x_{i-1,0}\xi^2 - x_{i-1,2} \\ x_{i0} & x_{i0}\xi - x_{i1} & x_{i0}\xi^2 - x_{i2} \\ x_{i+1,0} & x_{i+1,0}\xi - x_{i+1,1} & x_{i+1,0}\xi^2 - x_{i+1,2} \end{vmatrix},$$

and the absolute value of the last determinant is

$$\leqslant 6c^2 X_{i+1} X_i^{-\theta+2} X_{i+1}^{-\theta+2} = 6c^2 X_i^{-\theta+2} X_{i+1}^{-\theta+3},$$

by (28) and (30). On the other hand, this absolute value is at least 1, and (38) follows.

LEMMA 5. *For infinitely many i, the points $\boldsymbol{x}_{i-1}, \boldsymbol{x}_i, \boldsymbol{x}_{i+1}$ are linearly independent.*

Proof. In the contrary case, there would exist integers not all zero, such that

$$ax_{i0} + bx_{i1} + cx_{i2} = 0$$

for all sufficiently large i. But we have

$$x_{i1} = \xi x_{i0} + O(X_{i+1}^{-\theta+2}), \qquad x_{i2} = \xi^2 x_{i0} + O(X_{i+1}^{-\theta+2}),$$

and on substitution we obtain

$$x_{i0}(c\xi^2 + b\xi + a) \ll X_{i+1}^{-\theta+2}.$$

Since $c\xi^2+b\xi+a$ is distinct from zero, the left hand side tends to infinity and the right hand side to zero, and we obtain a contradiction.

Completion of the proof of Theorem 1a. By Lemma 3 we know that (37) holds for all large i, and by Lemmas 4 and 5 we know that (38) holds for infinitely many i. Together these inequalities give

$$X_i^{(\theta-2)^2} \leqslant (6c^2)^{\theta-2} X_{i+1}^{(3-\theta)(\theta-2)} \leqslant (6c^2)^{\theta-2}\left(2c(1+|\xi|)\right)^{3-\theta} X_i^{3-\theta}.$$

Since $(\theta-2)^2 = 3-\theta$, this is impossible if c is sufficiently small. This gives the contradiction asserted at the beginning of this section.

4. The case $n=4$. Before going on to the proofs of Theorems 2a and 2b, which involve different ideas from those encountered so far, we state and prove the result mentioned in § 1, relating to the case $n=4$.

THEOREM 4. *Suppose that ξ is neither rational nor a quadratic irrational. Then there are infinitely many algebraic integers a of degree at most 4 which satisfy*

$$0 < |\xi - a| \ll H(a)^{-3}.$$

THEOREM 4a. *Let ξ be as in Theorem 4. Then there are arbitrarily large values of X such that the inequalities*

$$|x_0| \leqslant X, \quad |x_0\xi - x_1| \leqslant cX^{-1/2},$$
$$|x_0\xi^2 - x_2| \leqslant cX^{-1/2}, \quad |x_0\xi^3 - x_3| \leqslant cX^{-1/2},$$

where c is a suitable positive number depending on ξ, have no solution in integers x_0, x_1, x_2, x_3 not all zero.

Theorem 4 follows from Theorem 4a in just the same way as Theorems 1 and 2 follow from Theorems 1a and 2a (see § 2); if $\lambda = \frac{1}{2}$ we have $1+\lambda^{-1} = 3$.

We begin the proof of Theorem 4a by defining the sequence of minimal points as in § 3, but of course taking now the least value of

$$\max(|x_0\xi - x_1|, |x_0\xi^2 - x_2|, |x_0\xi^3 - x_3|).$$

This least value may be attained at more than one point (if ξ is a cubic irrational). We make an arbitrary choice, but read (29) with $\leqslant$ in place of $<$. In place of (30) we have

$$|L_i| \leqslant cX_{i+1}^{-1/2}.$$

LEMMA 6. *For all sufficiently large i, the matrix*

$$\begin{pmatrix} x_{i0} & x_{i1} & x_{i2} \\ x_{i1} & x_{i2} & x_{i3} \end{pmatrix}$$

has rank 2.

Proof. The general lines are similar to that of Lemma 2. If the matrix had rank 1, we should have

$$x_{i0} = m^3, \quad x_{i1} = m^2 n, \quad x_{i2} = mn^2, \quad x_{i3} = n^3,$$

whence $|m| = X_i^{1/3}$. Since

$$|m^2(m\xi - n)| = |x_{i0}\xi - x_{i1}| \ll X_{i+1}^{-1/2},$$

we get

$$|m\xi - n| \ll X_i^{-2/3} X_{i+1}^{-1/2}.$$

We know that the matrix

$$\begin{pmatrix} x_{i-1,0} & x_{i-1,1} & x_{i-1,2} & x_{i-1,3} \\ x_{i0} & x_{i1} & x_{i2} & x_{i3} \end{pmatrix}$$

has rank 2. Since each column vector is distinct from zero, one of the three determinants

$$\begin{vmatrix} x_{i-1,0} & x_{i-1,1} \\ x_{i0} & x_{i1} \end{vmatrix}, \quad \begin{vmatrix} x_{i-1,1} & x_{i-1,2} \\ x_{i1} & x_{i2} \end{vmatrix}, \quad \begin{vmatrix} x_{i-1,2} & x_{i-1,3} \\ x_{i2} & x_{i3} \end{vmatrix}$$

is not 0. We shall suppose it is the first; the proof is essentially the same in the other cases. Thus we have

$$\begin{vmatrix} x_{i-1,0} & x_{i-1,1} \\ m & n \end{vmatrix} \neq 0.$$

Now $|x_{i-1,0}\xi - x_{i-1,1}| \ll X_i^{-1/2}$, and therefore

$$\begin{vmatrix} x_{i-1,0} & x_{i-1,0}\xi - x_{i-1,1} \\ m & m\xi - n \end{vmatrix} \ll X_i^{-2/3} X_{i+1}^{-1/2} X_{i-1} + X_i^{1/2} X_i^{-1/3} \ll X_i^{-1/6}.$$

This gives a contradiction.

We define vectors $\boldsymbol{y}_i, \boldsymbol{z}_i$ by

$$\boldsymbol{y}_i = (x_{i0}, x_{i1}, x_{i2}), \quad \boldsymbol{z}_i = (x_{i1}, x_{i2}, x_{i3}).$$

They are linearly independent by Lemma 6.

LEMMA 7. *$\boldsymbol{y}_{i+1}$ and $\boldsymbol{z}_{i+1}$ are linear combinations of $\boldsymbol{y}_i$ and $\boldsymbol{z}_i$.*

Proof. We have

$$\begin{vmatrix} x_{i0} & x_{i1} & x_{i2} \\ x_{i1} & x_{i2} & x_{i3} \\ x_{i+1,0} & x_{i+1,1} & x_{i+1,2} \end{vmatrix} = \begin{vmatrix} x_{i0} & x_{i0}\xi - x_{i1} & x_{i0}\xi^2 - x_{i2} \\ x_{i1} & x_{i1}\xi - x_{i2} & x_{i1}\xi^2 - x_{i3} \\ x_{i+1,0} & x_{i+1,0}\xi - x_{i+1,1} & x_{i+1,0}\xi^2 - x_{i+1,2} \end{vmatrix}$$
$$\ll c^2 X_{i+1} X_{i+1}^{-1/2} X_{i+1}^{-1/2} \ll c^2.$$

If c is sufficiently small, the determinant is zero, and therefore $\boldsymbol{y}_i, \boldsymbol{z}_i, \boldsymbol{y}_{i+1}$ are linearly dependent. Since $\boldsymbol{y}_i, \boldsymbol{z}_i$ are linearly independent, the result follows for $\boldsymbol{y}_{i+1}$, and similarly for $\boldsymbol{z}_{i+1}$.

Completion of the proof of Theorem 4a. By Lemma 7, the vectors

$$\boldsymbol{y}_i, \boldsymbol{z}_i, \boldsymbol{y}_{i+1}, \boldsymbol{z}_{i+1}, \ldots$$

all lie in the same rational linear subspace. Hence there exist integers a, b, c, not all 0, such that

$$ax_{i0} + bx_{i1} + cx_{i2} = 0$$

for all sufficiently large i. As in the proof of Lemma 5, this implies that

$$|x_{i0}|\,|c\xi^2 + b\xi + a| \ll X_{i+1}^{-1/2}.$$

Since ξ is neither rational nor a quadratic irrational, we obtain a contradiction on making $i \to \infty$.

5. A lemma on polynomials. Let $P = P(t)$ be a polynomial in one variable, not identically zero, with integer coefficients and of degree at most n. By Gauss's Lemma,

$$P = P_1 P_2 \ldots P_s,$$

where $P_1, \ldots, P_s$ have integer coefficients and are irreducible over the rationals. We recall the well known inequalities

$$\|P\| \ll \|P_1\| \ldots \|P_s\| \ll \|P\|, \tag{39}$$

where $\|P\|$ denotes the height of P. In particular there is a $K = K(n)$ such that every irreducible factor P_i of P satisfies

$$\|P_i\| \leqslant K(n)\, \|P\|. \tag{40}$$

LEMMA 8. *Let P, Q be polynomials, not identically zero, with integer coefficients and of degrees at most n. Suppose that P and Q have no common polynomial factor other than a constant. Then, for every ξ,*

$$\max\big(|P(\xi)|, |Q(\xi)|\big) \geqslant \big((2n)!\big)^{-1}(1+|\xi|^{n-1})^{-1}\big(\max(\|P\|, \|Q\|)\big)^{-2n+1}. \tag{41}$$

Proof. Write

$$P(t) = a_0 t^p + a_1 t^{p-1} + \ldots + a_p, \quad \text{where } a_0 \neq 0 \text{ and } p \leqslant n,$$
$$Q(t) = b_0 t^q + b_1 t^{q-1} + \ldots + b_q, \quad \text{where } b_0 \neq 0 \text{ and } q \leqslant n.$$

By hypothesis, the resultant of $P(t)$ and $Q(t)$ is not zero, and so has absolute value at least 1. The resultant is given by Sylvester's determinant:

$$\left.\begin{vmatrix}
a_0 & a_1 & \ldots & a_p & & \\
 & a_0 & \ldots & & a_p & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & a_0 & \ldots & & a_p \\
b_0 & b_1 & \ldots & b_q & & \\
 & b_0 & \ldots & & b_q & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & b_0 & \ldots & & b_q
\end{vmatrix}\right\} \begin{matrix} q \\ \\ p \end{matrix}$$

If we multiply the $(p+q-i)$th column by ξ^i for $i = 1, \ldots, p+q-1$ and add all these multiplied columns to the last column, the determinant becomes

$$\begin{vmatrix}
a_0 & a_1 & \ldots & a_p & & \xi^{q-1}P(\xi) \\
 & a_0 & \ldots & & a_p & \xi^{q-2}P(\xi) \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & a_0 & \ldots & & P(\xi) \\
b_0 & b_1 & \ldots & b_q & & \xi^{p-1}Q(\xi) \\
 & b_0 & \ldots & & b_q & \xi^{p-2}Q(\xi) \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & b_0 & \ldots & & Q(\xi)
\end{vmatrix}$$

The entries in the last column are at most $(1+|\xi|^{n-1})\max\big(|P(\xi)|, |Q(\xi)|\big)$, in absolute value, and the other entries have absolute values at most $\max(\|P\|, \|Q\|)$. Hence the determinant has absolute value at most

$$(2n)!\,(1+|\xi|^{n-1})\max\big(|P(\xi)|, |Q(\xi)|\big)\big(\max(\|P\|, \|Q\|)\big)^{2n-1}.$$

This proves (41).

6. Proof of Theorem 2b. Since the proof of Theorem 2b is much simpler than that of Theorem 2a we give it first. The basic idea of the proof is due to Wirsing [2].

It will be convenient to replace n by $n+1$ in the statement of the Theorem. Hence we suppose that $n \geqslant 1$ and that ξ is not algebraic of degree at most n. We shall assume that for every large X the inequalities

$$|x_0\xi^n + \ldots + x_n| \leqslant cX^{-2n+1}, \quad |x_m| \leqslant X \quad (m = 0, 1, \ldots, n-1) \tag{42}$$

have a solution in integers $x_0, \ldots, x_n$, not all zero. Eventually we shall reach a contradiction if c is sufficiently small.

In what follows, the constants implied by the notation $\ll$ will be independent of c. However, we shall assume that $0 < c < 1$, and therefore the estimate $c \ll 1$ will be valid.

The hypothesis can be reinterpreted as saying that for every large X there is a polynomial P_X, not identically zero, with integer coefficients and degree at most n, namely

$$P_X(t) = x_0 t^n + \ldots + x_n,$$

which satisfies

$$|P_X(\xi)| \leqslant cX^{-2n+1}, \quad \|P_X\| \ll X. \tag{43}$$

We shall prove that this is impossible if c is small. The proof is by induction on n, so we may assume that either $n=1$ or that $n>1$ and that the case $n-1$ has already been established.

We distinguish two cases.

Case 1. There exist finitely many polynomials $R_1, \ldots, R_h$ with integer coefficients such that all the polynomials P_X with X large are divisible by one at least of $R_1, \ldots, R_h$.

We then write

$$P_X = P_X^{(1)} P_X^{(2)},$$

where $P_X^{(1)}$ is a product of polynomials R_i and where $P_X^{(2)}$ has no factor R_i. There are only a finite number of possibilities for $P_X^{(1)}$, and since ξ is not algebraic of degree at most n, we deduce that

$$|P_X^{(1)}(\xi)| \geqslant f(c),$$

where $f(c)$ may depend on c. Hence

$$(44) \qquad |P_X^{(2)}(\xi)| \ll f(c)^{-1} X^{-2n+1}, \quad \|P_X^{(2)}\| \ll X.$$

For every large X there is a polynomial $P_X^{(2)}$, not identically zero and of degree at most $n-1$, which satisfies (44). This is obviously impossible if $n=1$, and it is contrary to the inductive hypothesis if $n>1$, since

$$f(c)^{-1} X^{-2n+1} = o(X^{-2(n-1)+1}).$$

This settles Case 1.

Case 2. Given any non-constant polynomials $R_1, \ldots, R_h$ with integer coefficients, there are arbitrarily large values of X for which P_X is not divisible by any of $R_1, \ldots, R_h$.

In this case there is a sequence $X^{(1)} < X^{(2)} < \ldots$ with limit infinity such that the polynomials $P_{X^{(1)}}, P_{X^{(2)}}, \ldots$ have no common factor. Let

$$P_{X^{(i)}} = P_1^{(i)} P_2^{(i)} \ldots$$

be the factorization of $P_{X^{(i)}}$ into irreducible polynomials with integer coefficients. By (39) and (43), one at least of the factors, say $P_1^{(i)}$, will have the property that

$$(45) \qquad |P_1^{(i)}(\xi)| \ll c^{1/n} \|P_1^{(i)}\|^{-2n+1}.$$

Hence there are infinitely many distinct irreducible polynomials $P_1^{(i)}$ satisfying (45). In particular, $\|P_1^{(i)}\| \to \infty$.

The inequality $\|P_X\| \ll X$ in (43) means that $\|P_X\| \leqslant c_3 X$ for some constant c_3 independent of c. Let i be large, and take

$$X = \tfrac{1}{2} c_3^{-1} K(n)^{-1} \|P_1^{(i)}\|.$$

There is a polynomial P_X with

$$(46) \qquad |P_X(\xi)| \ll c \|P_1^{(i)}\|^{-2n+1}, \quad \|P_X\| < K(n)^{-1} \|P_1^{(i)}\|.$$

It follows from the definition of $K(n)$ that $P_1^{(i)}$ is not a factor of P_X, and since $P_1^{(i)}$ is irreducible this means that $P_X, P_1^{(i)}$ have no common factor. We have

$$\max(\|P_1^{(i)}\|, \|P_X\|) = \|P_1^{(i)}\|,$$
$$\max(|P_1^{(i)}(\xi)|, |P_X(\xi)|) \ll c^{1/n} \|P_1^{(i)}\|^{-2n+1}.$$

This contradicts Lemma 8 if c is sufficiently small. The proof of Theorem 2b is now complete.

7. Beginning of the proof of Theorem 2a. As with Theorem 2b in the last section, it will be convenient to replace n by $n+1$ in the statement of the Theorem. Thus we suppose that $n \geqslant 2$ and that ξ is not algebraic of degree at most $\frac{1}{2}n$. We shall assume that for every large X the inequalities

$$(47) \qquad |x_0| \leqslant X, \quad |x_0 \xi^m - x_m| \leqslant c X^{-1/k} \quad (m = 1, \ldots, n),$$

where

$$(48) \qquad k = [\tfrac{1}{2} n],$$

have a solution in integers $x_0, \ldots, x_n$, not all zero. Eventually we shall reach a contradiction if c is sufficiently small.

Again the constants implied by the notation $\ll$ will be independent of c, and again we shall suppose that $0 < c < 1$.

We define a sequence of minimal points as in § 3. But now there may be more than one integer point in the region

$$1 \leqslant x_0 \leqslant X, \quad |x_0 \xi^m - x_m| \leqslant 1 \quad (m = 1, \ldots, n)$$

for which

$$\max(|x_0 \xi - x_1|, \ldots, |x_0 \xi^n - x_n|)$$

attains its least value. If so, we choose the one which comes first in some fixed ordering of the integer points, and call this the *minimal point corresponding to* X. There is a sequence of integers $X_1 < X_2 < \ldots$ such that $x_{i0} = X_i$, and $\boldsymbol{x}_i$ is the minimal point corresponding to every X in $X_i \leqslant X < X_{i+1}$. Putting

$$(49) \qquad L_i = \max(|x_{i0} \xi - x_{i1}|, \ldots, |x_{i0} \xi^n - x_{in}|),$$

we have $L_1 \geqslant L_2 \geqslant \ldots$ By our assumptions concerning (47), we have

$$(50) \qquad L_i \leqslant c X_{i+1}^{-1/k}$$

for large i. This is the analogue of (30).

Put

$$(51) \qquad k_i = \left\{ k \frac{\log X_i}{\log X_{i+1}} \right\},$$

where $\{a\}$ denotes the least integer $\geqslant a$. We have

(52) $$1 \leqslant k_i \leqslant k.$$

LEMMA 9. *Suppose that c is small in terms of n and ξ. Let i be large, and write $\boldsymbol{x}_i = \boldsymbol{y}$. Then the matrix*

(53) $$\begin{pmatrix} y_0 & y_1 & y_2 & \cdots & y_{n-k_i} \\ y_1 & y_2 & y_3 & \cdots & y_{n-k_i+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ y_{k_i} & y_{k_i+1} & y_{k_i+2} & \cdots & y_n \end{pmatrix}$$

has rank at most k_i.

Proof. We have to prove that every subdeterminant of order k_i+1 is zero. A typical such determinant is

$$\begin{vmatrix} y_{j_0} & y_{j_1} & \cdots & y_{j_{k_i}} \\ y_{j_0+1} & y_{j_1+1} & \cdots & y_{j_k+1} \\ \cdot & \cdot & \cdots & \cdot \\ y_{j_0+k_i} & y_{j_1+k_i} & \cdots & y_{j_{k_i}+k_i} \end{vmatrix}$$
$$= \pm \begin{vmatrix} y_{j_0} & \xi^{j_1-j_0}y_{j_0}-y_{j_1} & \cdots & \xi^{j_{k_i}-j_0}y_{j_0}-y_{j_{k_i}} \\ y_{j_0+1} & \xi^{j_1-j_0}y_{j_0}-y_{j_1+1} & \cdots & \xi^{j_{k_i}-j_0}y_{j_0+1}-y_{j_{k_i}+1} \\ \cdot & \cdot & \cdots & \cdot \\ y_{j_0+k_i} & \xi^{j_1-j_0}y_{j_0+k_i}-y_{j_1+k_i} & \cdots & \xi^{j_{k_i}-j_0}y_{j_0+k_i}-y_{j_{k_i}+k_i} \end{vmatrix}.$$

The entries in the first column have absolute values $\ll X_i$, while all the other entries are $\ll cX_{i+1}^{-1/k}$. It follows that the determinant has absolute value

$$\ll c^{k_i}X_iX_{i+1}^{-k_i/k} \ll c^{k_i},$$

by (51). If c is sufficiently small the determinant has absolute value less than 1 and is therefore zero.

8. A recurrence relation. In what follows, h_i denotes the least integer in $1 \leqslant h_i \leqslant k_i$ with the property that the matrix

(54) $$\begin{pmatrix} y_0 & y_1 & \cdots & y_{n-h_i} \\ y_1 & y_2 & \cdots & y_{n-h_i+1} \\ \cdot & \cdot & \cdots & \cdot \\ y_{h_i} & y_{h_i+1} & \cdots & y_n \end{pmatrix}$$

with $\boldsymbol{y} = \boldsymbol{x}_i$ has rank $\leqslant h_i$. We have

(55) $$1 \leqslant h_i \leqslant k_i \leqslant k.$$

It should perhaps be remarked that the matrix (54) is not in general a submatrix of (53).

The matrix

(56) $$\begin{pmatrix} y_0 & y_1 & y_2 & \cdots & y_{n-h_i+1} \\ y_1 & y_2 & y_3 & \cdots & y_{n-h_i+2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ y_{h_i-1} & y_{h_i} & y_{h_i+1} & \cdots & y_n \end{pmatrix}$$

has rank h_i. This follows from the minimal property of h_i if $h_i > 1$ and is obvious if $h_i = 1$. Define Z_i to be the maximum absolute value of any $h_i \times h_i$ subdeterminant of (56). The argument used in the proof of Lemma 9 shows that

(57) $$Z_i \ll X_iX_{i+1}^{-(h_i-1)/k} \ll X_i^{1-(h_i-1)/k}.$$

Since the matrix (54) has rank $\leqslant h_i$, its rows are linearly dependent. Hence there exist integers $a_0^{(i)}, a_1^{(i)}, \ldots, a_{h_i}^{(i)}$, not all zero and with no common factor, such that

(58) $$a_0^{(i)}y_j + a_1^{(i)}y_{j+1} + \ldots + a_{h_i}^{(i)}y_{j+h_i} = 0 \quad \text{for} \quad 0 \leqslant j \leqslant n-h_i.$$

Write

$$a^{(i)} = \max(|a_0^{(i)}|, \ldots, |a_{h_i}^{(i)}|).$$

LEMMA 10. $a^{(i)} \ll X_i^{1/n}$.

Proof. Denote the column vectors of the matrix (56) by $\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_m$, where $m = n-h_i+1$. By (58) we have

$$a_0^{(i)}\boldsymbol{w}_j + a_1^{(i)}\boldsymbol{w}_{j+1} + \ldots + a_{h_i}^{(i)}\boldsymbol{w}_{j+h_i} = 0 \quad (0 \leqslant j \leqslant m-h_i).$$

Since $h_i \leqslant k = [\frac{1}{2}n]$, we have $m-h_i \geqslant 1$.

By Theorem 3, which will be proved in §§ 11 and 12, we have

$$a^{(i)} \ll Z_i^{1/(m-h_i+1)} = Z_i^{1/(n-2h_i+2)},$$

and this is

$$\ll X_i^{(1-(h_i-1)/k)/(n-2h_i+2)},$$

by (57).

Since $1 \leqslant h_i \leqslant k \leqslant \frac{1}{2}n$, the function

$$\frac{1-(h_i-1)/k}{n-2h_i+2},$$

as a function of h_i, is decreasing. Hence it attains its maximum when $h_i = 1$, and this maximum is $1/n$. Thus

$$a^{(i)} \ll X_i^{1/n},$$

and this proves Lemma 10.

9. The construction of certain polynomials. The equation
$$a_0^{(i)}y_0+a_1^{(i)}y_1+\ldots+a_{h_i}^{(i)}y_{h_i}=0$$
is a particular case of (58). Since $\boldsymbol{y}$ stands for $\boldsymbol{x}_i$, we obtain
$$a_0^{(i)}x_{i0}+a_1^{(i)}x_{i1}+\ldots+a_{h_i}^{(i)}x_{ih_i}=0.$$
Substituting
$$x_{im}=\xi^m x_{i0}+O(X_{i+1}^{-1/k}),\qquad 1\leqslant m\leqslant h_i,$$
as in earlier arguments, we obtain
$$|x_{i0}|(a_{h_i}^{(i)}\xi^{h_i}+\ldots+a_0^{(i)})\ll a^{(i)}X_{i+1}^{-1/k}.$$
From Lemma 10 and the fact that $|x_{i0}|\gg X_i$ we get
$$|a_{h_i}^{(i)}\xi^{h_i}+\ldots+a_0^{(i)}|\ll X_i^{-1+1/n}X_{i+1}^{-1/k}. \tag{59}$$
We have $X_i\geqslant X_{i+1}^{(k_i-1)/k}$ from the definition of k_i in (51). Thus (59) gives
$$|a_{h_i}^{(i)}\xi^{h_i}+\ldots+a_0^{(i)}|\ll\min(X_i^{-1+(1/n)-(1/k)},X_{i+1}^{-(nk_i-k_i+1)/nk}).$$
Putting
$$Y_i=X_i^{1/n} \tag{60}$$
and defining a polynomial $P_i(t)$ by
$$P_i(t)=a_{h_i}^{(i)}t^{h_i}+\ldots+a_0^{(i)}, \tag{61}$$
we can restate the result in the following form.

LEMMA 11. *For every i there is a polynomial P_i, not identically zero, with integer coefficients and of degree at most k_i, which satisfies*
$$|P_i(\xi)|\ll\min(Y_i^{-n+1-n/k},Y_{i+1}^{-(nk_i-k_i+1)/k}) \tag{62}$$
and
$$\|P_i\|\ll Y_i. \tag{63}$$

10. Completion of the proof of Theorem 2a. We shall prove that the situation described in Lemma 11 leads to a contradiction.

Let k_0 be the least integer in $1\leqslant k_0\leqslant k$ such that
$$k_i=k_0 \tag{64}$$
infinitely often. For every large Y there is a subscript i with $Y_i\leqslant Y<Y_{i+1}$, and on putting $Q=P_i$ we obtain the following as an immediate consequence of Lemma 11.

LEMMA 12. *For every large Y there is an integer $k(Y)$ in*
$$k_0\leqslant k(Y)\leqslant k \tag{65}$$
and a polynomial Q, not identically zero, with integer coefficients and of degree at most $k(Y)$, which satisfies
$$|Q(\xi)|\ll Y^{-(nk(Y)-k(Y)+1)/k},\qquad \|Q\|\ll Y. \tag{66}$$

LEMMA 13. *There are infinitely many irreducible polynomials P with integer coefficients, not identically zero and of degree at most k_0 which satisfy*
$$|P(\xi)|\ll\|P\|^{-n+2-n/k}. \tag{67}$$

Proof. If this were false, all the irreducible polynomials P of degree at most k_0 would have
$$|P(\xi)|\gg\|P\|^{-n+2-n/k}.$$
This holds for the possible finite number of exceptions because $k_0\leqslant k\leqslant\frac12 n$ and ξ is not algebraic of degree at most $\frac12 n$.

The polynomials P_i of Lemma 11 with $k_i=k_0$ are products of irreducible polynomials:
$$P_i=P_{i1}P_{i2}\ldots$$
We have
$$|P_i(\xi)|=|P_{i1}(\xi)P_{i2}(\xi)\ldots|\gg(\|P_{i1}\|\,\|P_{i2}\|\ldots)^{-n+2-1/k}\gg\|P_i\|^{-n+2-n/k}.$$
On the other hand, (62) and (63) imply that
$$|P_i(\xi)|\ll\|P_i\|^{-n+1-n/k},$$
and we have a contradiction (with a large margin).

We can now complete the proof of Theorem 2a. Let P be an irreducible polynomial with large height $\|P\|$ which satisfies (67), and put
$$Y=\tfrac12 c_4^{-1}K(n)^{-1}\|P\|,$$
where c_4 is sufficiently large in relation to the implied constant in the inequality $\|Q\|\ll Y$ in (66). By Lemma 12 there is a polynomial Q of degree
$$k(Y)=k',$$
say, with
$$|Q(\xi)|\ll\|P\|^{-(nk'-k'+1)/k},\qquad \|Q\|<K(n)^{-1}\|P\|.$$
By the definition of $K(n)$, the polynomial P is not a factor of Q, and therefore P, Q have no common factor. We have
$$\max(\|P\|,\|Q\|)=\|P\|,$$
and since
$$n-2+n/k\geqslant(n-1)(k'/k)-1+n/k\geqslant(n-1)(k'/k)-(1/k),$$

we have

$$\max(|P(\xi)|,\ |Q(\xi)|) \ll \|P\|^{-(nk'-k'+1)/k}.$$

Now $k' \leqslant k \leqslant \frac{1}{2}n$, and hence

$$2k'-1-(nk'-k'+1)/k \leqslant 2k'-1-2(nk'-k'+1)/n = (2k'-n-2)/n < 0.$$

We therefore obtain

$$(\max(\|P\|,\ \|Q\|))^{2k'-1}\max(|P(\xi)|,\ |Q(\xi)|) \ll \|P\|^{2k'-1-(nk'-k'+1)/k} = o(1).$$

Since both P and Q have degree at most k', this contradicts Lemma 8. The proof of Theorem 2a is now complete, apart from the need to prove Theorem 3.

11. Identities involving determinants. We recall the hypotheses of Theorem 3. $\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ are integer vectors in h dimensional space which span the whole space, and $1 \leqslant h \leqslant m$. They satisfy the recurrence relation

$$a_0\boldsymbol{y}_i + a_1\boldsymbol{y}_{i+1} + \ldots + a_h\boldsymbol{y}_{i+h} = 0 \quad (0 \leqslant i \leqslant m-h),$$

where $a_0, \ldots, a_h$ are integers with no common factor throughout.

If only one of the a_i is different from zero, this one must have the value ± 1, and the conclusion is trivially true. We may therefore assume that at least two are non-zero. We define l to be the largest suffix for which $a_l \neq 0$, and have $1 \leqslant l \leqslant h$. By the recurrence relation, $\boldsymbol{y}_l$ lies in the space generated by $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{l-1}$. Further, by repetition, we see that $\boldsymbol{y}_l, \boldsymbol{y}_{l+1}, \ldots, \boldsymbol{y}_{m-h+l}$ lie in the space generated by $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{l-1}$.

It follows that if $l = h$ the whole space is generated by $\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{h-1}$. On the other hand, if $l < h$ the vectors

$$\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{l-1}, \boldsymbol{y}_{m-h+l+1}, \ldots, \boldsymbol{y}_m$$

generate the whole space.

We define the determinant

$$D(\nu_1, \ldots, \nu_l)$$

of order h as follows. If $l = h$ it is the determinant formed with the column vectors $\boldsymbol{y}_{\nu_1}, \ldots, \boldsymbol{y}_{\nu_l}$; if $l < h$ it is formed with the column vectors

$$\boldsymbol{y}_{\nu_1}, \ldots, \boldsymbol{y}_{\nu_l}, \boldsymbol{y}_{m-h+l+1}, \ldots, \boldsymbol{y}_m.$$

In either case we have

$$D(0, 1, \ldots, l-1) \neq 0. \tag{68}$$

We shall investigate $D(\nu_1, \ldots, \nu_l)$ only in the range

$$0 \leqslant \nu_j \leqslant m-h+l,$$

and in fact it vanishes outside this range.

LEMMA 14. *Write $\nu = \max(\nu_1, \ldots, \nu_l)$. We have*

$$a_l^{\nu-l+1}D(\nu_1, \ldots, \nu_l) = P(\nu_1, \ldots, \nu_l | a_0, \ldots, a_l)D(0, 1, \ldots, l-1), \tag{69}$$

where $P(\nu_1, \ldots, \nu_l | a_0, \ldots, a_l)$, for fixed $\nu_1, \ldots, \nu_l$, is a polynomial in $a_0, \ldots, a_l$ with integer coefficients and every monomial

$$a_0^{j_0} \ldots a_l^{j_l}$$

in P with non-zero coefficient satisfies

$$j_0 + j_1 + \ldots + j_l = \nu - (l-1) \tag{70}$$

and

$$lj_0 + (l-1)j_1 + \ldots + j_{l-1} = (\nu_1 + \ldots + \nu_l) - (0 + \ldots + (l-1)). \tag{71}$$

This polynomial is identically zero if two of the ν_i are equal, and it changes into itself or minus itself if $\nu_1, \ldots, \nu_l$ undergo an even or an odd permutation.

Proof. We proceed by induction on $\nu = \max(\nu_1, \ldots, \nu_l)$. There is nothing to prove if $\nu \leqslant l-1$. Suppose the result has been proved for $\nu \leqslant t$ where t is an integer $\geqslant l-1$. We have to prove the result when $\nu = t+1$; and we may clearly assume that

$$\nu_1 < \nu_2 < \ldots < \nu_l = t+1.$$

By the recurrence relation,

$$a_l\boldsymbol{y}_{\nu_l} = -a_{l-1}\boldsymbol{y}_{\nu_l-1} - \ldots - a_0\boldsymbol{y}_{\nu_l-l}.$$

Hence

$$a_l^{\nu-l+1}D(\nu_1, \ldots, \nu_l) = -a_{l-1}a_l^{t+1-l}D(\nu_1, \ldots, \nu_{l-1}, \nu_l-1) - \ldots - a_0a_l^{t+1-l}D(\nu_1, \ldots, \nu_{l-1}, \nu_l-l).$$

Using the inductive hypothesis, we can substitute for each D on the right from (69), and this gives us the form of $P(\nu_1, \ldots, \nu_l | a_0, \ldots, a_l)$. It is a sum of terms

$$-a_{l-i}a_l^{t-\max(\nu_1, \ldots, \nu_{l-1}, \nu_l-i)}P(\nu_1, \ldots, \nu_{l-1}, \nu_l-i | \ldots), \tag{72}$$

where i goes from 1 to l. By induction, each term is homogeneous of degree

$$1+t-\max(\nu_1, \ldots, \nu_{l-1}, \nu_l-i)+\max(\nu_1, \ldots, \nu_{l-1}, \nu_l-i)-(l-1) = \nu-(l-1).$$

Hence (70) holds. Further, a typical term in (72) has

$$lj_0+(l-1)j_1+\ldots+j_{l-1}$$
$$=\big(l-(l-i)\big)\cdot 1+(\nu_1+\ldots+\nu_{l-1}+\nu_l-i)-\big(0+1+\ldots+(l-1)\big)$$
$$=\nu_1+\ldots+\nu_l-\big(0+1+\ldots+(l-1)\big).$$

Hence (71) holds. The final clauses of the lemma merely reflect the properties of determinants.

Lemma 15. *Suppose that $0\leqslant i<l$. The polynomial*(²)

$$P_i=P(\overbrace{0,1,\ldots,i-1},\overbrace{m-h+i+1,\ldots,m-h+l})$$

is of the type

(73) $$P_i=\pm a_i^{m+1-h}+Q_i,$$

where Q_i is homogeneous of degree $m+1-h$ and each term in Q_i contains one at least of the variables $a_{i+1},\ldots,a_l$.

Proof. It is easily seen that

$$P(\overbrace{0,1,\ldots,i-1},\overbrace{i+1,\ldots,l})=-a_i;$$

this follows from the inductive relation applied to the lth column of the determinant. Now consider

(74) $$P(\overbrace{0,1,\ldots,i-1},\overbrace{i+s,\ldots,l+s-1}).$$

The same consideration shows that this is equal to

$$-a_iP(\overbrace{0,1,\ldots,i-1},\overbrace{i+s-1,\ldots,l+s-2})$$

plus terms which are multiples of one of $a_0,\ldots,a_{i-1}$. It follows by induction on s that (74) has a term $\pm a_i^s$ and that the other terms are multiples of one of $a_0,\ldots,a_{i-1}$. By (70) and (71) these terms are also multiples of one of $a_{i+1},\ldots,a_l$. The lemma follows on taking $s=m+1-h$.

12. Proof of Theorem 3. The proof of Theorem 3 is now completed as follows. By the two preceding lemmas we have

(75) $$a_l^{m+1-h}D(\overbrace{0,1,\ldots,i-1},\overbrace{m-h+i+1,\ldots,m-h+l}) = P_i(a_0,\ldots,a_l)D(0,\ldots,l-1)$$

for $0\leqslant i<l$, where P_i is of the form stated in (73).

(²) The bracket indicates a block of consecutive integers.

It follows that the integers a_l^{m-h+1} divides each of

$$P_i(a_0,\ldots,a_l)D(0,\ldots,l-1)$$

for $i=0,\ldots,l-1$. This implies that it divides $D(0,\ldots,l-1)$. For suppose that p is a prime such that

$$p\,|\,a_l,\quad \ldots,\quad p\,|\,a_{i+1},\quad \text{but}\quad p\nmid a_i.$$

Then p divides $Q_i(a_0,\ldots,a_l)$ but not a_i^{m+1-h}, and hence cannot divide $P_i(a_0,\ldots,a_l)$.

We have proved that

(76) $$a_l^{m+1-h}|D(0,\ldots,l-1),$$

and this implies that

(77) $$|a_l|^{m+1-h}\leqslant Z,$$

since Z is an upper bound for the absolute value of the determinant D and this determinant is not 0, by (68).

(75) in conjunction with (77) gives

$$|P_i(a_0,\ldots,a_l)|\leqslant|D(\overbrace{0,\ldots,i-1},\overbrace{m-h+i+1,\ldots,m-h+l})|\leqslant Z,$$

whence

(78) $$|a_i|^{m+1-h}\leqslant Z+|Q_i(a_0,\ldots,a_l)|.$$

If all of $|a_0|,\ldots,|a_l|$ are at most $Z^{1/(m+1-h)}$, there is nothing to prove. Otherwise, let e_1 be the largest subscript in $0\leqslant e_1<l$ with

$$|a_{e_1}|^{m+1-h}>Z,$$

let e_2 be the largest subscript with

$$|a_{e_2}|>|a_{e_1}|,$$

and so on. We have $e_1>e_2>\ldots>e_t$ and

$$|a_{e_t}|>\ldots>|a_{e_1}|>Z^{1/(m+1-h)},$$
$$|a_{e_t}|=\max(|a_0|,\ldots,|a_l|).$$

Now apply (78) with $i=e_j$ where $j>1$. We obtain

$$|a_{e_j}|^{m+1-h}\ll Z+|Q_{e_j}(a_0,\ldots,a_l)|\ll Z+|a_{e_{j-1}}|\,|a_{e_t}|^{m-h},$$

whence

(79) $$|a_{e_j}|^{m+1-h}\ll|a_{e_{j-1}}|\,|a_{e_t}|^{m-h}.$$

Finally apply (78) with $i = e_1 < l$. We obtain

$$|a_{e_1}|^{m+1-h} \ll Z^{1/(m+1-h)}|a_{e_t}|^{m-h}. \tag{80}$$

The inequality (79) with $j = t$ gives $|a_{e_t}| \ll |a_{e_{t-1}}|$. The same inequality with $j = t-1$, together with $|a_{e_t}| \ll |a_{e_{t-1}}|$, gives $|a_{e_{t-1}}| \ll |a_{e_{t-2}}|$. Continuing in this way we obtain

$$|a_{e_t}| \ll |a_{e_{t-1}}| \ll \ldots \ll |a_{e_1}|.$$

Using this in conjunction with (80) we find that $|a_{e_i}| \ll Z^{1/(m+1-h)}$.

This completes the proof of Theorem 3.

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