ACTA ARITHMETICA
XV (1969)

On the differences of primes in arithmetical progressions

by

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1. Introduction. It is a well-known consequence of the prime number theorem that $p_{n+1}-p_n$, where p_n denotes the nth prime number, is on average about $\log p_n$; in particular we have

$$E = \liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} \leqslant 1.$$

In 1940, after earlier results by Hardy and Littlewood and Rankin [6], II, depending on a conjecture concerning the zeros of Dirichlet's L-functions, Erdös [4] succeeded in obtaining the first unconditional improvement on this inequality; he showed in fact that $E<1$. The stronger result $E<42/43$ was subsequently obtained by Rankin [6], III, IV, and this was sharpened by Ricci [7] in 1954 to $E<15/16$.

In a recent paper Bombieri and Davenport [2] succeeded in proving a substantially better inequality; by means of an improved version of the famous 'large sieve' of Linnik and Renyi, they showed that

$$E \leqslant \tfrac{1}{8}(2+\sqrt{3}) = .46650\ldots$$

Further, they considered the limit

$$E_r = \liminf_{n\to\infty} \frac{p_{n+r}-p_n}{\log p_n} \tag{1.1}$$

and proved that $E_r \leqslant r-\frac{1}{2}$ for all r.

The first object of the present paper is to establish analogues of the results of Bombieri and Davenport for the primes in an arithmetical progression. Let k, l be given relatively prime positive integers with k even, and for each positive integer r let E_r be defined in analogy to (1.1) by

$$E_r = \liminf_{n\to\infty} \frac{p_{n+r}-p_n}{\varphi(k)\log p_n}, \tag{1.2}$$

where p_n now denotes the nth prime in the arithmetical progression with first term l and common difference k. Our principal results may be stated thus:

THEOREM 1. *For each positive integer r we have*

(1.3) $$E_r \leqslant \tfrac{1}{2}(r-\tfrac{1}{2})\left(1+\left(\frac{4r-1}{4r}\right)^{1/2}\right),$$

and also

(1.4) $$E_r \leqslant \frac{5r}{8}-\frac{3}{8}+\frac{1}{16}\left\{9(2r-1)^2-\frac{3(4r^2-17r+11)}{r+1}\right\}^{1/2}.$$

We can restate (1.3) for large r as

(1.5) $$E_r \leqslant r-\tfrac{9}{16}+O(1/r),$$

and (1.4) as

(1.6) $$E_r \leqslant r-\tfrac{5}{8}+O(1/r),$$

so that (1.4) is seen to be an improvement on (1.3) for large r. In fact for $r \geqslant 3$ (1.4) is better; for $r = 1$ both results give the constant $.46650\ldots$ found by Bombieri and Davenport, and for $r = 2$ (1.3) gives

$$E_2 \leqslant 1.451\ldots$$

and (1.4)

$$E_2 \leqslant 1.461\ldots,$$

so that (1.3) improves on (1.4) only when $r = 2$. We state it generally as it is the natural extension of the result of Bombieri and Davenport to the case $r > 1$.

The proof of (1.4) uses three different combinatorial ideas, two of which have been isolated as Lemma 15 below. The first section of the proof uses the methods of Hardy, Littlewood and Rankin, as also employed by Bombieri and Davenport, but the exponential sum is formed with the indices being every rth prime in the arithmetical progression with first term l and common difference k. The argument proceeds in a similar manner to [2], but the evaluation of the 'singular series' presents greater difficulty, and a large part of the present paper is devoted to this end. I am grateful to Dr. Bombieri for the observation that the use made in [2] of the Hardy–Littlewood 'circle method' to obtain a lower bound for a certain sum over primes is now a special case of the 'large sieve' developed in [1] and [3], and the result required may be quoted from the results of [3].

Finally we discuss the problem of Erdös mentioned by Bombieri and Davenport in [2]. By suitable modifications of our arguments we can prove, for example, that

$$\liminf_{n\to\infty} \max_{1\leqslant s\leqslant r} \frac{p_{n+s}-p_{n+s-1}}{\varphi(k)\log p_n} \leqslant \frac{2r-1}{4}+\frac{\sqrt{3}}{8}\{r(3r-2)\}^{1/2} < \frac{7r}{8},$$

a result better than the trivial upper bound afforded by E_r.

The author wishes to express his thanks to Professor Davenport for suggesting this problem.

2. Notation. We number the primes congruent to $l(\mathrm{mod}\,k)$ as $p_1, p_2, p_3, \ldots$ in ascending order, and divide them into r classes, of which the ith will consist of

$$p_i,\ p_{i+r},\ p_{i+2r},\ \ldots$$

Any sum or product with (i) as a superscript will be understood to be restricted to primes in the ith class. p will denote a general prime. We put

$$H = \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right),$$

and note that the product converges, and for each positive integer n we write

(2.1) $$H(n) = H\prod_{\substack{p\mid n\\ p>2}}\left(\frac{p-1}{p-2}\right).$$

We denote by $\mu(n)$ and $\varphi(n)$ the usual Möbius and Euler functions, and by $d(n)$ the number of divisors of n. $\mu^2(n)$ will denote $\{\mu(n)\}^2$ and other functions will be written similarly. (m, n) and $\{m, n\}$ denote the highest common factor and lowest common multiple of m, n respectively. $m|n$ means 'm divides n'. We shall write as usual $[x]$ for the integer part of the real number x.

Let N denote a large positive integer. Writing $L = \log N$ for brevity, we suppose that $L > k$. We define $X = N^{1/2}L^{-100}$. Putting $e(x) = \exp 2\pi i x$, we introduce the exponential sums

$$S^{(i)}(a) = \sum_{p\leqslant N}^{(i)} \log p\, e(pa)$$

and

$$U(a) = \sum_{n=0}^{h} e(nka),$$

where h is a positive integer not exceeding L^2 to be chosen later. We note that $|U(a)|^2$ is a Fejér kernel:

(2.2) $$|U(a)|^2 = \sum_{n=-h}^{h}(h-|n|)e(nka) = \frac{\sin^2\pi hka}{\sin^2\pi ka}.$$

For any integer n we shall write

$$e_q(n) = e\left(\frac{n}{q}\right),$$

and use the Ramanujan sum

$$c_q(n) = \sum_{\substack{a=1\\(a,q)=1}}^{q} e_q(an).$$

We define

$$Z_r(nk) = \sum_{i=1}^{r} \sum_{p,p'}^{(i)} \log p \log p',$$

for each positive integer n or for $n = 0$, where the summation is over all pairs p, p' both in the ith class with $p'-p = nk$ and $p' \leqslant N$; and

$$Y_r(nk) = \sum_{i=1}^{r} \sum_{p,p'}^{(i)} 1$$

over the same range of summation with the additional condition $p > NL^{-4}$. We shall write $Y(nk), Z(nk)$ for $Y_1(nk), Z_1(nk)$ respectively.

We remark finally that the constants implied by the O-signs and by Vinogradov symbol $\ll$ will always be absolute.

3. Lemmas. We now prove fifteen lemmas in preparation for the proof of Theorem 1. Of these, the first ten constitute an application of the large sieve method, the next three apply the upper bound sieve method of Selberg as modified by Bombieri and Davenport in [2], and the last two are combinatorial in character.

LEMMA 1. *Let a, q denote relatively prime positive integers. Put $q' = (q, k)$, $q'' = q/q'$. Let $\omega_{a,q} = 0$ if $(q', q'') > 1$ and $\omega_{a,q} = e_q(atq'+al)$ if $(q', q'') = 1$, where t denotes a solution of the congruence $tq' \equiv -l \pmod{q''}$. Then*

$$\sum_{m}^{*} e_q(am) = \mu(q'')\,\omega_{a,q}, \tag{3.1}$$

where the asterisk denotes summation over all m satisfying the conditions

$$1 \leqslant m \leqslant q, \quad (m, q) = 1 \quad \text{and} \quad m \equiv l \pmod{q'}. \tag{*}$$

Proof. Writing $m = nq'+l$, we see that the sum on the left of (3.1) is given by $\sum_{n} e_q(anq'+al)$, where the summation is over all n with $1 \leqslant n \leqslant q''$, $(q, nq'+l) = 1$. The second condition can be written as $(q'', nq'+l) = 1$, since $(q', l) = 1$. By the properties of the Möbius function, the sum is given by

$$\sum_{n} \sum_{d|(q'',nq'+l)} \mu(d) e_q(anq'+al) = \sum_{d|q''} \sum_{n} \mu(d) e_q(anq'+al),$$

where the second summation is over all n with $1 \leqslant n \leqslant q''$ and $nq'+l \equiv 0 \pmod{d}$. Since $(q', l) = 1$, the congruence is soluble for n if and only if $(q', d) = 1$, and in the latter case, if n' denotes any solution, all solutions can be written as $n'+n''d$, where n'' runs through a complete system of residues modulo q''/d. Thus we can write the double sum in the form

$$e_q(an'q'+al) \sum_{\substack{d|q''\\(q',d)=1}} \mu(d) \sum_{n''} e_q(an''dq').$$

Now the second sum is given by $\sum_{n''} e_{q/d}(n'')$, since $(a, q''/d) = 1$, and this is 0 if $q'' \neq d$, and 1 if $q'' = d$. This implies the required result.

DEFINITIONS. Let a, q be any pair of relatively prime positive integers, and let m be any integer satisfying the conditions (*) of Lemma 1. We define

$$\varrho_{m,q}^{(i)}(n) = \log n - \frac{1}{r\varphi(q''k)},$$

if n is a prime congruent to $m \pmod{q}$, and in the ith class; and

$$\varrho_{m,q}^{(i)}(n) = -\frac{1}{r\varphi(q''k)},$$

if n is not a prime of the ith class, or does not satisfy the congruence conditions. We write

$$\varrho_{m,q}(n) = \sum_{i=1}^{r} \varrho_{m,q}^{(i)}(n),$$

and put

$$S_{a,q}^{(i)} = \sum_{m}^{*} e_q(am) \sum_{n=1}^{N} \varrho_{m,q}^{(i)}(n),$$

$$S_{a,q} = \sum_{i=1}^{r} S_{a,q}^{(i)},$$

where the asterisk denotes summation over m satisfying the conditions (*) of Lemma 1. We define

$$A^{(i)} = \sum_{q \leqslant X} \frac{\mu(q'')}{\varphi(q''k)} \sum_{\substack{a=1\\(a,q)=1}}^{q} \overline{\omega}_{a,q} S_{a,q}^{(i)} \left| U\left(\frac{a}{q}\right)\right|^2,$$

$$A = \sum_{i=1}^{r} A^{(i)},$$

the bar denoting complex conjugation. Finally we define the 'singular series' $\mathfrak{S}$ by

$$\mathfrak{S} = \sum_{q\leqslant X} \frac{\mu^2(q'')\,\omega(q)}{\varphi^2(q''k)} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left|U\left(\frac{a}{q}\right)\right|^2, \tag{3.2}$$

where

$$\omega(q) = |\omega_{a,q}|,$$

a number clearly independent of the choice of reduced residue a.

LEMMA 2. *We have*

$$\big(N+O(NL^{-100})\big)\sum_{i=1}^{r}\int_0^1 |S^{(i)}(\alpha)|^2\,|U(\alpha)|^2\,d\alpha \tag{3.3}$$

$$\geqslant \frac{N^2\mathfrak{S}}{r} + \frac{2N\Re A}{r} + O(NL^8).$$

Proof. Theorem 1 of Bombieri and Davenport [3], with the numbers x_r taken to be the rationals a/q with $q\leqslant X$ in $(0,1]$, implies that

$$\sum_{q\leqslant X}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\sum_{n=N_1+1}^{N_1+H} a_n e\left(\frac{na}{q}\right)\right|^2 \leqslant (H^{1/2}+X)^2\int_0^1\left|\sum_{n=N_1+1}^{N_1+H} a_n e(n\alpha)\right|^2 d\alpha.$$

Now $S^{(i)}(\alpha)\,U(\alpha)$ is a sum of the form considered here, with $H\leqslant N+2hk$. On recalling that $x = N^{1/2}L^{-100}$, we obtain

$$\sum_{q\leqslant X}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|U\left(\frac{a}{q}\right)S^{(i)}\left(\frac{a}{q}\right)\right|^2$$

$$\leqslant \big((N+2hk)^{1/2}+N^{1/2}L^{-100}\big)^2\int_0^1 |S^{(i)}(\alpha)\,U(\alpha)|^2\,d\alpha.$$

Dividing the primes of class i less than or equal to N which do not divide q into residue classes to the modulus q, and noting that any non-empty class corresponds to a residue class congruent to $l(\bmod q')$, we obtain

$$S^{(i)}\left(\frac{a}{q}\right) = \sum_m{}^{*}\sum_p{}^{(i)}\log p\, e_q(ap) + O\Big(\sum_{p|q}{}^{(i)}\log p\Big),$$

where the second sum is over all primes of class i not exceeding N which are congruent to $m(\bmod q)$. By the definitions above, we see that the double sum may be written as

$$\sum_m{}^{*}\sum_{n=1}^{N} e_q(an)\left(\varrho_{m,q}^{(i)}(n)+\frac{1}{r\varphi(q''k)}\right),$$

and by (3.1) of Lemma 1 we obtain

$$S^{(i)}\left(\frac{a}{q}\right) = S_{a,q}^{(i)} + N\Omega_{a,q},$$

where for brevity we have written

$$\Omega_{a,q} = \frac{\mu(q'')\,\omega_{a,q}}{r\varphi(q''k)}.$$

This gives

$$\left|S^{(i)}\left(\frac{a}{q}\right)\right|^2 = |S_{a,q}^{(i)}|^2 + 2N\Re S_{a,q}^{(i)}\,\overline{\Omega}_{a,q} + N^2|\Omega_{a,q}|^2 + O\left(\frac{NL^2}{r}\right),$$

on noting that

$$\left|S^{(i)}\left(\frac{a}{q}\right)\right| \leqslant \frac{NL}{r}$$

and

$$\log q < L.$$

Thus by the definitions above we have

$$\left|U\left(\frac{a}{q}\right)S^{(i)}\left(\frac{a}{q}\right)\right|^2 \geqslant \frac{N^2\mu^2(q'')\,\omega(q)}{r^2\varphi^2(q''k)} + \frac{2N}{r}\Re A^{(i)} + O\left(\frac{NL^8}{r}\right),$$

since $h<L^2$, $k<L$. Summing from $i=1$ to r, we have (3.3).

DEFINITIONS. For each pair of relatively prime positive integers m, q satisfying $(*)$ of Lemma 1 we define

$$E_{m,q} = \sum_p \log p - \frac{N}{\varphi(q''k)},$$

where the sum is over all primes p with $p\leqslant N$, $p\equiv m(\bmod q)$, and $p\equiv l(\bmod k)$. We write

$$E_q = \max_m{}^{*}|E_{m,q}|,$$

where the maximum is over all m satisfying $(*)$ of Lemma 1.

LEMMA 3. *We have*

$$|A| < \sum_{\substack{q\leqslant X\\(q'',k)=1}} \frac{h^2 q''\mu^2(q'')\,d(q'')\,E_q}{\varphi(q''k)}. \tag{3.4}$$

Proof. Noting that

$$S_{a,q} = \sum_m{}^* e_q(am) E_{m,q}$$

we can write

$$A = \sum_{q \leqslant X} \frac{\mu(q'')}{\varphi(q''k)} \sum_m{}^* \sum_{\substack{a=1\\(a,q)=1}}^{q} \overline{\omega}_{a,q} e_q(am) \left| U\left(\frac{a}{q}\right)\right|^2 E_{m,q}.$$

Going back to the definitions of $\omega_{a,q}$ in Lemma 1 and to (2.2), we see that the coefficient of $E_{m,q}$ is 0 if $(q', q'') > 1$, and if $(q', q'') = 1$ it is

$$\sum_{n=-h}^{h} (h-|n|) c_q(m - tq' - l + nk),$$

where t is as in Lemma 1. Now for given n and for each m satisfying $1 \leqslant m \leqslant q$, $(m, q) = 1$, $m \equiv l (\operatorname{mod} q')$, we have

$$m - tq' - l + nk \equiv uq' (\operatorname{mod} q),$$

where $1 \leqslant u \leqslant q''$, and distinct u correspond to distinct m. Thus the sum over m is in modulus at most

$$\sum_{u=1}^{q''} |c_{q''}(u)| h^2.$$

The well-known formula (c.f. [5], § 16.6)

$$c_q(u) = \sum_{d|(q,u)} d\mu\left(\frac{q}{d}\right)$$

now gives

$$\sum_{u=1}^{q''} |c_{q''}(u)| h^2 < h^2 \sum_{u=1}^{q''} \sum_{d|(q,u)} d = h^2 \sum_{d|q''} d \sum_{\substack{u=1\\u\equiv 0 (\operatorname{mod} d)}}^{q''} 1 < h^2 q'' d(q''),$$

which gives (3.4) on noting that $|E_{m,q}| \leqslant E_q$.

LEMMA 4. *We have the bound*

$$\sum_{\substack{q \leqslant X\\(q'',k)=1}} E_q \ll NL^{-24}. \tag{3.5}$$

Proof. If $m \equiv l (\operatorname{mod} q')$, the congruences $x \equiv m (\operatorname{mod} q)$ and $x \equiv l (\operatorname{mod} k)$ posess a solution $x = a$, and the equations $(m, q) = 1$, $(k, l) = 1$ imply that $(a, q''k) = 1$, also that a complete set of solutions is given by the congruence class $a (\operatorname{mod} q''k)$. Thus for given m satisfying $(*)$ of Lemma 1, if $(q'', k) = 1$, then $E_{m,q}$ can be written in the form

$$\sum_{\substack{p \equiv a (\operatorname{mod} q''k)\\p \leqslant N}} \log p - \frac{N}{\varphi(q''k)}.$$

We now divide all $q \leqslant X$ satisfying $(q'', k) = 1$ into sets so that two q occur in the same set if and only if the corresponding q'' are equal. It is easily seen that there are at most $d(k)$ elements in any one set. Hence, noting that $q''k \leqslant Xk \leqslant N^{1/2} L^{-99}$, and applying Theorem 4 of Bombieri [1] with $A = 25$, $B = 98$ and $Z = N$, we see that the number on the left of (3.5) is at most $d(k) NL^{-25}$. The required result follows since $d(k) \leqslant k \leqslant L$.

LEMMA 5. *We have*

$$|A| \ll NL^{-9}. \tag{3.6}$$

Proof. Since $h \leqslant L^2$ we have from (3.4)

$$|A| \ll \frac{L^4}{\varphi(k)} \sum_{\substack{q \leqslant X\\(q'',k)=1}} \frac{q^2 \mu(q'') d(q'') E_q}{\varphi(q'')}.$$

By Cauchy's inequality, the sum over q is at most

$$\left\{\sum_{q \leqslant X} \frac{q''^2 d^2(q'')}{\varphi^3(q'')}\right\}^{1/2} \left\{\sum_{\substack{q \leqslant X\\(q'',k)=1}} \varphi(q'')(E_q)^2\right\}^{1/2}.$$

Now observing the trivial estimate

$$E_q \ll \frac{NL}{\varphi(q''k)},$$

we deduce from (3.5) that the second sum over q is at most $N^2 L^{-23}/\varphi(k)$. For the first sum we use the estimate $d(k)(\log\log X)^3 (\log X)^4$, which is easily proved. Combining these estimates we have (3.6).

The next four lemmas are concerned with the evaluation of the 'singular series' $\mathfrak{S}$ and with the arithmetical function $H(nk)$. We prove first

LEMMA 6. *Suppose R is a positive integer. Then*

$$\sum_{\substack{r=1\\(r,k)=1}}^{R} \frac{\mu^2(r)}{\varphi(r)} = \frac{\varphi(k)}{k}\left(\log R + O(\log k)\right). \tag{3.7}$$

Proof. We define a multiplicative function b_n by $b_1 = 1$ and

$$b_n = \begin{cases} \dfrac{1}{n(n-1)} & \text{if } n \text{ is a prime and } n \nmid k,\\ 0 & \text{if } n \text{ is a prime dividing } k,\\ -b_p & \text{if } n \text{ is the square of a prime } p,\\ 0 & \text{if } n \text{ is divisible by the cube of a prime number.} \end{cases}$$

Then we have, as is easily verified,

$$\sum_{d|n} db_d = \frac{n\mu^2(n)}{\varphi(n)} \quad \text{if} \quad (n, k) = 1,$$

and also

$$\sum_{r>u} b_r = O\left(\frac{\log\log u}{u^{1/2}}\right) = O(u^{-1/3}).$$

Hence writing $B = \sum_{r=1}^{\infty} b_r$, we have

$$\sum_{\substack{r=1\\(r,k)=1}}^{R} \frac{\mu^2(r)}{\varphi(r)} = \sum_{\substack{r_1,r_2\leqslant R\\(r_1r_2,k)=1}} \frac{b_{r_1}}{r_2} = \sum_{\substack{(u,k)=1\\u\leqslant R}} \frac{1}{u}\left(B+O\left(\frac{u^{1/3}}{R^{1/3}}\right)\right)$$

$$= B\sum_{\substack{u\leqslant R\\(u,k)=1}} \frac{1}{u} + O(1) = B\sum_{d|k}\mu(d)\sum_{\substack{u=1\\u\equiv 0 (\bmod d)}}^{R} \frac{1}{u} + O(1)$$

$$= B\sum_{d|k}\frac{\mu(d)}{d}\left(\log\frac{R}{d}+O(1)\right)+O(1)$$

$$= B\frac{\varphi(k)}{k}\log R + O(\log\log k) - \sum_{d|k}\frac{\mu(d)\log d}{d}$$

$$= B\frac{\varphi(k)}{k}\left(\log R + O(\log k)\right).$$

By considering the Euler product for B, we see that $B = 1$, and the result follows.

Lemma 7. *We have*

$$\sum_{q\leqslant X}\frac{\mu^2(q'')\varphi(q)\omega(q)}{\varphi^2(q''k)} = \frac{\log X + O(\log k)}{\varphi(k)}. \tag{3.8}$$

Proof. Since $\omega(q)$ is zero unless $(q'', k) = 1$, we may assume that this latter condition holds. The sum on the left of (3.8) is now

$$\sum_{q'|k}\sum_{q''}\frac{\mu^2(q'')\varphi(q'q'')}{\varphi^2(q''k)},$$

where the second summation is over all $q'' \leqslant X/q'$ with $(q'', k) = 1$. From (3.7) the inner sum is given by

$$\frac{\varphi(q')\varphi(k)}{\varphi^2(k)k}\left(\log\frac{X}{q'}+O(\log k)\right),$$

and the lemma follows since

$$\sum_{q'|k}\varphi(q') = k.$$

Lemma 8. *For each positive integer u we have*

$$\sum_{n=1}^{u} H(nk) = \frac{1}{2}\cdot\frac{uk}{\varphi(k)} + O\left(\frac{k\log u(\log\log u)^2}{\varphi(k)}\right). \tag{3.9}$$

Proof. Since k is supposed even, the definition (2.1) of $H(nk)$ give

$$H(nk) = H(k)\prod_{\substack{p|n\\(p,k)=1}}\left(\frac{p-1}{p-2}\right) = H(k)\sum_{d|n}\tau(d),$$

where we have written $\tau(d)$ for the multiplicative function

$$\tau(d) = \begin{cases} \mu^2(d)\prod_{p|d}\dfrac{1}{(p-2)} & \text{if} \quad (d, k) = 1,\\ 0 & \text{if} \quad (d, k) > 1.\end{cases}$$

Now we have

$$\sum_{n=1}^{u}\sum_{d|n}\tau(d) = \sum_{d\leqslant u}\tau(d)\sum_{\substack{n=1\\n\equiv 0 (\bmod d)}}^{u} 1 = \sum_{d\leqslant u}\tau(d)\left(\frac{u}{d}+O(1)\right).$$

We have

$$\sum_{d\leqslant u}\tau(d) \ll \sum_{d\leqslant u}\frac{(\log\log d)^2}{d} \ll \log u(\log\log u)^2,$$

$$\sum_{d\leqslant u}\frac{\tau(d)}{d} = T + O\left(\frac{(\log\log u)^2}{u}\right),$$

where $T = \sum_{d=1}^{\infty}\tau(d)/d$ can be expressed as

$$\prod_{(p,k)=1}\left(1+\frac{1}{p(p-2)}\right) = \prod_{p\nmid k}\frac{(p-1)^2}{p(p-2)} = \frac{k}{2H(k)\varphi(k)},$$

using (2.1) again. The equation (3.9) is now easily verified.

Lemma 9. *We have*

$$\mathfrak{S} = \frac{1}{2}\cdot\frac{hL}{\varphi(k)} + \frac{h^2k}{\varphi^2(k)} + O\left(\frac{hk(\log L)^2}{\varphi^2(k)}\right). \tag{3.10}$$

Proof. From (2.2) we see that

$$\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|U\left(\frac{a}{q}\right)\right|^2=\sum_{n=-h}^{h}(h-|n|)\sum_{\substack{a=1\\(a,q)=1}}^{q}e_q(nka)=h\varphi(q)+2\sum_{n=1}^{h}(h-n)c_q(nk).$$

Also for $1\leqslant n\leqslant h$ we have

$$\sum_{q\leqslant X}\frac{\mu^2(q)\,\omega(q)\,c_q(nk)}{\varphi^2(q''k)}=\frac{1}{\varphi^2(k)}\left\{\sum_{\substack{q=1\\(q',q'')=1}}^{\infty}\frac{\mu^2(q'')\,c_q(nk)}{\varphi^2(q'')}+O\left(\sum_{q>X}\frac{nk}{\varphi^2(q)}\right)\right\}.$$

The error term here is plainly

$$O\left(hk\frac{(\log\log X)^2}{X}\right).$$

Further, the infinite series is given by

$$\sum_{\substack{q''=1\\(q'',k)=1}}^{\infty}\frac{\mu^2(q'')\,c_{q''}(nk)}{\varphi^2(q'')}\sum_{q'|k}c_{q'}(nk).$$

Since $c_{q'}(nk)=\varphi(q')$ if $q'\,|\,nk$, the second factor is

$$\sum_{q'|k}\varphi(q')=k.$$

Since also $c_p(nk)$ is given by $p-1$ if $p\,|\,n$ but $p\nmid k$, and by -1 if $p\nmid nk$, it follows by virtue of the arithmetical properties of these multiplicative functions that the first factor is

$$\prod_{p\nmid k}\left(1+\frac{c_p(nk)}{(p-1)^2}\right)=\prod_{\substack{p\nmid k\\p|n}}\left(1+\frac{1}{p-1}\right)\prod_{p\nmid nk}\left(1-\frac{1}{(p-1)^2}\right)=\frac{2\varphi(k)H(nk)}{k}.$$

The sum of the infinite series is therefore $2\varphi(k)H(nk)$, and substituting in the definition (3.2) of $\mathfrak{S}$, we obtain

$$\mathfrak{S}=\sum_{\substack{q\leqslant X\\(q',q'')=1}}\frac{\mu^2(q'')\varphi(q)\,\omega(q)}{\varphi^2(q''k)}+2\sum_{u=1}^{h-1}\sum_{n=1}^{u}\sum_{q\leqslant X}\frac{\mu^2(q'')c_q(nk)}{\varphi^2(q''k)}$$
$$=\frac{h}{\varphi(k)}\big(\log X+O(\log k)\big)+4\sum_{u=1}^{h-1}\sum_{n=1}^{u}\frac{H(nk)}{k\varphi(k)}$$

by (3.8). The result is now easily verified using (3.9), since

$$\log X=\tfrac{1}{2}L+O(\log L).$$

LEMMA 10. *We have*

$$2\sum_{n=1}^{h}(h-n)Z_r(nk)\geqslant\left(\lambda^2-(r-\tfrac{1}{2})\lambda+O\left(\frac{(\log L)^2}{L}\right)\right)\frac{NL^2}{rk},\tag{3.11}$$

where λ *is defined by*

$$hk=\lambda\varphi(k)L.$$

Proof. Since $\int_0^1 e(m\alpha)\,d\alpha$ is 1 if $m=0$ and 0 otherwise, we have

$$\sum_{i=1}^{r}\int_0^1|S^{(i)}(\alpha)|^2|U(\alpha)|^2d\alpha=hZ_r(0)+2\sum_{n=1}^{h}(h-n)Z_r(nk).$$

By the prime number theorem for arithmetical progressions we have

$$Z_r(0)=\sum_p(\log p)^2+O(\log^2N)=\frac{NL}{\varphi(k)}+O\left(\frac{N}{\varphi(k)}\right),$$

the sum being over primes $p\leqslant N$ with $p\equiv l\,(\mathrm{mod}\,k)$. By (3.10)

$$\mathfrak{S}=\left(\lambda^2+\tfrac{1}{2}\lambda+O\left(\frac{(\log L)^2}{L}\right)\right)\frac{L^2}{k},$$

and (3.3) and (3.6) give

$$\big(1+O(L^{-100})\big)\sum_{i=1}^{r}\int_0^1|S^{(i)}(\alpha)|^2|U(\alpha)|^2d\alpha\geqslant\frac{N\mathfrak{S}}{r}+O\left(\frac{N}{rL^9}\right).$$

On rearrangement we have (3.11).

The reader will note that Lemma 10, which is the analogue of Theorem 1 of [2], already implies $E_r\leqslant r-\frac{1}{2}$, since for $\lambda>r-\frac{1}{2}$ the right-hand member of (3.11) is positive for all large N, and so the sum on the left cannot be $O(L^2)$ as $N\to\infty$ which it would be if $p_{n+r}-p_n<(r-\frac{1}{2})\varphi(k)L$ only finitely often.

We now apply the sieve method of Selberg, as modified in [2], to obtain an upper bound for $Y(nk)$, defined in Section 2, where n denotes any integer with $1\leqslant n\leqslant 2h$. We shall require the following notation:

DEFINITIONS. For each positive integer m we write

$$\psi(m)=\sum_{d|m}\mu(d)\varphi\left(\frac{m}{d}\right).$$

We put $R=N^{1/4}L^{-100}$ and define

$$T=\sum_r{}'\,\mu^2(r)/\psi(r),$$

where the accent denotes summation over all positive integers r with $r \leqslant R$ and $(r, nk) = 1$. Further we write

$$\lambda(r) = \frac{\varphi(r)}{T} \sum_{m} \frac{\mu(m)\mu(mr)}{\varphi(mr)},$$

the summation being over all m for which $m \leqslant R/r$ and $(mr, nk) = 1$. Note that $\lambda(r)$ is defined for all positive integers r, the sum over m being taken as 0 if $r > R$ or if $(r, nk) > 1$.

For each positive integer q we define

$$E'_{a,q} = -\frac{\operatorname{li} N}{\varphi(q)} + \sum_{p} 1,$$

the summation being taken over all primes $p \leqslant N$ with $p \equiv a \pmod{q}$, and $\operatorname{li} N$ denoting the usual logarithmic integral function. We define further

$$E'_q = \max_a |E'_{a,q}|,$$

the maximum being over all integers a with $(a, q) = 1$.

Lemma 11. *We have*

$$Y(nk) \leqslant \operatorname{li} N \sum_{d_1} \sum_{d_2} \frac{\lambda(d_1)\lambda(d_2)}{\varphi(dk)} + B, \tag{3.12}$$

where $d = \{d_1, d_2\}$ *and*

$$B = \Big| \sum_{d_1} \sum_{d_2} \lambda(d_1)\lambda(d_2) \Big| E'_{dk}.$$

Proof. We observe first that for each prime $p > NL^{-4}$ $(> R)$ we have

$$\sum_{d|p} \lambda(d) = \lambda(1) = 1.$$

Then by definition

$$Y(nk) = \sum_{p} \Big(\sum_{d|p} \lambda(d) \Big)^2,$$

the summation being over all primes p with $NL^{-4} < p \leqslant N$, $p \equiv l \pmod{k}$ for which $p - nk$ is a prime p' satisfying $NL^{-4} < p' < p$. This gives

$$Y(nk) \leqslant \sum_{m} \Big(\sum_{d|m} \lambda(d) \Big)^2,$$

where the summation is over all positive integers m with $m \leqslant N + nk$, $m \equiv l \pmod{k}$ and $m - nk$ prime. The sum can be written as

$$\sum_{m} \sum_{d_1|m} \sum_{d_2|m} \lambda(d_1)\lambda(d_2) = \sum_{d_1}{}' \sum_{d_2}{}' \lambda(d_1)\lambda(d_2) \sum_{\substack{m \\ m \equiv 0 \pmod{d}}} 1.$$

Since $(d, nk) = 1$, and $m - nk$ is a prime congruent both to $-nk \pmod{d}$ and to $l \pmod{k}$, the final sum over m is the number of primes in a certain residue class to the modulus dk which do not exceed N; hence at most it is

$$\frac{\operatorname{li} N}{\varphi(dk)} + |E'_{dk}|.$$

This proves (3.12).

Lemma 12. *We have*

$$\sum_{d_1} \sum_{d_2} \frac{\lambda(d_1)\lambda(d_2)}{\varphi(d)} = \frac{1}{T}, \tag{3.13}$$

and

$$T = \frac{\log R}{2H(nk)} + O\big((\log\log nk)^2\big). \tag{3.14}$$

Proof. The equations are given by (53) and (54) of Bombieri and Davenport [2] on taking $\frac{1}{2}nk$ for n, R for z and noting that $\frac{1}{2}nk \leqslant L^3 \leqslant R^5$ and $H(\frac{1}{2}nk) = H(nk)$.

Lemma 13. *We have*

$$Y(nk) < \frac{NH(nk)}{\varphi(k)L^2} \left(8 + O\left(\frac{\log L}{L} \right) \right) \tag{3.15}$$

and for each integer u with $1 \leqslant u \leqslant h$

$$\sum_{n=u}^{h} (h-n) Y(nk) < \frac{2(h-u)^2 kN}{\varphi^2(k)L^2} + O\left(\frac{h^2 kN \log L}{\varphi^2(k)L^3} \right). \tag{3.16}$$

Proof. As in Lemma 15 of [2], noting that $kR^2 \leqslant N^{1/2}L^{-99}$, we obtain $B \ll NL^{-26}$. Hence by (3.12), (3.13) and (3.14), and the inequalities $n \leqslant 2h \leqslant 2L^2$, $k < L$ and

$$H(nk) \ll \frac{nk}{\varphi(nk)} \ll \log\log nk,$$

we have

$$Y(nk) \leqslant \frac{H(nk) \operatorname{li} N}{\varphi(k) \log R} \left(2 + O\left(\frac{(\log\log L)^3}{L} \right) \right),$$

and (3.15) follows since

$$\log R = \tfrac{1}{4}L + O(\log L)$$

and

$$\operatorname{li} N = \frac{N}{L} + O\left(\frac{N}{L^2} \right).$$

Now (3.16) is easily obtained by means of (3.15), (3.9) and the identity

$$\sum_{n=u}^{h}(h-n)H(nk) = \sum_{\nu=u}^{h-1}\sum_{n=u}^{\nu}H(nk).$$

We now have all the machinery required for the proof of (1.3). For (1.4) we need further arguments of a combinatorial nature.

LEMMA 14. *Suppose that for some $c>0$ we have*

$$Y(nk) < \frac{cNH(nk)}{\varphi(k)L^2}. \tag{3.17}$$

Suppose that we have a set Π of pairs of primes, the number of pairs in the set not exceeding $mN/\{\varphi(k)L\}$, with both members of any pair (p, p') satisfying $p\equiv p'\equiv l(\operatorname{mod} k)$ and $NL^{-4}<p<p'\leqslant N$. Let $Y'(nk)$ be the number of pairs with $p'-p=nk$. Then if $h_ik=\lambda_i\varphi(k)L$ for $i=0,1,2$, where $h_0\geqslant h_2\geqslant h_1$ and

$$\lambda_2-\lambda_1\geqslant\frac{2m}{c}, \tag{3.18}$$

we have

$$\sum_{n=h_1}^{h_2}(h_0-n)Y'(nk)\leqslant\left\{(\lambda_0-\lambda_1)m-\frac{m^2}{c}+O\left(\frac{(\log L)^2}{L}\right)\right\}\frac{N}{k}. \tag{3.19}$$

Proof. We have

$$\sum_{n=h_1}^{h_2}(h_0-n)Y'(nk) = \sum_{u=h_1}^{h_2-1}\sum_{n=h_1}^{u}Y'(nk)+(h_0-h_2)\sum_{n=h_1}^{h_2}Y'(nk).$$

We divide the range for u in the first sum into $u<h_3$ and $u\geqslant h_3$, where the integer h_3 is given by

$$h_3=\left[\left(\lambda_1+\frac{2m}{c}\right)\frac{\varphi(k)L}{k}\right],$$

corresponding to

$$\lambda_3=\lambda_1+\frac{2m}{c}+O\left(\frac{k}{\varphi(k)L}\right),$$

so that by (3.18), λ_3 is less than λ_2. For values of u less than h_3 we observe that $Y'(nk)$ does not exceed $Y(nk)$ and use (3.17), so that an upper bound for this part of the sum can be deduced from one for the corresponding sum with $Y'(nk)$ replaced by $H(nk)$. By (3.9) we have

$$\sum_{u=h_1}^{h_3-1}\sum_{n=h_1}^{u}H(nk) < \sum_{u=h_1}^{h_3-1}\frac{k}{2\varphi(k)}\{u-h_1+1+O(\log u(\log\log u)^2)\}$$
$$<\frac{k}{2\varphi(k)}\cdot\frac{1}{2}(h_3-h_1)^2\left(1+O\left(\frac{\log h_1(\log\log h_1)^2}{h_1}\right)\right).$$

For $u\geqslant h_3$ and for the second sum we use

$$\sum_{n=h_1}^{u}Y'(nk)\leqslant \text{cardinal of } \Pi\leqslant\frac{mN}{\varphi(k)L}.$$

We may replace each h_i by $\lambda_i\varphi(k)L/k$ with an error at most $O\big((\log L)^2/L\big)$ times the sum in which it occurs. Hence combining these results, we have (3.19).

LEMMA 15. *Suppose that we have (3.17) and also*

$$2\varphi(k)L<ch_1k, \tag{3.20}$$

and for $NL^{-4}<p_t<p_{t+r}\leqslant N$ we have

$$p_{t+r}-p_t\geqslant h_1k. \tag{3.21}$$

Then for any $h\leqslant L^2$ we have

$$2\sum_{n=1}^{h_1}(h-n)Y(nk) \tag{3.22}$$
$$\leqslant\frac{(r-1)(2h-h_1)N}{\varphi(k)L}-\frac{(r-1)^2N}{ck(r+1)}+O\left(\frac{hrN(\log L)^2}{kL^2}\right).$$

Proof. From (3.21) we have

$$\sum_{n=1}^{h_1}(h_1-n)Y(nk)=\sum_t\sum_u\max{}^*\left(0,\,h_1-\frac{p_u-p_t}{k}\right),$$

where the asterisk denotes that the maximum is replaced by zero unless both the primes p_t, p_u satisfy $NL^{-4}<p_t<p_u\leqslant N$. Since $p_u-p_t\geqslant h_1k$ if $u-t\geqslant r$ when this condition holds, we may write the sum as

$$\sum_{s=1}^{r-1}\sum_t\max{}^*\left(0,\,h_1-\frac{p_{t+s}-p_t}{k}\right).$$

We now group the terms in pairs, taking s with $r-s$, so that the whole sum becomes

$$\tfrac{1}{2}\sum_{s=1}^{r-1}\sum_t M_{s,t},$$

where

$$M_{s,t}=\max{}^*\left(0,\,h_1-\frac{p_{t+s}-p_t}{k}\right)+\max{}^*\left(0,\,h_1-\frac{p_{t+r}-p_{t+s}}{k}\right).$$

Clearly $M_{s,t}$ is an integer between 0 and h_1 inclusive. If $M_{s,t}=h_1-m$ where $m>0$, then one of three possibilities holds: that the second term is zero and $p_{t+r}-p_t=mk$, or that the first term is zero and $p_{t+r}-p_{t+s}=mk$, or that neither is zero and $p_{t+r}-p_t=(h_1+m)k$. The total number of pairs (s,t) for which $M_{s,t}=h_1-m$ we call $\nu(m)$, so that for $0<m<h_1$ we have $\nu(m)\leqslant(r-1)Y(h_1k+mk)+2Y(mk)$, and also

$$\sum_{m=1}^{h_1}\nu(m)\leqslant\sum_{s=1}^{r-1}\sum_{t=1}^{T}1\leqslant(r-1)T,$$

where T is the number of primes congruent to $l(\operatorname{mod}k)$ which do not exceed N. We put

$$u=\left[\frac{2L}{c}\cdot\frac{\varphi(k)}{k}\left(\frac{r-1}{r+1}\right)\right],$$

noting that by (3.20) we have $u<h_1$. For $n\leqslant u$ we use (3.17) and (3.9) as in Lemma 14, so that

$$\sum_{m=1}^{n}\nu(m)<\frac{(r+1)cknN}{2\varphi^2(k)L^2}\left(1+O\left(\frac{(\log L)^2}{L}\right)\right).$$

For $u<n<h_1$ we use

$$\sum_{m=1}^{n}\nu(m)\leqslant(r-1)T\leqslant\frac{(r-1)N}{\varphi(k)L}\left(1+O\left(\frac{1}{L}\right)\right)$$

by the prime number theorem in arithmetical progressions. We now have

$$\sum_{s=1}^{r-1}\sum_{t=1}^{T}M_{s,t}=\sum_{m=1}^{h_1}(h_1-m)\nu(m)=\sum_{n=1}^{h_1-1}\sum_{m=1}^{n}\nu(m)$$

$$\leqslant\sum_{n=1}^{u}\frac{(r+1)cknN}{2\varphi^2(k)L^2}\left(1+O\left(\frac{(\log L)^2}{L}\right)\right)+\sum_{n=u+1}^{h_1-1}\frac{(r-1)N}{\varphi(k)L}\left(1+O\left(\frac{1}{L}\right)\right).$$

We now sum over n and note that the errors involved in replacing $u+1$ by u and h_1-1 by h_1 can be absorbed into the error term, so that our upper bound is

$$\leqslant\frac{u^2(r+1)ckN}{4\varphi^2(k)L^2}\left(1+O\left(\frac{(\log L)^2}{L}\right)\right)+\frac{(r-1)(h_1-u)}{\varphi(k)L}\left(\frac{r-1}{r+1}\right)\left(1+O\left(\frac{\log L}{L}\right)\right).$$

Substituting

$$u=\frac{2L}{c}\cdot\frac{\varphi(k)}{k}\left(\frac{r-1}{r+1}\right)\left(1+O\left(\frac{\log L}{L}\right)\right),$$

we have (3.22) in the case $h_1=h$. The general case follows since

$$\sum_{n=1}^{h_1}(h-h_1)Y(nk)\leqslant(h-h_1)(r-1)T.$$

4. Proof of Theorem 1. First we prove (1.3). We suppose that

$$p_{n+r}-p_n\geqslant h_1k$$

whenever $NL^{-4}<p_n<p_{n+r}\leqslant N$. We first make the trivial observation

$$Z_r(nk)=L^2Y_r(nk)\left(1+O\left(\frac{\log L}{L}\right)\right)+O\left(\frac{N}{L^2}\right). \tag{4.1}$$

We now have from (3.11)

$$\left(\lambda^2-(r-\tfrac{1}{2})\lambda+O\left(\frac{(\log L)^2}{L}\right)\right)\frac{NL^2}{rk}\leqslant2\sum_{n=1}^{h}(h-n)Z_r(nk)$$

$$\leqslant2\sum_{n=1}^{h}(h-n)\left\{Y_r(nk)\left(1+O\left(\frac{\log L}{L}\right)\right)+O\left(\frac{N}{L^2}\right)\right\}L^2.$$

We note that $Y_r(nk)=0$ whenever $1\leqslant n\leqslant h_1-1$ and use (3.16) to estimate the sum from h_1 to h, so that the right-hand side above does not exceed

$$\frac{4(h-h_1)^2kN}{\varphi^2(k)}\left(1+O\left(\frac{\log L}{L}\right)\right)+O\left(\frac{h^2N}{L^2}\right).$$

Hence writing $hk=\lambda\varphi(k)L$, $h_1k=\lambda_1\varphi(k)L$ we have

$$4r(\lambda-\lambda_1)^2\geqslant\lambda^2-(r-\tfrac{1}{2})\lambda+O\left(\frac{(\log L)^2}{L}\right). \tag{4.2}$$

Choosing the integer h to make

$$\lambda=\frac{1}{2}(r-\tfrac{1}{2})\left(1+\left(\frac{4r}{4r-1}\right)^{1/2}\right)+O\left(\frac{k}{\varphi(k)L}\right),$$

a choice which ensures that the right-hand member of (3.11) is positive, so that we know that $\lambda_1\leqslant\lambda$, we can now deduce

$$\lambda_1\leqslant\frac{1}{2}(r-\tfrac{1}{2})\left(1+\left(\frac{4r-1}{4r}\right)^{1/2}\right)+O\left(\frac{(\log L)^2}{L}\right)$$

from the quadratic inequality (4.2). Since

$$E_r\leqslant\limsup_{N\to\infty}\lambda_1,$$

we have (1.3).

We obtain (1.4) similarly, using the case $r=1$ of (3.11):

$$\left(\lambda^2-\frac{1}{2}\lambda+O\left(\frac{(\log L)^2}{L}\right)\right)\frac{NL^2}{k}\leqslant2\sum_{n=1}^{h}Z(nk)(h-n)$$

and by virtue of (4.1) we may replace the right-hand side by

$$2L^2\sum_{n=1}^{h}(h-n)\,Y(nk)\left(1+O\left(\frac{\log L}{L}\right)\right)+O\left(\frac{h^2N}{L^2}\right).$$

We suppose that (3.21) is satisfied for some h_1, which we may choose to be smaller than the integer h which makes

$$\lambda=\frac{2r-1}{4}+\frac{1}{4}\left((2r-1)^2-\frac{4r^2-17r+11}{3(r+1)}\right)^{1/2}+O\left(\frac{k}{\varphi(k)L}\right).$$

Suppose first that h_1 satisfies (3.20). We split the sum over n as above into the ranges 1 to h_1-1 and h_1 to h. Noting that by (3.15) we may take the c of (3.22) to be $8+O(\log L/L)$, we apply (3.22) to the first and (3.16) to the second range for n to obtain the following upper bound for the sum over n:

$$\frac{(r-1)(2h-h_1)NL}{\varphi(k)}-\frac{(r-1)^2NL^2}{8(r+1)k}+\frac{4(h-h_1)^2kN}{\varphi^2(k)}+$$
$$+O\left(\frac{hN(\log L)^2}{k}\right)+O\left(\frac{h^2kN\log L}{\varphi^2(k)L}\right)+O\left(\frac{h^2N}{L^2}\right).$$

This becomes

$$\frac{NL^2}{k}\left(4(\lambda-\lambda_1)^2+(r-1)(2\lambda-\lambda_1)-\frac{(r-1)^2}{8(r+1)}+O\left(\frac{(\log L)^2}{L}\right)\right)$$

when we substitute for h_1 and h. As above we substitute the value of λ to get a quadratic inequality. Noting that $\lambda-\lambda_1$ is positive when we take the square root, we have

$$(4.3)\quad \lambda_1\leqslant\frac{5r-3}{8}+\frac{1}{16}\left(9(2r-1)^2-\frac{3(4r^2-17r+11)}{r+1}\right)^{1/2}+O\left(\frac{(\log L)^2}{L}\right).$$

If on the other hand (3.20) is satisfied for no such h_1, we have the stronger inequality

$$\lambda_1\leqslant\frac{1}{4}+O\left(\frac{\log L}{L}\right).$$

We conclude that if h_1 is chosen so large that (4.3) does not hold, then (3.21) must be false for this h_1. Hence our choosing $h_1\leqslant h$ was no restriction, and

$$E_r\leqslant\limsup_{N\to\infty}\lambda_1,$$

which gives (1.4).

5. A problem of Erdös. Let

$$F_r=\liminf_{n\to\infty}\max_{1\leqslant s\leqslant r}\frac{p_{n+s}-p_{n+s-1}}{\varphi(k)\log p_n}.$$

It is clear that

$$(5.1)\qquad F_r\leqslant E_r-(r-1)E_1,$$

and we consider the problem of obtaining numerical upper bounds for F_r. In the case $r=2$ this represents a problem of Erdös. As is indicated by (5.3) below, stronger results can be obtained conditional on the hypothesis $E_1>0$. There is a wealth of combinatorial arguments available. Theorem 2 below gives two results connecting F_r and E_1; relations can be obtained between F_r and any other F_i or E_i if desired. Of the results below, (5.2) is the better in the case when E_1 is small, but (5.3), being linear in E_1, is stronger for certain larger values of E_1. In particular (5.2) implies $F_2\leqslant 1.3624\ldots$

THEOREM 2. *We have*

$$(5.2)\qquad F_r\leqslant\frac{2r-1}{4}+\frac{\sqrt{3}}{8}\{r(3r-2)-16(r-1)E_1\}^{1/2},$$

and if $E_1>\frac{2}{9}$ *we have*

$$(5.3)\quad F_r\leqslant\frac{5r-3}{8}+\frac{1}{8}\left\{3r(3r-2)-\frac{4(r-1)(2r-1)}{r}\right\}^{1/2}-(r-1)E_1.$$

Proof. We prove first (5.2). Suppose that out of every r consecutive differences $p_{n+1}-p_n,\ldots,p_{n+r}-p_{n+r-1}$, at least one exceeds h_2k if $p_n>NL^{-4}$. Consider the set Π of pairs $(p_t,p_{t'})$ of primes with $NL^{-4}<p_t<p_{t'}\leqslant N$ and $p_{t'}-p_t\leqslant h_2k$. Then for $(p_t,p_{t'})\in\Pi$, we have $t'-t\leqslant r-1$, and for $s=1,2,\ldots,r-1$, the number of pairs (p_n,p_{n+s}) which belong to Π is at most a proportion $1-s/r+o(1)$ of the number of possible values of n. Hence by the prime number theorem in arithmetical progressions, Π has at most

$$\sum_{s=1}^{r-1}\left(1-\frac{s}{r}+o(1)\right)\frac{N}{\varphi(k)L}=\left(\frac{r-1}{2}+o(1)\right)\frac{N}{\varphi(k)L}$$

members. In applying Lemma 14 we can therefore take

$$m=\tfrac{1}{2}(r-1)+o(1).$$

If h_1k is the minimum difference $p_{t'}-p_t$ of the members of such a pair, then $h_1k=\lambda_1\varphi(k)L$, where λ_1 may be taken to be $E_1+o(1)$. After (1.3) we have $E_1<\frac{1}{2}$, so that (3.18) is certainly satisfied if

$$\lambda_2\geqslant(r+3)/8,$$

noting that by (3.15) we may take $c = 8+o(1)$ in (3.17). Choosing the integer h_0 so that $h_0 = \lambda_0\varphi(k)L$, where

$$\lambda_0 = \tfrac{1}{2}(r-\tfrac{1}{2})+\tfrac{1}{6}\sqrt{3}\{r(3r-2)-16(r-1)\lambda_1\}^{1/2}+O(\log L/L),$$

and noting that for $r \geqslant 2$ we have $\lambda_0 \geqslant \lambda_2$, we have satisfied all the hypotheses of Lemma 14, and hence (3.19) gives

$$\sum_{n=1}^{h_2}(h_0-n)Y(nk) = \sum_{n=h_1}^{h_2}(h_0-n)Y(nk)$$
$$\leqslant \left(\frac{(\lambda_0-\lambda_1)(r-1)}{2}-\frac{(r-1)^2}{32}+O\left(\frac{(\log L)^2}{L}\right)\right)\frac{N}{k}.$$

We use (3.16) with $u = h_2+1$, $h = h_0$, so that

$$\sum_{n=h_2+1}^{h_0}(h_0-n)Y(nk) < \big(2(\lambda_0-\lambda_2)^2+o(1)\big)\frac{N}{k}.$$

We now apply (3.11) with $r = 1$ and $h = h_0$:

$$\big(\lambda_0^2-\tfrac{1}{2}\lambda_0+o(1)\big)\frac{NL^2}{k} \leqslant 2\sum_{n=1}^{h_0}(h_0-n)Z(nk)$$
$$\leqslant 2\sum_{n=1}^{h_0}(h_0-n)L^2Y(nk)\big(1+o(1)\big)+O(h_0^2NL^{-2}),$$

where we have used (4.1) to transform to the sum over $Y(nk)$. Substituting for the sum over n, we see that

$$\lambda_0^2-\tfrac{1}{2}\lambda_0+o(1) \leqslant 4(\lambda_0-\lambda_2)^2+(r-1)(\lambda_0-\lambda_1)-(r-1)^2/16+o(1).$$

If we substitute the values of λ_0 and λ_1, we see that any choice of h_2 which makes

$$(r+3)/8 < \lambda_2 \leqslant \lambda_0$$

must make $\lambda_2 \leqslant F_r(N)$, where

$$F_r(N) = \tfrac{1}{2}(r-\tfrac{1}{2})+\tfrac{1}{8}\sqrt{3}\{r(3r-2)-16(r-1)\lambda_1\}^{1/2}+o(1).$$

Since $\lambda_2 \leqslant (r+3)/8$ certainly makes $\lambda_2 \leqslant F_r(N)$ if $r \geqslant 2$, we have

$$F_r \leqslant \limsup_{N\to\infty} F_r(N),$$

which gives (5.2).

For (5.3) we suppose first that $p_{n+1}-p_n$ is at least u_1k for $p_n > NL^{-4}$; clearly we may assume $u_1k = \big(E_1+o(1)\big)\varphi(k)L$, and that out of every r consecutive differences $p_{n+1}-p_n, \ldots, p_{n+r}-p_{n+r-1}$ with $p_n > NL^{-4}$, at least one exceeds $u_2k = \mu\varphi(k)L$. Suppose now that $1 \leqslant s \leqslant r-1$.

The proportion of differences $p_{n+s}-p_n$ which are less than $u_2k+(s-1)u_1k$ out of the pairs (p_n, p_{n+s}) with $NL^{-4} < p_n < p_{n+s} \leqslant N$ is at most $1-s/r+o(1)$, and for each of these $p_{n+s}-p_n$ exceeds su_1k. We can therefore divide those pairs with $p_{n+s}-p_n \leqslant hk$, where $hk = \lambda\varphi(k)L$ (we shall discuss the choice of λ below) into two classes, of which the first, $\Pi_1(s)$, has at most

$$\left(1-\frac{s}{r}+o(1)\right)\frac{N}{\varphi(k)L}$$

members, for each of which

$$p_{n+s}-p_n \geqslant su_1k,$$

and the second, $\Pi_2(s)$, has at most

$$\left(\frac{s}{r}+o(1)\right)\frac{N}{\varphi(k)L}$$

members, for each of which

$$p_{n+s}-p_n \geqslant (s-1)u_1k+u_2k.$$

If now

(5.4) $$\lambda-\mu-(r-1)E_1 \geqslant \frac{2}{c}\left(1-\frac{1}{r}\right),$$

we can satisfy (3.18) for each class $\Pi_1(s)$ and $\Pi_2(s)$, and for $1 \leqslant s \leqslant r-1$ we have (in the notation used in the proof of Lemma 15)

$$\sum_n \max{}^*\left(0, h-\frac{p_{n+s}-p_n}{k}\right)$$
$$\leqslant \frac{N}{k}\left\{(\lambda-sE_1)\left(1-\frac{s}{r}\right)-\frac{1}{c}\left(1-\frac{s}{r}\right)^2+\big(\lambda-(s-1)E_1-\mu\big)\frac{s}{r}-\frac{1}{c}\cdot\frac{s^2}{r^2}+o(1)\right\}$$
$$\leqslant \frac{N}{k}\left\{\lambda-\frac{s\nu}{r}-\frac{r^2-2rs-2s^2}{cr^2}+o(1)\right\},$$

where we have written

$$\nu = \mu+(r-1)E_1.$$

We note that any other difference $p_{t'}-p_t$ with $NL^{-4} < p_t < p_{t'} \leqslant N$ and $t'-t \geqslant r$ must exceed $u_3k = \big(E_r+o(1)\big)\varphi(k)L$. Clearly we have

$$E_r \geqslant \mu+(r-1)E_1+o(1).$$

By (3.16) of Lemma 13 we have

$$\sum_{t}\sum_{t'}\max{}^*\Big(0,\,h-\frac{p_{t'}-p_t}{k}\Big)\leqslant\sum_{n=u_3}^{h}(h-n)Y(nk)$$
$$<\frac{2(h-u_3)^2kN}{\varphi^2(k)L^2}\big(1+o(1)\big)$$
$$\leqslant\frac{2N}{k}\{(\lambda-E_r)^2+o(1)\},$$

the result being trivially true also if $u_3>h$, that is, if $E_r>\lambda$. We now apply (3.11) with $r=1$, so that

$$\big(\lambda^2-\tfrac{1}{2}\lambda+o(1)\big)\frac{NL^2}{k}\leqslant 2\sum_{n=1}^{h}(h-n)Z(nk),$$

and by (4.1), the right-hand side is at most

$$2\sum_{n=1}^{h}(h-n)L^2Y(nk)\big(1+o(1)\big)+O(h^2NL^{-2}).$$

Now since

$$\sum_{n=1}^{h}(h-n)Y(nk)=\sum_{t}\sum_{t'}\max{}^*\Big(0,\,h-\frac{p_{t'}-p_t}{k}\Big),$$

the right-hand side is at most

$$\frac{NL^2}{k}\Big\{2\sum_{s=1}^{r-1}\Big(\lambda-\frac{s\nu}{r}-\frac{r^2-2rs+2s^2}{cr^2}\Big)+4(\lambda-E_r)^2+o(1)\Big\}$$
$$=\frac{NL^2}{k}\Big\{(r-1)(2\lambda-\nu)-\frac{2(r-1)(2r-1)}{3cr^2}+4(\lambda-E_r)^2+o(1)\Big\}.$$

After (3.15) we may take c to be $8+o(1)$. We now have the inequality

$$(5.5)\qquad \lambda^2-(r-\tfrac{1}{2})\lambda\leqslant 4(\lambda-E_r)^2+(r-1)(\lambda-\nu)-\frac{(r-1)(2r-1)}{12r}+o(1).$$

Since $\nu=(r-1)E_1+\mu$, we may regard it as a parameter at our disposal, since μ is subject only to

$$\mu\varphi(k)L\leqslant\inf_{NL^{-4}<p_n}\max(p_{n+1}-p_n,\,\ldots,\,p_{n+r}-p_{n+r-1}).$$

We take λ to be

$$(5.6)\qquad r-\frac{3}{4}+\Big\{\frac{4}{3}\Big((r-\tfrac{3}{4})^2-(r-1)\nu-\frac{(r-1)(2r-1)}{12r}\Big)\Big\}^{1/2}+o(1),$$

and deduce that

$$E_r\leqslant r-\frac{3}{4}+\Big\{\frac{3}{4}\Big((r-\tfrac{3}{4})^2-(r-1)\nu-\frac{(r-1)(2r-1)}{12r}\Big)\Big\}^{1/2}+o(1).$$

Since $\nu\leqslant E_r$ we have

$$\nu\leqslant r-\frac{3}{4}+\Big\{\frac{3}{4}\Big((r-\tfrac{3}{4})^2-(r-1)\nu-\frac{(r-1)(2r-1)}{12r}\Big)\Big\}^{1/2}+o(1),$$

which gives $\nu\leqslant\nu_0$, where

$$(5.7)\qquad \nu_0=\frac{5r-3}{8}+\frac{1}{8}\Big(9r^2-6r-\frac{4(r-1)(2r-1)}{r}\Big)^{1/2}+o(1).$$

We note that since we can write ν_0 as

$$\frac{5r-3}{8}+\frac{1}{8}\Big((3r-\tfrac{7}{3})^2+\frac{923}{81}-\frac{4}{r}\Big)^{1/2}+o(1),$$

we have

$$\nu_0>r-\tfrac{2}{3},$$

and thus

$$\nu_0-(r-\tfrac{3}{4})>\tfrac{1}{12}.$$

Hence by (5.6) and (5.7) we have

$$\lambda-\nu_0=\frac{1}{3}\Big(\frac{3}{4}\Big((r-\tfrac{3}{4})^2-(r-1)\nu_0-\frac{(r-1)(2r-1)}{12r}\Big)\Big)^{1/2}+o(1)$$
$$=\tfrac{1}{3}\big(\nu_0-(r-\tfrac{3}{4})\big)+o(1)>\tfrac{1}{36}+o(1)>\tfrac{1}{4}\Big(1-\frac{1}{r}\Big)-E_1,$$

provided that $E_1\geqslant 2/9$, so that (5.4) holds with $\nu=\nu_0$. It follows that if ν is a little larger than ν_0, then (5.5) fails for the corresponding value of λ, but (5.4) still holds. We conclude that

$$F_r\leqslant\nu_0+o(1),$$

and hence deduce (5.3).

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Reçu par la Rédaction le 30. 7. 1968

ACTA ARITHMETICA
XV (1969)

Approximation to real numbers by algebraic integers

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1. Introduction. The problem of approximating to a real numbet ξ by algebraic numbers was investigated by Wirsing [2]. He proved thae if $n > 1$ and ξ is not an algebraic number of degree at most n, there arr infinitely many algebraic numbers α of degree at most n which satisfy (1)

$$(1) \qquad |\xi-\alpha| \ll H(\alpha)^{-(n+3)/2},$$

where $H(\alpha)$ denotes the height of α. The constant implied by the notation $\ll$ depends on n and ξ, but its dependence on ξ is of a simple nature. In the case $n = 2$ we showed [1] that the result holds with $H(\alpha)^{-3}$ on the right, and this is best possible.

In the present paper we investigate approximation to a real number ξ by real algebraic *integers* α.

It is instructive to consider first the case $n = 2$, even though this is every simple. In the first place, if ξ is rational there are infinitely many quadratic integers α satisfying

$$(2) \qquad 0 < |\xi-\alpha| \ll H(\alpha)^{-1}.$$

For there are infinitely many integer pairs x, y satisfying

$$0 < |\xi^2+\xi x+y| \leqslant 1,$$

and if we put $t^2+tx+y = (t-\alpha)(t-\alpha')$ we have

$$(\alpha-\alpha')^2 = x^2-4y = (2\xi+x)^2+O(1).$$

Without loss of generality we can suppose that

$$|\xi-\alpha'| \gg |2\xi+x| \gg |x|,$$

and as $0 < |(\xi-\alpha)(\xi-\alpha')| \leqslant 1$, this implies that

$$0 < |\xi-\alpha| \ll |x|^{-1},$$

(1) Wirsing also gave an exponent slightly better than that in (1).