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Extreme copositive quadratic forms

by

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1. Introduction. A real quadratic form $Q = Q(x_1, \ldots, x_n) = \sum a_{ij}x_ix$ $(a_{ij} = a_{ji}, 1 \leqslant i, j \leqslant n)$ is called *copositive* if $Q(x_1, \ldots, x_n) \geqslant 0$ whenever $x_1 \geqslant 0, \ldots, x_n \geqslant 0$. A copositive quadratic form Q is *extreme* when $Q = Q_1 + Q_2$, where Q_1 and Q_2 are copositive, implies $Q_1 = \lambda Q, Q_2 = (1-\lambda)Q$, $0 \leqslant \lambda \leqslant 1$. An extreme copositive quadratic form is *basic* if no two rows (or columns) of its matrix are identical. If S is the class of positive semi-definite quadratic forms and P the class of quadratic forms with non-negative coefficients then clearly any form expressible as a sum of elements of P and S is necessarily copositive. Hall and Newman [4] have determined the extreme copositive quadratic forms belonging to the class $P+S$ so interest now centres on the extreme copositive quadratic forms not in $P+S$; the Horn form

$$(x_1 - x_2 + x_3 + x_4 - x_5)^2 + 4x_2x_4 + 4x_3(x_5 - x_4),$$

constructed by Horn [5], shows that such forms do in fact exist. The Horn form has the property $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant n)$ where $n = 5$ and there are extreme copositive quadratic forms with this property for each $n \geqslant 5$; this is immediate from the following theorem of Baumert [1].

THEOREM. *If Q_n is an extreme copositive quadratic form in $n \geqslant 3$ variables $x_1, \ldots, x_n$ then replacing any x_i by $x_i + x_{n+1}$ in Q_n yields a new copositive form Q_{n+1} which is extreme.*

However Baumert [2] has also shown that there are no basic extreme copositive quadratic forms with $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant n)$ for $n = 6$ and 7.

In this paper we obtain simple necessary and sufficient conditions for a quadratic form with $|a_{ij}| = 1$ to be an extreme copositive form. From these conditions we obtain a stronger form of the Hall and Newman Theorem 4.1 in [4] for basic extreme copositive quadratic forms with $|a_{ij}| = 1$ and $n \geqslant 5$, which shows that the Horn form plays a fundamental role in every form of this type and which also enables us to prove the conjecture 4.1 of Baumert [2] for such forms. Finally we show that basic extreme copositive quadratic forms with $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant n)$ exist for all integers $n \geqslant 8$ and then obtain some of their properties.

2. In this section we obtain necessary and sufficient conditions for a quadratic form $Q = \sum a_{ij} x_i x_j$ with $a_{ij} = a_{ji}$, $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant n)$ to be an extreme copositive form. For copositivity we must clearly have $a_{ii} = 1$ $(1 \leqslant i \leqslant n)$.

LEMMA 2.1. *Let* $Q(x_1, \ldots, x_n) = \sum a_{ij} x_i x_j$ *where* $a_{ij} = a_{ji}$, $|a_{ij}| = 1$ *and* $a_{ii} = 1$ $(1 \leqslant i, j \leqslant n)$ *then* Q *is copositive if and only if there is no triple* (r, s, t) *such that* $a_{rs} = -1 = a_{rt} = a_{st}$.

Proof. Necessity. If there is such a triple (r, s, t) we may clearly suppose that $(r, s, t) = (1, 2, 3)$ and then $Q(1, 1, 1, 0, 0, \ldots, 0) = -3$ so that Q is not copositive.

Sufficiency. If $n = 1$ or 2, Q is clearly copositive so we may assume $n \geqslant 3$. Suppose Q satisfies the condition but is not copositive. Since $n \geqslant 3$ there exists a pair (r, s) with $r \neq s$ such that $a_{rs} = 1$ for, if not, the condition is violated; we may clearly suppose $r = 1$, $s = 2$. Let T be the set of t such that $a_{1t} = -a_{2t}$, then

$$(1) \qquad Q(x_1, x_2, \ldots, x_n) = Q(0, x_1 + x_2, x_3, \ldots, x_n) + 4x_1 \sum_{t \in T} a_{1t} x_t$$

and

$$(2) \qquad Q(x_1, x_2, \ldots, x_n) = Q(x_1 + x_2, 0, x_3, \ldots, x_n) + 4x_2 \sum_{t \in T} a_{2t} x_t$$
$$= Q(x_1 + x_2, 0, x_3, \ldots, x_n) - 4x_2 \sum_{t \in T} a_{1t} x_t.$$

Since Q is not copositive there exists $(u_1, \ldots, u_n)$ with $u_i \geqslant 0$ $(1 \leqslant i \leqslant n)$ such that $Q(u_1, \ldots, u_n) < 0$. If $\sum_{t \in T} a_{1t} u_t \geqslant 0$, $Q(0, u_1 + u_2, u_3, \ldots, u_n) < 0$ from (1) whilst if $\sum_{t \in T} a_{1t} u_t < 0$ $Q(u_1 + u_2, 0, u_3, \ldots, u_n) < 0$ from (2). Thus if $Q(x_1, \ldots, x_n)$ is not copositive it contains a subform of $(n-1)$ variables which is also not copositive; further this subform satisfies the condition since Q does. If $n \geqslant 4$ repeat the argument on this subform and continue the process until one obtains that Q has a subform of two variables which is not copositive. We now have a contradiction and the lemma is established.

LEMMA 2.2. *A copositive quadratic form* $Q(x_1, \ldots, x_n) = \sum a_{ij} x_i x_j$ *where* $a_{ij} = a_{ji}$, $|a_{ij}| = 1$ *and* $a_{ii} = 1$ $(1 \leqslant i, j \leqslant n)$ *is extreme if and only if for each pair* (r, s) *with* $r \neq s$ *and* $a_{rs} = 1$ *there exists a* t *such that* $a_{rt} = -1 = a_{st}$.

Proof. Necessity. Suppose there is a pair (r, s) with $r \neq s$ and $a_{rs} = 1$ such that there is no t with $a_{rt} = -1 = a_{st}$. We may clearly assume $r = 1, s = 2$. Let V be the set of $v > 2$ such that $a_{1v} = a_{2v}$ then

$$P(x_1, x_2, \ldots, x_n) = Q(x_1, x_2, \ldots, x_n) - 4x_1 x_2$$
$$(1) \qquad = Q(x_1 - x_2, 0, x_3, \ldots, x_n) + 4x_2 \sum_{v \in V} a_{1v} x_v$$
$$(2) \qquad = Q(0, x_2 - x_1, x_3, \ldots, x_n) + 4x_1 \sum_{v \in V} a_{1v} x_v.$$

Now $a_{1v} = 1$ for $v \in V$ since there is no t such that $a_{1t} = -1 = a_{2t}$ and further Q is copositive so $P \geqslant 0$ for $x_i \geqslant 0$ $(1 \leqslant i \leqslant n)$ and $x_1 \geqslant x_2$ from (1) whilst $P \geqslant 0$ for $x_i \geqslant 0$ $(1 \leqslant i \leqslant n)$ and $x_2 > x_1$ from (2). Thus P is copositive and, since $Q = P + 4x_1 x_2$, Q is not extreme.

Sufficiency. The result is trivial for $n = 1$ and $n = 2$ so we may assume $n \geqslant 3$. Suppose Q satisfies the condition and $Q = Q_1 + Q_2$ where Q_1 and Q_2 are copositive. Consider the term a_{rs} with $r \neq s$ then either (i) $a_{rs} = -1$ or (ii) $a_{rs} = 1$.

(i) If $a_{rs} = -1$, on replacing all the x_i except x_r and x_s by zero, Q reduces to $(x_r - x_s)^2$ which is extreme so Q_1 must reduce to $\lambda_{rs}(x_r - x_s)^2$ and Q_2 to $(1 - \lambda_{rs})(x_r - x_s)^2$, $0 \leqslant \lambda_{rs} \leqslant 1$.

(ii) If $a_{rs} = 1$ then, since $r \neq s$, by the condition there is a t such that $a_{rt} = -1 = a_{st}$. Thus, on replacing all the x_i except x_r, x_s and x_t by zero, Q reduces to $(x_r + x_s - x_t)^2$ which is extreme so Q_1 must reduce to $\lambda_{rs}(x_r + x_s - x_t)^2$ and Q_2 to $(1 - \lambda_{rs})(x_r + x_s - x_t)^2$, $0 \leqslant \lambda_{rs} \leqslant 1$.

Thus, in Q_1, the coefficient of both x_r^2 and x_s^2 is λ_{rs} and the coefficient of $x_r x_s$ is $2\lambda_{rs} a_{rs}$. Taking u with $u \neq r$, $u \neq s$ and $1 \leqslant u \leqslant n$ we obtain similarly that the coefficient in Q_1 of both x_r^2 and x_u^2 is λ_{ru} so we must have $\lambda_{ru} = \lambda_{rs} = \lambda_r$, say. Clearly $\lambda_{ij} = \lambda_{ji}$ so $\lambda_i = \lambda_j = \lambda$ say, i.e. $\lambda_{rs} = \lambda$ for $1 \leqslant r, s \leqslant n$. Hence $Q_1 = \lambda Q$, $Q_2 = (1 - \lambda) Q$, $0 \leqslant \lambda \leqslant 1$ and the result is established.

Combining the results of Lemmas 2.1 and 2.2 we have:

THEOREM 2.3. *Let* $Q(x_1, \ldots, x_n) = \sum a_{ij} x_i x_j$ *where* $a_{ij} = a_{ji}$ *and* $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant n)$ *then* Q *is a copositive extreme if and only if the following conditions hold:*

(i) $a_{ii} = 1$, $i = 1, \ldots, n$,

(ii) *there is no triple* (i, j, k) *such that* $a_{ij} = -1 = a_{ik} = a_{jk}$,

(iii) *for each pair* (r, s) *with* $r \neq s$ *and* $a_{rs} = 1$ *there exists a* t *such that* $a_{rt} = -1 = a_{st}$.

3. Hall and Newman [4] have shown that extreme copositive forms, except those of the type $bx_i x_j$, are "locally" semi-definite in the sense that the form becomes semi-definite if appropriate variables are replaced by zero and that such replacements exist which leave any two specified variables unchanged. In a similar sense we now show that basic extreme copositive forms $Q(x_1, \ldots, x_n)$ with $|a_{ij}| = 1$ and $n \geqslant 5$ are "locally" equivalent to the Horn form and also that they satisfy the following conjecture by Baumert [2].

CONJECTURE. *If $Q(x_1, \ldots, x_n)$, $n \geqslant 3$, is an extreme copositive quadratic form, then for every index pair i, j $(1 \leqslant i, j \leqslant n)$, Q has a non-negative component zero u with $u_i u_j > 0$.*

We firstly require:

LEMMA 3.1. *If $Q = \sum a_{ij} x_i x_j$ where $a_{ij} = a_{ji}$, $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant n)$ is a basic extreme copositive quadratic form then given any pair (i, j) there exists a pair (u, v) such that $a_{iu} = -a_{ju} = 1$ and $a_{iv} = -a_{jv} = -1$.*

Proof. Since Q is basic given any pair (i, j) there is a u such that $a_{iu} = -a_{ju}$. We may clearly suppose $a_{iu} = 1$ and also $u \neq i$, for otherwise, the result is established, then by Lemma 2.2 with $(r, s) = (i, u)$ there is a v with $a_{iv} = -1 = a_{uv}$ so by Lemma 2.1 $a_{jv} = 1$ and the lemma is proved.

THEOREM 3.2. *Let $Q = \sum a_{ij} x_i x_j$ where $a_{ij} = a_{ji}$, $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant n)$ be a basic extreme copositive quadratic form in $n \geqslant 5$ variables. Let x_r, x_s be any two of the variables $x_1, \ldots, x_n$ then, upon suitably replacing all but five of $x_1, \ldots, x_n$ by zero but neither x_r nor x_s, Q reduces to a form equivalent to the Horn form.*

Proof. We may clearly suppose that the two variables are x_1 and x_2. Two cases arise either (i) $a_{12} = 1$ or (ii) $a_{12} = -1$.

(i) If $a_{12} = 1$ then by Lemma 2.2 there is a t, which we may assume to be 3, such that $a_{13} = -1 = a_{23}$. Since the form is basic we may suppose without loss of generality that $a_{14} = -a_{24} = 1$. By Lemma 2.1 $a_{34} = 1$ so by Lemma 2.2 with $(r, s) = (1, 4)$ there is a u, which we may assume to be 5, such that $a_{15} = -1 = a_{45}$. Lemma 2.1 now gives $a_{25} = 1 = a_{35}$ and, on replacing the variables $6, \ldots, n$ by zero, we have a form equivalent to the Horn form.

(ii) If $a_{12} = -1$, by Lemma 2.1 $a_{j1} = -1 = a_{j2}$ for j one of $3, 4, \ldots, n$ is impossible so we may clearly assume $a_{13} = 1$. By Lemma 3.1 there is a u, which we may suppose to be 4, such that $a_{14} = 1 = -a_{34}$. Now by Lemma 2.1 we cannot have $a_{32} = a_{42} = -1$ so, without loss of generality, we may assume $a_{32} = 1$. We may further suppose $a_{42} = -1$ for, if not, by Lemma 2.2 there is a t such that $a_{2t} = -1 = a_{3t}$ and then by Lemma 2.1 $a_{1t} = 1$. By Lemma 2.2 with $(r, s) = (1, 3)$ we may assume $a_{15} = -1 = a_{35}$. By Lemma 2.1 we must now have $a_{45} = 1 = a_{25}$ and, on replacing the variables $6, \ldots, n$ by zero, we have a form equivalent to the Horn form.

THEOREM 3.3. *If $Q = \sum a_{ij} x_i x_j$ where $a_{ij} = a_{ji}$, $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant n)$ and $n \geqslant 3$ is an extreme copositive quadratic form then for every index pair i, j $(1 \leqslant i, j \leqslant n)$, Q has a non-negative component zero u with $u_i u_j > 0$.*

Proof. From Baumert [2] the result is true if Q is the Horn form or if Q is in $P + S$ where S is the class of positive semi-definite quadratic forms and P is the class of quadratic forms all of whose coefficients are non-negative. Hence we may further suppose that Q is not semi-definite.

If Q is not basic we may clearly express it in the form $Q(x_1, \ldots, x_n) = q(y_1, \ldots, y_m) = \sum b_{ij} y_i y_j$ say, where $q(y_1, \ldots, y_m)$ is a basic extreme copositive quadratic form with $|b_{ij}| = 1$ and $y_j = x_{j1} + x_{j2} + \ldots + x_{jk_j}$ $(j = 1, \ldots, m)$ where $x_{jt} = x_r$ for some r $(t = 1, 2, \ldots, k_j)$ and $\sum_{j=1}^{m} y_j = \sum_{r=1}^{n} x_r$. Since Q is not semi-definite neither is q and further Q obviously satisfies the theorem if $q(y_1, \ldots, y_m)$ does since $m \geqslant 2$ because Q is extreme.

Thus we may assume that Q is also basic. Diananda [3] has shown that a copositive quadratic form in $n \leqslant 4$ variables is in $P + S$ so from the above we need only consider the case $n \geqslant 5$. Hence, from Theorem 3.2, given any pair (r, s) with $1 \leqslant r, s \leqslant n$ upon suitably replacing all but five of $x_1, \ldots, x_n$ by zero but neither x_r nor x_s Q reduces to a form equivalent to the Horn form. Since the Horn form satisfies the theorem Q has a non-negative component zero u with $u_r u_s > 0$.

4. We now show that basic extreme copositive quadratic forms with $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant n)$ exist in n variables for every integer $n \geqslant 8$. We also consider a few of the properties of these forms.

For $p \geqslant 3$ let $q_{3p}(x_1, \ldots, x_{3p})$ be the quadratic form whose coefficients a_{ij} are defined by $a_{ij} = a_{ji}$ and $a_{ij} = 1$ for $i \leqslant j$ except for the following:

$$a_{1j} = -1 \quad (j = 2, 3, \ldots, p+2),$$

$$a_{i,p+2i-1} = a_{i,p+2i} = -1 \quad (i = 2, 3, \ldots, p),$$

$$a_{p+2i-1,p+2i+2r} = -1 = a_{p+2i,p+2i+2r-1}$$
$$(i = 1, 2, \ldots, p-1, r = 1, 2, \ldots, p-i).$$

For $p \geqslant 4$ define $q_{3p-1}(x_1, \ldots, x_{3p-1})$ and $q_{3p-2}(x_1, \ldots, x_{3p-2})$ by

$$q_{3p-1}(y_1, \ldots, y_{3p-1}) = q_{3p}(y_1, y_2, \ldots, y_{p+1}, 0, y_{p+2}, \ldots, y_{3p-1}),$$

$$q_{3p-2}(z_1, \ldots, z_{3p-2}) = q_{3p}(z_1, z_2, \ldots, z_p, 0, 0, z_{p+1}, \ldots, z_{3p-2}).$$

THEOREM 4.1. *The quadratic forms $q_r(x_1, \ldots, x_r)$, $r \geqslant 9$, defined above are basic copositive extreme forms with $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant r)$.*

Proof. By inspection the forms are clearly basic and $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant r)$.

Copositivity. From the definitions q_{3p-1} and q_{3p-2} are copositive if q_{3p} is so it is sufficient to show that q_{3p} $(p \geqslant 3)$ is copositive. Thus consider q_{3p} $(p \geqslant 3)$; we shall show that there is no triple (r, s, t) with the property that $a_{rs} = -1 = a_{st} = a_{rt}$ and copositivity will then follow by Lemma 2.1. We need clearly only consider those triples (r, s, t) with $r < s < t$.

If $a_{1s} = -1 = a_{1t}$ with $s < t$ then $t \leqslant p+2$. If $s = p+1$, then $t = p+2$ and $a_{st} = 1$. Otherwise $2 \leqslant s \leqslant p$ so that for $s < k$ $a_{sk} = -1$ only if $k = p+2s-1$ or $p+2s$ and since $t \leqslant p+2$, we must therefore have $a_{st} = 1$.

For fixed r with $2 \leqslant r \leqslant p$, $a_{rs} = -1 = a_{rt}$ with $r < s < t$ implies $s = p+2r-1 = t-1$ and then $a_{st} = a_{p+2r-1,p+2r} = 1$.

For fixed i with $1 \leqslant i \leqslant p-1$, $a_{p+2i-1,s} = -1 = a_{p+2i-1,t}$ with $p+2i-1 < s < t$ implies $s = p+2i+2r_1$, $t = p+2i+2r_2$ for some r_1, r_2 with $1 \leqslant r_1 < r_2 \leqslant p-i$ and then $a_{st} = a_{p+2i+2r_1,p+2i+2r_1+2(r_2-r_1)} = 1$.

For fixed i with $1 \leqslant i \leqslant p-1$ $a_{p+2i,s} = -1 = a_{p+2i,t}$ with $p+2i < s < t$ implies $s = p+2i+2r_1-1$, $t = p+2i+2r_2-1$ for some r_1, r_2 with $1 \leqslant r_1 < r_2 \leqslant p-i$ and then $a_{st} = a_{p+2i+2r_1-1,p+2i+2r_1+2(r_2-r_1)-1} = 1$.

Hence there is no triple (r, s, t) such that $a_{rs} = -1 = a_{rt} = a_{st}$ and copositivity is established.

Extremity. To prove that q_{3p} $(p \geqslant 3)$ is extreme, it is sufficient, by Lemma 2.2, to show that for each pair (r, s) with $r \neq s$ and $a_{rs} = 1$ there is a t such that $a_{rt} = -1 = a_{st}$. In our proof we shall show in addition that, if $p \geqslant 4$, we can always find such a t with $t \neq p+1, t \neq p+2$. From this result and their definitions, q_{3p-1} and q_{3p-2} will then be extreme by Lemma 2.2. Clearly we need only consider those $a_{rs} = 1$ for which $r < s$, so let us examine such a_{rs} in q_{3p} $(p \geqslant 3)$.

If $a_{1s} = 1$ with $1 < s$ then $p+3 \leqslant s \leqslant 3p$; let $t = \left[\dfrac{s-p+1}{2}\right]$, where $[x]$ denotes the greatest integer not greater than x, then $2 \leqslant t \leqslant p$ so $a_{1t} = -1$ and also $s = 2t+p$ or $s = 2t+p-1$ so that $a_{st} = a_{ts} = -1$.

For fixed r with $2 \leqslant r \leqslant p$ consider the s with $r < s$ for which $a_{rs} = 1$. If $s \leqslant p+2$ then $a_{r1} = -1 = a_{s1}$ so consider $s > p+2$;

(i) If $s = p+2v-1$ let $t = p+2r \geqslant p+4$ then $a_{rt} = -1$ and, since $r \neq v$, either $2 \leqslant v \leqslant r-1$ in which case $a_{st} = a_{p+2v-1,p+2v+2(r-v)} = -1$ or $r+1 \leqslant v \leqslant p$ in which case $a_{st} = a_{ts} = a_{p+2r,p+2r+2(v-r)-1} = -1$;

(ii) If $s = p+2v$ let $t = p+2r-1 \geqslant p+3$ then $a_{rt} = -1$ and, since $v \neq r$, either $2 \leqslant v \leqslant r-1$ in which case $a_{st} = a_{p+2v,p+2v+2(r-v)-1} = -1$ or $r+1 \leqslant v \leqslant p$ in which case $a_{st} = a_{ts} = a_{p+2r-1,p+2r+2(v-r)} = -1$.

For fixed i with $1 \leqslant i \leqslant p-1$ consider the s with $p+2i-1 < s$ for which $a_{p+2i-1,s} = 1$;

If $s = p+2i$ choose $t = i < p$ so

$$a_{p+2i-1,t} = a_{p+2i-1,i} = -1 = a_{p+2i,i} = a_{st}.$$

If $s = p+2i+2u-1$ with $1 \leqslant u \leqslant p-i-1$ choose $t = p+2i+2u+2 > p+2$ then

$$a_{p+2i-1,t} = a_{p+2i-1,p+2i+2(u+1)} = -1 = a_{p+2i+2u-1,p+2i+2u+2} = a_{st}.$$

If $s = 3p-1$ choose $t = 3p-2 = p+2i+2(p-i-1) > p+2$ if $i \leqslant p-2$ and $t = 3p-4$ if $i = p-1$ (note that $t > p+2$ for $p \geqslant 4$), then $a_{st} = -1 = a_{p+2i-1,t}$.

For fixed i with $1 \leqslant i \leqslant p-1$ consider the s with $p+2i < s$ for which $a_{p+2i,s} = 1$, then $s = p+2i+2v$ for some v with $1 \leqslant v \leqslant p-i$. For $1 \leqslant v \leqslant p-i-1$ choose $t = p+2i+2v+1 > p+2$ then $a_{st} = -1 = a_{p+2i,t}$. For $v = p-i$ choose $t = p+2i+1 > p+2$ if $i \leqslant p-2$ and $t = 3p-5 > p+2$ for $p \geqslant 4$ if $i = p-1$, then $a_{p+2i,t} = -1 = a_{t,3p}$.

The only case $a_{rs} = 1$ with $r < s$ remaining for consideration is $a_{3p-1,3p} = 1$ in which case we choose $t = p$ so that $a_{3p-1,t} = a_{p,3p-1} = -1 = a_{p,3p} = a_{3p,t}$.

The extremity of q_r $(r \geqslant 9)$ now follows from the above comments.

In the class of basic extreme copositive quadratic forms with $|a_{ij}| = 1$ the Horn form is:

(i) the only one in 5 variables,

(ii) the only one whose matrix has at least one row containing exactly two -1's.

Condition (i) is immediate from Theorem 3.2 and has already been proved by Baumert [2]. Since the only basic extreme copositive quadratic forms with $|a_{ij}| = 1$ in less than five variables are x_1^2 and $(x_1-x_2)^2$, condition (ii) is immediate from (i) and the following lemma.

LEMMA 4.2. *If $Q_n = \sum a_{ij}x_ix_j$, $a_{ij} = a_{ji}$, $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant n)$ is a basic extreme copositive quadratic form and $n \geqslant 6$ then, for each i $(1 \leqslant i \leqslant n)$, there are at least three values of j in $1 \leqslant j \leqslant n$ such that $a_{ij} = -1$.*

Proof. Suppose the result is false then we may suppose that at most two of the a_{1j}, $2 \leqslant j \leqslant n$, are -1's. However by Theorem 3.2 we may assume that $Q_n(x_1, x_2, \ldots, x_5, 0, 0, \ldots, 0)$ is the Horn form so the leading 5×5 minor of the matrix associated with Q_n is of the form

$$\begin{vmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \end{vmatrix}$$

Thus $a_{1j} = 1$ for $j \geqslant 4$. By symmetry and Lemma 2.2 we may assume $a_{62} = -1$ so by Lemma 3.1 $a_{63} = 1$ and by Lemma 2.1 $a_{64} = 1$. From rows 4 and 6, Lemma 3.1 says we may assume $a_{47} = 1 = -a_{67}$. By Lemma 2.1 $a_{27} = 1$ and by Lemma 2.2 $a_{37} = -1$ so by Lemma 2.1 $a_{57} = 1$. From rows 4 and 7 Lemma 2.2 says we may assume $a_{48} = -1 = a_{78}$. By Lemma 2.1 $a_{28} = 1 = a_{38}$ and rows 1 and 8 now show that we have a contradiction to Lemma 2.2.

We now require:

LEMMA 4.3. *If* $Q_n = \sum a_{ij}x_ix_j$, $a_{ij} = a_{ji}$, $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant n)$ *is a basic extreme copositive quadratic form such that, for at least one fixed* i *in* $1 \leqslant i \leqslant n$, *there are at least three values of* j *in* $1 \leqslant j \leqslant n$ *with* $a_{ij} = -1$, *then* $n \geqslant 8$ *and if* $n = 8$ *the form is unique apart from interchange of variables.*

Proof. We may suppose that $a_{rr} = 1$ $(1 \leqslant r \leqslant n)$ and $a_{12} = -1 = a_{13} = a_{14}$ then by Lemma 2.1 $a_{rs} = 1$ $(r, s = 2, 3, 4)$. Lemma 3.1 with $(i, j) = (2, 3)$ implies that $n \geqslant 6$ so by Lemma 4.2 at least six of the a_{rs} $(r = 2, 3, 4, s = 5, 6, \ldots, n)$ must be -1. Hence if $n \leqslant 8$ there is a $j \geqslant 5$ such that at least two of a_{2j}, a_{3j} and a_{4j} are -1's; we may assume $a_{25} = -1 = a_{35}$. By Lemma 3.1 we may suppose $a_{26} = 1 = -a_{36}$, $a_{27} = -1 = -a_{37}$ then $a_{15} = a_{16} = a_{17} = 1 = a_{57} = a_{56}$ by Lemma 2.2. By Lemma 3.1 we may suppose $a_{58} = -1 = -a_{18}$ so, since $n \leqslant 8$ $a_{54} = 1$. By Lemma 2.1 $a_{28} = 1 = a_{38}$. By Lemma 2.2 with (r, s) equal to $(2, 6)$ and $(4, 5)$ we have respectively $a_{67} = -1$ and $a_{48} = -1$. At least one of a_{86} and a_{87} is -1 by Lemma 4.2 and by symmetry we may suppose $a_{86} = -1$ then by Lemma 2.1 $a_{87} = 1 = a_{64}$. Thus by Lemma 4.2 $a_{47} = -1$.

The form obtained is clearly basic and by suitably renumbering the variables it becomes cyclic, i.e.

$$Q_8 = \Big(\sum_{i=1}^{8} x_i\Big)^2 - 2\sum_{i=1}^{8} x_i(x_{i+1} + x_{i+4} + x_{i+7}),$$

where $x_{r+8} = x_r$. That the form is in fact copositive and extreme follows from:

THEOREM 4.4.

$$\Big(\sum_{i=1}^{3m+2} x_i\Big)^2 - 2\sum_{i=1}^{3m+2} x_i(x_{i+1} + x_{i+4} + x_{i+7} + \ldots + x_{i+3m+1})$$

where $x_{r+3m+2} = x_r$ *is an extreme copositive quadratic form for each* $m \geqslant 1$.

Proof. Since the forms are cyclic the proof is particularly simple. Employing Lemma 2.1 to prove copositivity, it is sufficient to show that if $a_{1s} = -1 = a_{1t}$ then $a_{st} = 1$. However if $a_{1s} = -1 = a_{1t}$ with $s < t$ then $t = s + 3u$ for some u and $a_{st} = a_{s,s+3u} = 1$. Employing Lemma 2.2 to prove extremity, it is sufficient to show that if $a_{1s} = 1$ $(s > 1)$ then there is a t such that $a_{1t} = -1 = a_{st}$. However if $a_{1s} = 1$ $(s > 1)$, then either $s = 3u$ or $3u+1$ for some $u \geqslant 1$.

If $s = 3u$ let $t = 3u-1$ then $a_{1t} = a_{1,1+3u-2} = -1 = a_{3u-1,3u} = a_{ts} = a_{st}$.

If $s = 3u+1$ let $t = 3u+2$ then $a_{1t} = a_{1,1+3u+1} = -1 = a_{3u+1,3u+2} = a_{st}$.

Note that when $m = 1$ we have the Horn form. We also remark that, when $m \geqslant 3$, the above forms cannot be obtained from the q_{3m+2} of Theorem 4.1 by renumbering of variables.

From Lemmas 4.2 and 4.3 we have:

LEMMA 4.5. *There is no basic extreme copositive quadratic form* $Q_n = \sum a_{ij}x_ix_j$, $a_{ij} = a_{ji}$, $|a_{ij}| = 1$ $(1 \leqslant i, j \leqslant n)$ *for* $n = 6$ *or* 7 *and there is a unique such form (apart from interchange of variables) when* $n = 8$.

The first part of Lemma 4.5 was proved by Baumert in [2].

From Theorem 4.1 and Lemma 4.3 we finally have:

THEOREM 4.6. *There exist basic extreme copositive quadratic forms*

$$Q_n = \sum a_{ij}x_ix_j, \quad a_{ij} = a_{ji}, \ |a_{ij}| = 1 \ (1 \leqslant i, j \leqslant n)$$

for $n \geqslant 8$.

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