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L-functions and character sums for quadratic forms (II)*

by

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1. Let $Q(x, y) = ax^2 + bxy + cy^2$ be a positive definite binary quadratic form with integral coefficients and discriminant $d = b^2 - 4ac < 0$, and let χ be a character $(\bmod k)$. Let

(1) $$L(s, \chi, Q) = \tfrac{1}{2} \sum_{x,y \neq 0,0} \chi\big(Q(x, y)\big) Q(x, y)^{-s}.$$

The series converges to an analytic function of s for $\operatorname{Re} s > 1$. The function in (1) is a special case of the functions considered in [7] where quadratic forms in n variables were considered. As shown in [7], if $(k, d) = 1$ and χ is a primitive character $(\bmod k)$, then $L(s, \chi, Q)$ can be extended to an entire function of s satisfying a functional equation (in [7], it was convenient to call $-d$ the *discriminant* of Q; this will account for the sign changes between certain equations in [7] and here). In this paper we present an expansion of $L(s, \chi, Q)$ which is very rapidly convergent in the neighborhood of $s = 1$. Similar expansions have been known for the Epstein zeta function for some time [1], [2] and certain cases of this expansion have been considered in [5] and [6] (k a prime, χ real, and $k = 8$ or 12, χ real respectively). However the expansion in general and the functional equation both depend on a character identity quoted below as Theorem 1.

2. Notation and statement of results. It will be assumed throughout that χ is a primitive character $(\bmod k)$ and $k > 1$. As noted in [7], this means that $k \not\equiv 2 \pmod 4$. However χ^2 is not necessarily a primitive character $(\bmod k)$. Thus we put

(2) $$\chi^2 = \chi_0 \chi_1$$

where χ_0 is the principal character $(\bmod k)$ and χ_1 is a primitive character $(\bmod k_1)$. We set $k = k_0 k_1$ and note that we do allow $k_1 = 1$. In any event

(3) $$\chi_1(-1) = 1.$$

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Set

(4) $$\tau(\chi) = \sum_{j=1}^{k} \chi(j) e_k(j), \qquad \tau(\chi_1) = \sum_{j=1}^{k_1} \chi_1(j) e_{k_1}(j)$$

where for convenience we write

$$e_r(j) = e^{2\pi i j/r}.$$

Because of (3) and the fact that χ_1 is primitive $(\mathrm{mod}\, k_1)$ ([3], p. 70),

(5) $$\tau(\chi_1)\tau(\bar{\chi}_1) = k_1.$$

Set

(6) $$\chi_2(j) = \left(\frac{k'}{j}\right), \quad k' = \begin{cases} (-1)^{(k-1)/2} k & \text{if } k \text{ is odd}, \\ -k & \text{if } k \equiv 0 (\mathrm{mod}\ 4). \end{cases}$$

Here we have used the Kronecker symbol. Now let

(7) $$\varepsilon = \begin{cases} 1 & \text{if } k \equiv 1 (\mathrm{mod}\ 4), \\ i & \text{if } k \equiv 0 \text{ or } 3 (\mathrm{mod}\ 4) \end{cases}$$

and

(8) $$a = \frac{\varepsilon^2 \chi(d) \chi_2(-d) \tau(\chi)}{\tau(\bar{\chi})}.$$

We will use the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

and the Dirichlet L-functions

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

In addition we will use the modified Bessel function of the second kind

(9) $$K_s(x) = \int_0^{\infty} e^{-x\cosh t} \cosh st\, dt$$

defined for all s and $x > 0$. In fact $K_s(x)$ is an entire function of s and

(10) $$K_s(x) = K_{-s}(x)$$

for all s and $x > 0$.

THEOREM 1. *If* $(d, k) = 1$ *and* χ *is a primitive character* $(\mathrm{mod}\, k)$ *then*

(11) $$\sum_{x=1}^{k} \chi\big(Q(x, y)\big) e_k(xz) = a \sum_{x=1}^{k} \bar{\chi}\big(Q(x, z)\big) e_k(xy).$$

THEOREM 2. *Under the hypothesis of Theorem* 1,

(12) $$a^s L(s, \chi, Q) = \chi(a) L(2s, \chi_1) \prod_{\substack{p \mid k_0 \\ p \text{ prime}}} \left(1 - \frac{\chi_1(p)}{p^{2s}}\right) + \\ + a\bar{\chi}(a) \frac{\tau(\bar{\chi}_1)}{k_1} \left(\frac{k\sqrt{|d|}}{2ak_1}\right)^{1-2s} \frac{\sqrt{\pi}\,\Gamma(s-1/2)}{\Gamma(s)} L(2s-1, \bar{\chi}_1) \prod_{\substack{p \mid k_0 \\ p \text{ prime}}} \left(1 - \frac{\bar{\chi}_1(p)}{p^{2-2s}}\right) + \\ + \frac{1}{\Gamma(s)} \left(\frac{k\sqrt{|d|}}{2a\pi}\right)^{-s} H(s)$$

where

(13) $$H(s) = H(s, \chi) = 2\left(\frac{\sqrt{|d|}}{2ak}\right)^{1/2} \sum_{n \neq 0} e_{2ak}(bn) K_{s-1/2}\left(\frac{\pi |n| \sqrt{|d|}}{ak}\right) \times \\ \times |n|^{s-1/2} \sum_{\substack{y \mid n \\ y > 0}} y^{1-2s} \sum_{j=1}^{k} \chi\big(Q(j, y)\big) e_k(jn/y)$$

is an entire function of s *and*

(14) $$H(s, \chi) = aH(1-s, \bar{\chi}).$$

COROLLARY 1. *Under the hypothesis of Theorem* 1,

(15) $$\left(\frac{k\sqrt{|d|}}{2\pi}\right)^{s} \Gamma(s) L(s, \chi, Q) = a \left(\frac{k\sqrt{|d|}}{2\pi}\right)^{1-s} \Gamma(1-s) \Gamma(1-s, \bar{\chi}, Q).$$

COROLLARY 2. *If* χ *is a real primitive character* $(\mathrm{mod}\, k)$ *and* $(k, d) = 1$,

(16) $$a^s L(s, \chi, Q) = \chi(a) \zeta(2s) \prod_{\substack{p \mid k \\ p \text{ prime}}} (1 - p^{-2s}) + \\ + \chi(a) \left(\frac{k\sqrt{|d|}}{2a}\right)^{1-2s} \frac{\sqrt{\pi}\,\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1) \prod_{\substack{p \mid k \\ p \text{ prime}}} (1 - p^{2s-2}) + \\ + \frac{1}{\Gamma(s)} \left(\frac{k\sqrt{|d|}}{2a\pi}\right)^{-s} H(s)$$

where now $H(s)$ *simplifies to*

(17) $$H(s) = 4\left(\frac{\sqrt{|d|}}{2ak}\right)^{1/2} \sum_{n=1}^{\infty} K_{s-1/2}\left(\frac{\pi n \sqrt{|d|}}{ak}\right) n^{s-1/2} \sum_{y \mid n} y^{1-2s} \times \\ \times \mathrm{Re}\left\{\sum_{j=1}^{k} \chi\big(Q(j, y)\big) e_k(jn/y) e_{2ak}(bn)\right\}$$

and if $a = 1$ *then the quantity in* { } *is already a real number.*

Remark 1. The restriction that Q be positive definite is unnecessary in Theorem 1.

Remark 2. Most of the expansion in Theorem 2 is independent of the conditions that χ be primitive $(\mathrm{mod}\, k)$ and $(k, d) = 1$ (see Lemma 3). These conditions are used only in applications of Theorem 1. In fact Theorem 1 is used only to prove (14) and evaluate the sum

$$\sum_{n=1}^{\infty}\sum_{j=1}^{k}\chi\big(Q(j, n)\big)n^{1-2s}$$

in Lemma 5. It is interesting to note that when χ is real, Ramanujan's sum is involved in this last sum in an unfamiliar form.

Remark 3. Corollary 1 was proved in [7]. It is included here as an application of the expansion of Theorem 2.

3. Proof of Theorem 1 and lemmas.

For all practical purposes, Theorem 1 was proved in [7]. If we put

$$\bar{Q}(x, y) = cx^2 - bxy + ay^2 = Q(y, -x)$$

then Theorem 1′ of [7] says that

$$\sum_{x=1}^{k}\chi\big(Q(x, y)\big)e_k(xz) = a\sum_{x=1}^{k}\bar{\chi}\big(\bar{Q}(-z, x)\big)e_k(xy) = a\sum_{x=1}^{k}\bar{\chi}\big(Q(x, z)\big)e_k(xy)$$

which is Theorem 1. Theorem 1′ of [7] is valid for indefinite forms and thus so is Theorem 1 here.

Lemma 1. *If x is real and $\mathrm{Re}\, s > 1/2$,*

$$\int_{-\infty}^{\infty}\frac{e^{ixu}}{(u^2+1)^s}\,du = \begin{cases}\dfrac{\sqrt{\pi}\,\Gamma(s-1/2)}{\Gamma(s)} & \text{if} \quad x = 0,\\[2ex] \dfrac{2\sqrt{\pi}}{\Gamma(s)}\left(\dfrac{|x|}{2}\right)^{s-1/2}K_{s-1/2}(|x|) & \text{if} \quad x \neq 0.\end{cases}$$

Proof. The first part is a well known result on gamma and beta functions (see for example, the last 3 lines of [1], p. 369) and the second is contained in Lemma 1 of [1].

Lemma 2. *If $x \geqslant x_0 > 0$ and s is in a bounded region B of the s plane then there is a real number c which depends only on B and x_0 such that*

$$|K_s(x)| < ce^{-x}.$$

Proof. This is a special case of Lemma 2 of [1].

Lemma 3. *Whether or not χ is primitive $(\mathrm{mod}\, k)$ and whether or not $(k, d) = 1$, we still have for $\mathrm{Re}\, s > 1$,*

$$a^sL(s, \chi, Q) = \chi(a)L(2s, \chi_1)\prod_{\substack{p|k_0\\ p\ \text{prime}}}\big(1-\chi_1(p)p^{-2s}\big)+$$
$$+\frac{1}{k}\left(\frac{\sqrt{|d|}}{2a}\right)^{1-2s}\frac{\sqrt{\pi}\,\Gamma(s-1/2)}{\Gamma(s)}\sum_{y=1}^{\infty}\sum_{j=1}^{k}\chi\big(Q(j, y)\big)y^{1-2s}+\frac{1}{\Gamma(s)}\left(\frac{k\sqrt{|d|}}{2a\pi}\right)^{-s}H(s)$$

where $H(s)$ is given by (13) and is an entire function of s.

Proof. For $\mathrm{Re}\, s > 1$,

$$\text{(18)} \qquad a^sL(s, \chi, Q) = \frac{a^s}{2}\sum_{x,y\neq 0,0}\chi\big(Q(x, y)\big)Q(x, y)^{-s}$$
$$= \chi(a)\sum_{x=1}^{\infty}\chi(x^2)x^{-2s}+a^s\sum_{y=1}^{\infty}\sum_{x}\chi\big(Q(x, y)\big)Q(x, y)^{-s}$$
$$= \chi(a)L(2s, \chi^2)+a^s\sum_{y=1}^{\infty}\sum_{j=1}^{k}\chi\big(Q(j, y)\big)\sum_{z}Q(j+kz, y)^{-s}$$
$$= \chi(a)L(2s, \chi_1)\prod_{\substack{p|k_0\\ p\ \text{prime}}}\big(1-\chi_1(p)/p^{2s}\big)+$$
$$+a^s\sum_{y=1}^{\infty}\sum_{j=1}^{k}\chi\big(Q(j, y)\big)\sum_{x}\int_{-\infty}^{\infty}Q(j+kz, y)^{-s}e^{-2\pi ixz}dz,$$

where we have used Poisson's summation formula to evaluate the sum over z, this being allowable for $\mathrm{Re}\, s > 1$. If we let

$$\text{(19)} \qquad R = \frac{\sqrt{|d|}}{2a}$$

and make the substitution

$$j+kz+\frac{b}{2a}y = Ryu$$

in the integral, we get

$$a^s\int_{-\infty}^{\infty}Q(j+kz, y)^{-s}e^{-2\pi ixz}dz = \frac{1}{k}(Ry)^{1-2s}e_k\left(xj+\frac{b}{2a}xy\right)\int_{-\infty}^{\infty}\frac{e_k(-xyRu)}{(u^2+1)^s}du.$$

Therefore

$$a^s\sum_{y=1}^{\infty}\sum_{j=1}^{k}\chi(Q(j,y))\sum_{x}\int_{-\infty}^{\infty}Q(j+kz,y)^{-s}e^{-2\pi ixz}dz \tag{20}$$

$$=\frac{1}{k}R^{1-2s}\sum_{y=1}^{\infty}\sum_{x}y^{1-2s}\sum_{j=1}^{k}\chi(Q(j,y))e_k\left(xj+\frac{b}{2a}xy\right)\int_{-\infty}^{\infty}\frac{e_k(-xyRu)}{(u^2+1)^s}du$$

$$=\frac{1}{k}R^{1-2s}\sum_{n}e_{2ak}(bn)\int_{-\infty}^{\infty}\frac{e_k(-nRu)}{(u^2+1)^s}du\sum_{\substack{y|n\\ y>0}}y^{1-2s}\sum_{j=1}^{k}\chi(Q(j,y))e_k(jn/y).$$

By Lemma 1, the $n=0$ term in (20) is

$$\frac{1}{k}R^{1-2s}\frac{\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}\sum_{y=1}^{\infty}y^{1-2s}\sum_{j=1}^{k}\chi(Q(j,y))$$

and the $n\neq 0$ terms combine to give

$$\frac{1}{\Gamma(s)}\left(\frac{kR}{\pi}\right)^{-s}H(s).$$

Thus the expansion in Lemma 3 follows from (18), (19) and (20).

It remains to show that $H(s)$ is entire. By Lemma 2, if s is in a bounded region B of the s plane and $n\neq 0$,

$$\left|K_{s-1/2}\left(\frac{\pi|n|\sqrt{|d|}}{ak}\right)\right|<ce^{-\pi|n|\sqrt{|d|}/ak}$$

where c is a positive real number depending on B but not on n. Therefore the series in (13) converges absolutely and uniformly on B and thus $H(s)$ is an analytic function on B. Since B is arbitrary, $H(s)$ is an entire function.

LEMMA 4.

$$\sum_{j=1}^{k}\bar{\chi}^2(j)e_k(nj)=\tau(\bar{\chi}_1)\sum_{f|(n,k_0)}f\mu\left(\frac{k_0}{f}\right)\bar{\chi}_1\left(\frac{k_0}{f}\right)\chi_1\left(\frac{n}{f}\right)$$

where $\mu(n)$ is the Möbius function. When χ is a real character, this is a well known formula for Ramanujan's sum ([4], p. 237).

Proof. Let

$$S(n)=\sum_{f|(n,k_0)}f\mu\left(\frac{k_0}{f}\right)\bar{\chi}_1\left(\frac{k_0}{f}\right)\chi_1\left(\frac{n}{f}\right). \tag{21}$$

We note that $f|(n,k_0)$ if and only if $f|(n+k,k_0)$. Also, if $f|(n,k_0)$ then

$$\chi_1\left(\frac{n+k}{f}\right)=\chi_1\left(\frac{n}{f}+\frac{k_0}{f}k_1\right)=\chi_1\left(\frac{n}{f}\right)$$

and thus

$$S(n+k)=S(n).$$

Thus we may expand $S(n)$ in a finite Fourier series,

$$S(n)=\sum_{j=1}^{k}a_je_k(nj) \tag{22}$$

where

$$a_j=\frac{1}{k}\sum_{n=1}^{k}S(n)e_k(-nj) \tag{23}$$

$$=\frac{1}{k}\sum_{n=1}^{k}\sum_{\substack{f|n\\ f|k_0}}f\mu\left(\frac{k_0}{f}\right)\bar{\chi}_1\left(\frac{k_0}{f}\right)\chi_1\left(\frac{n}{f}\right)e_k(-nj)$$

$$=\frac{1}{k}\sum_{f|k_0}f\mu\left(\frac{k_0}{f}\right)\bar{\chi}_1\left(\frac{k_0}{f}\right)\sum_{\substack{n=1\\ f|n}}^{k}\chi_1\left(\frac{n}{f}\right)e_k(-nj)$$

$$=\frac{1}{k_1}\sum_{f|k_0}\frac{1}{f}\mu(f)\bar{\chi}_1(f)\sum_{m=1}^{k_1f}\chi_1(m)e_{k_1f}(-mj)$$

and we have replaced f by k_0/f in the last step. But if we set

$$m=r+vk_1,\quad 0\leqslant v\leqslant f-1,\quad 1\leqslant r\leqslant k_1$$

then

$$\sum_{m=1}^{k_1f}\chi_1(m)e_{k_1f}(-mj)=\sum_{r=1}^{k_1}\chi_1(r)e_{k_1f}(-rj)\sum_{v=0}^{f-1}e_f(-vj)$$

$$=\begin{cases} f\sum_{r=1}^{k_1}\chi_1(r)e_{k_1}(-rj/f) & \text{if } f|j,\\ 0 & \text{if } f\nmid j.\end{cases}$$

Therefore

$$(24)\qquad a_j = \frac{1}{k_1}\sum_{\substack{f|k_0\\ f|j}}\mu(f)\bar{\chi}_1(f)\sum_{r=1}^{k_1}\chi_1(r)e_{k_1}(-rj/f)$$

$$= \frac{\tau(\chi_1)}{k_1}\sum_{f|(k_0,j)}\mu(f)\bar{\chi}_1(f)\bar{\chi}_1(-j/f) = \frac{\bar{\chi}_1(-j)\tau(\chi_1)}{k_1}\sum_{f|(k_0,j)}\mu(f)$$

$$= \frac{\bar{\chi}_1(-j)\tau(\chi_1)}{k_1}\cdot\begin{cases}0 & \text{if}\quad (j,k_0)>1\\ 1 & \text{if}\quad (j,k_0)=1\end{cases}$$

$$= \frac{\bar{\chi}_1(-j)\tau(\chi_1)}{k_1}\cdot\begin{cases}0 & \text{if}\quad (j,k)>1\\ 1 & \text{if}\quad (j,k)=1\end{cases}$$

$$= \frac{\bar{\chi}_1(-j)\tau(\chi_1)}{k_1}\chi_0(j) = \frac{\tau(\chi_1)}{k_1}\bar{\chi}^2(j)$$

by (2) and (3). The lemmas follows from (5), (21), (22) and (24).

Lemma 5. *If χ is a primitive character* (mod k) *and* $(k,d)=1$ *then for* $\operatorname{Re}s>1$,

$$\sum_{y=1}^{\infty}y^{1-2s}\sum_{j=1}^{k}\chi\big(Q(j,y)\big) = a\bar{\chi}(a)\tau(\bar{\chi}_1)k_0^{2-2s}L(2s-1,\chi_1)\prod_{\substack{p|k_0\\ p\text{ prime}}}\big(1-\bar{\chi}_1(p)p^{2s-2}\big).$$

Proof. By Theorem 1 and Lemma 4,

$$\sum_{y=1}^{\infty}y^{1-2s}\sum_{j=1}^{k}\chi\big(Q(j,y)\big) = a\sum_{y=1}^{\infty}y^{1-2s}\sum_{j=1}^{k}\bar{\chi}\big(Q(j,0)\big)e_k(jy)$$

$$= a\bar{\chi}(a)\sum_{y=1}^{\infty}y^{1-2s}\sum_{j=1}^{k}\bar{\chi}^2(j)e_k(jy)$$

$$= a\bar{\chi}(a)\tau(\bar{\chi}_1)\sum_{y=1}^{\infty}y^{1-2s}\sum_{f|(y,k_0)}f\mu\Big(\frac{k_0}{f}\Big)\bar{\chi}_1\Big(\frac{k_0}{f}\Big)\chi_1\Big(\frac{y}{f}\Big).$$

Let

$$g(f) = \begin{cases}1 & \text{if}\quad f\,|\,k_0,\\ 0 & \text{if}\quad f\nmid k_0.\end{cases}$$

Then

$$\sum_{y=1}^{\infty}y^{1-2s}\sum_{j=1}^{k}\chi\big(Q(j,y)\big) = a\bar{\chi}(a)\tau(\bar{\chi}_1)\sum_{y=1}^{\infty}y^{1-2s}\sum_{f|y}\chi_1\Big(\frac{y}{f}\Big)\cdot f\mu\Big(\frac{k_0}{f}\Big)\bar{\chi}_1\Big(\frac{k_0}{f}\Big)g(f)$$

$$= a\bar{\chi}(a)\tau(\bar{\chi}_1)L(2s-1,\chi_1)\sum_{f=1}^{\infty}f^{1-2s}f\mu\Big(\frac{k_0}{f}\Big)\bar{\chi}_1\Big(\frac{k_0}{f}\Big)g(f)$$

and

$$\sum_{f=1}^{\infty}f^{2-2s}\mu\Big(\frac{k_0}{f}\Big)\bar{\chi}_1\Big(\frac{k_0}{f}\Big)g(f) = \sum_{f|k_0}f^{2-2s}\mu\Big(\frac{k_0}{f}\Big)\bar{\chi}_1\Big(\frac{k_0}{f}\Big) = k_0^{2-2s}\sum_{f|k_0}\frac{\mu(f)\bar{\chi}_1(f)}{f^{2-2s}}$$

$$= k_0^{2-2s}\prod_{\substack{p|k_0\\ p\text{ prime}}}\big(1-\bar{\chi}_1(p)p^{2s-2}\big).$$

4. Proof of Theorem 2 and corollaries. The expansion (12) is an immediate consequence of Lemmas 3 and 5 and we have shown that $H(s)$ is entire in Lemma 3. We use Theorem 1 to prove (14). It follows from (10) that for all $x>0$,

$$(25)\qquad K_{s-1/2}(x) = K_{(1-s)-1/2}(x).$$

By Theorem 1, for $n>0$,

$$(26)\qquad n^{s-1/2}\sum_{\substack{y|n\\ y>0}}y^{1-2s}\sum_{j=1}^{k}\chi\big(Q(j,y)\big)e_k(jn/y)$$

$$= an^{s-1/2}\sum_{\substack{y|n\\ y>0}}y^{1-2s}\sum_{j=1}^{k}\bar{\chi}\big(Q(j,n/y)\big)e_k(jy)$$

$$= an^{(1-s)-1/2}\sum_{\substack{y|n\\ y>0}}y^{1-2(1-s)}\sum_{j=1}^{k}\bar{\chi}\big(Q(j,y)\big)e_k(jn/y),$$

where we have replaced y by n/y in the last step. By Theorem 1, for $n<0$,

$$(27)\qquad |n|^{s-1/2}\sum_{\substack{y|n\\ y>0}}y^{1-2s}\sum_{j=1}^{k}\chi\big(Q(j,y)\big)e_k(jn/y)$$

$$= a|n|^{s-1/2}\sum_{\substack{y|n\\ y>0}}y^{1-2s}\sum_{j=1}^{k}\chi\big(Q(j,n/y)\big)e_k(jy)$$

$$= a|n|^{1/2-s}\sum_{\substack{y|n\\ y>0}}y^{2s-1}\sum_{j=1}^{k}\chi\big(Q(j,-y)\big)e_k(-jn/y)$$

$$= a|n|^{(1-s)-1/2}\sum_{\substack{y|n\\ y>0}}y^{1-2(1-s)}\sum_{j=1}^{k}\chi\big(Q(j,y)\big)e_k(jn/y),$$

where we have replaced y by $-n/y$ and j by $-j$ in the last two steps. Equation (14) follows from (13), (25), (26), (27). This completes the proof of Theorem 2.

The functional equation of $L(s, \chi_1)$ is ([3], p. 71; recall $\chi_1(-1) = 1$),

$$\left(\frac{k_1}{\pi}\right)^{(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) L(1-s, \bar{\chi}_1) = \frac{\tau(\bar{\chi}_1)}{k_1^{1/2}} \left(\frac{k_1}{\pi}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi_1).$$

When we replace s in this functional equation by $2s-1$, we get

$$\left(\frac{k_1}{\pi}\right)^{1-s} \Gamma(1-s) L(2-2s, \bar{\chi}_1) = \tau(\bar{\chi}_1) \pi^{1/2-s} k_1^{s-1} \Gamma\left(s-\frac{1}{2}\right) L(2s-1, \chi_1).$$

It follows from this and Theorem 2 that

$$(28) \quad \left(\frac{k\sqrt{|d|}}{2\pi}\right)^s \Gamma(s) L(s, \chi, Q) = \chi(a)\left(\frac{k\sqrt{|d|}}{2a\pi}\right)^s \Gamma(s) L(2s, \chi_1) \prod_{\substack{p|k_0 \\ p \text{ prime}}} (1-\chi_1(p)p^{-2s}) + a\bar{\chi}(a)\left(\frac{k\sqrt{|d|}}{2a\pi}\right)^{1-s} \Gamma(1-s) L(2-2s, \bar{\chi}_1) \prod_{\substack{p|k_0 \\ p \text{ prime}}} (1-\bar{\chi}_1(p)p^{2s-2}) + H(s, \chi).$$

Now if we replace s by $1-s$ and χ by $\bar{\chi}$ in (28) and then multiply both sides by a, then the right side of the new equation is identical to the right side of (28). This is because of (14) and the fact that $a\bar{a} = 1$ where $\bar{a}$ is not only the complex conjugate of a but is also the number defined in (8) when χ is replaced by $\bar{\chi}$. This proves Corollary 1.

The proof of Corollary 2 is also simple. First, when χ is a real primitive character then $a = 1$ always. This follows from the fact that the only primitive real characters are the Kronecker symbols

$$\left(\frac{q'}{j}\right), \quad \left(\frac{-4q'}{j}\right), \quad \left(\frac{8q}{j}\right), \quad \left(\frac{-8q}{j}\right) \qquad q' = (-1)^{(q-1)/2}q,$$

where q is an odd positive square-free integer and the corresponding moduli are q, $4q$, $8q$ and $8q$ respectively ([3], p. 42). When χ is real, (17) follows instantly from (13). Lastly, if χ is real, $y|n$ and $a = 1$ then

$$\sum_{j=1}^{k} \chi(Q(j, y)) e_k(jn/y) e_{2k}(bn) = \sum_{j=1}^{k} \chi(Q(j-by, y)) e_k\left(\frac{n(j-by)}{y}\right) e_{2k}(bn) = \sum_{j=1}^{k} \chi(Q(-j, y)) e_k(jn/y) e_{2k}(-bn) = \sum_{j=1}^{k} \chi(Q(j, y)) e_k(-jn/y) e_{2k}(-bn)$$

so that $\sum_{j=1}^{k} \chi(Q(j, y)) e_k(jn/y) e_{2k}(bn)$ is equal to its own complex conjugate and is hence real. This last fact was discovered accidentally in [6] in certain cases with $k = 8$ and 12.

5. Concluding remarks. The expansion of Theorem 2 has already proved very useful. In particular, the point $s = 1$ plays an important role and we should expect an analogue of Kronecker's limit formula when χ is real. Such a formula does in fact exist with $L(1, \chi, Q)$ coming out in terms of logarithms of algebraic numbers. In another vein, after Lemma 3 we should not be surprised to learn that Theorems 1 and 2 may be generalized to other cases where $(k, d) \neq 1$ or χ is not primitive (mod k). This is indeed possible although the form of (11), (12), (14), and (15) change in the generalizations. These results will appear in future papers.

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Added in proof. Besides the application to [6], further uses of the expansions in Theorem 2 and Corollary 2 will appear shortly in [8], [9], [10]. In particular, we see in [9] that (16) is perhaps even more valuable with composite k than with prime k.

Additional references

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