and so this has only the non-negative integer solutions

$$x_0 = 0,\ x_1 = 1,\ x_2 = 2.$$

It might be of interest to find similar equations with four or more solutions.

An instance when $k = 2$ is given by

$$(17)\quad y^2+2l^2 = \big((8p+2)x^2-8q-3\big)(rx^2-s),\quad p \geqslant 0,\ q \geqslant 0,\ r > 0,\ s > 0,$$

where we suppose l has no prime factors $\equiv 5, 7 \pmod 8$. The first factor if positive excludes both $x \equiv 0 \pmod 2$ and $x \equiv 1 \pmod 2$.

If $(x, y) = (0, y_0), (1, y_1)$ are solutions, then

$$y_0^2+2l^2 = (8q+3)s,\quad y_1^2+2l^2 = (8q-8p+1)(s-r).$$

Hence

$$s = \frac{y_0^2+2l^2}{8q+3},\quad r = s - \frac{y_1^2+2l^2}{8q-8p+1}.$$

Take $l = 1, p = q = 0, y_0 = 8, s = 22, r = 20-y_1^2$. Then

$$y^2+2 = (2x^2-3)\big((20-y_1^2)x^2-22\big)$$

has only the solutions $(x, y) = (0, \pm 8), (\pm 1, \pm y_1)$.

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On ratio sets of sets of natural numbers

by

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Let us denote by N (C and R^+ respectively) the set of all natural numbers (all integral numbers and all positive rational numbers respectively). If $A \subset N$, $A \neq \emptyset$, then we put

$$D(A) = \{x \in C;\ \underset{c,d \in A}{\exists}\ x = c-d\},$$

$$R(A) = \Big\{x \in R^+;\ \underset{c,d \in A}{\exists}\ x = \frac{c}{d}\Big\}.$$

$D(A)$ is the set of differences of numbers of the set A and $R(A)$ is the ratio set of the set A.

In the paper [3] it is proved that $D(A) = C$ if the upper asymptotic density of the set A is greater than 1/2. It is even proved in that paper that in this case (that is if the upper asymptotic density of A is greater than 1/2) the following holds: for each $x \in C$ there exists an infinite number of pairs (c, d) of numbers of the set A such that $x = c-d$.

Let us remark that the condition $\delta_2(A) > 1/2$ ($\delta_2(A)$ denotes the upper asymptotic density of the set A) it is only a sufficient condition for the equality $D(A) = C$ to be true. E.g. if $A = \{1, 2, 4, \ldots, 2n, \ldots\}$, then we have obviously $\delta_2(A) = 1/2$ ($= \delta(A)$, $\delta(A)$ denotes the asymptotic density of the set A) and simultaneously $D(A) = C$.

We shall prove in this paper a theorem on the ratio sets which is analogous to the above mentioned theorem of Professor W. Sierpiński (see Theorem 1) and then we shall study some properties of $A \subset N$ which quarantee the density of $R(A)$ in the interval $\langle 0, +\infty)$.

THEOREM 1. *Let $\delta_2(A) = 1$. Then for each $x \in R^+$ there exists an infinite number of pairs (c, d) of numbers of the set A such that $x = c/d$.*

COROLLARY. *If $\delta_2(A) = 1$, then $R(A) = R^+$.*

Proof of the theorem. Let $\delta_2(A) = 1$. Let us suppose that the assertion of the theorem is not true. Then there exists a positive rational number $r = \frac{p}{q} \neq 1$, $(p, q) = 1$ such that $r = \frac{c}{d}$ only for a finite number of pairs (c, d) of numbers of the set A.

Let (c_i, d_i) $(i = 1, 2, \ldots, m)$ be all the pairs of numbers of the set A for which $r = \frac{c_i}{d_i}$ $(i = 1, 2, \ldots, m)$. Let us put $a = \max(c_1, \ldots, c_m, d_1, \ldots \ldots, d_m)$. Let us form the sequence

$$(1) \qquad a+1, a+2, \ldots, n \quad (n > a).$$

It follows from the definition of the number a that the quotient of any two numbers of the set A belonging to sequence (1) is different from r.

To sequence (1) belong all the multiples lp of the number p, where $\frac{a}{p} < l \leqslant \frac{n}{p}$, and all the multiples sq of the number q, where $\frac{a}{q} < s \leqslant \frac{n}{q}$. Let us put $d = \max(p, q)$, $d' = \min(p, q)$. Then the numbers ip, iq belong to (1) if

$$(2) \qquad \frac{a}{d'} < i \leqslant \frac{n}{d}.$$

Because the quotient of each two numbers of A belonging to (1) is different from r and $\frac{ip}{iq} = r$, at least one of the numbers ip, iq need not belong to A if i fulfils the inequalities (2).

Let us denote by M_1 (M_2) the set of all numbers i which fulfil inequalities (2) and for which simultaneously $ip \notin A$ $(iq \notin A)$. Hence we have

$$(3) \qquad P(M_1)+P(M_2) \geqslant \left[\frac{n}{d}\right] - \left[\frac{a}{d'}\right],$$

where $P(M_j)$ $(j = 1, 2)$ denotes the number of elements of the set M_j. It follows from (3) that at least one of the numbers $P(M_1)$, $P(M_2)$ is not smaller than $\frac{1}{2}\left(\left[\frac{n}{d}\right] - \left[\frac{a}{d'}\right]\right)$ and so from the definition of the sets M_1, M_2 we obtain for $A(n) = \sum_{l \in A, l \leqslant n} 1$ the inequality

$$A(n) \leqslant n - \frac{1}{2}\left(\left[\frac{n}{d}\right] - \left[\frac{a}{d'}\right]\right) \leqslant n - \frac{n}{2d} + \frac{1}{2}\left[\frac{a}{d'}\right] + \frac{1}{2};$$

from this we get

$$\delta_2(A) = \limsup_{n\to\infty} \frac{A(n)}{n} \leqslant 1 - \frac{1}{2d} < 1.$$

This is a contradiction of the assumption of the theorem. The proof is complete.

Let us remark that the assumption $\delta_2(A) = 1$ is only a sufficient condition for the equality $R(A) = R^+$ to be true. E.g. let $A = \{2, 4, \ldots \ldots, 2n, \ldots\}$. Then $\delta_2(A) = 1/2$ and simultaneously we have $R(A) = R^+$. There even exist sets $A \subset N$ of asymptotic density 0 such that $R(A) = R^+$. Such a set is the set of the terms of the sequence $\{\varphi(n)\}_{n=1}^{\infty}$, φ being Euler's function (see [4], pp. 235–236, [2]).

We shall show now that number 1 in the assumption of the foregoing theorem is the best possible; it cannot be replaced by any smaller number.

THEOREM 2. *For each* ε, $0 < \varepsilon < 1$, *there exists a set* $A \subset N$ *such that* $\delta_2(A) > 1-\varepsilon$ *and simultaneously there exists an interval* $I \subset (0, +\infty)$ *such that* $I \cap R(A) = \emptyset$.

Proof. Let $0 < \varepsilon < 1$. Let us choose a natural number s for which $1/s < \varepsilon$. Let us put $A = \bigcup_{k=s}^{\infty} A_k$, where

$$A_k = \{(2k+1)^{2k+1}+1, (2k+1)^{2k+1}+2, \ldots, s(2k+1)^{2k+1}\} \qquad (k = s, s+1, \ldots).$$

Then we obviously have

$$A\left(s(2k+1)^{2k+1}\right) \geqslant (s-1)(2k+1)^{2k+1} \qquad (k = s, s+1, \ldots)$$

and so

$$\frac{A\left(s(2k+1)^{2k+1}\right)}{s(2k+1)^{2k+1}} \geqslant \frac{s-1}{s} > 1-\varepsilon \qquad (k = s, s+1, \ldots).$$

This requires that $\delta_2(A) \geqslant 1 - \frac{1}{s} > 1-\varepsilon$.

Let $I = \left(s, \frac{(2s+3)^2}{s}\right)$. We prove that $I \cap R(A) = \emptyset$. Let $c, d \in A$, $c \geqslant d$. Then we have the following two possibilities:

(a) There exists a number $k \geqslant s$ such that $c, d \in A_k$.

(b) $c \in A_l$, $d \in A_j$, $l \neq j$.

Ad (a). Obviously we have

$$\frac{c}{d} \leqslant \frac{s(2k+1)^{2k+1}}{(2k+1)^{2k+1}} = s.$$

Ad (b). Because $c \geqslant d$, we must have $l > j$ and so $l \geqslant j+1$. But then we have

$$\frac{c}{d} \geqslant \frac{(2j+3)^{2j+3}}{s(2j+1)^{2j+1}} \geqslant \frac{(2s+3)^2}{s}.$$

The author thanks Professor P. Erdös for the remark that the following, slightly weaker theorem, can easily be proved.

THEOREM 2'. *For each* ε, $0 < \varepsilon < 1$, *there exists a set* $A \subset N$ *such that* $\delta_2(A) > 1-\varepsilon$ *and* $R(A) \neq R^+$.

Proof. Let $0 < \varepsilon < 1$ and let p be a prime number with $1/p < \varepsilon$. Let A denote the set of all natural numbers which are not divisible by the prime number p. Then A has the asymptotic density $1-\frac{1}{p} > 1-\varepsilon$ and obviously $R(A) \neq R^+$.

Let us remark that the set $R(A)$ of Theorem 2' is dense in $\langle 0, +\infty)$, so that we cannot find in this case any interval $I \subset (0, +\infty)$ with $I \cap R(A) = \emptyset$ (see Theorem 2).

In what follows we shall study some sufficient conditions for the density of the set $R(A)$ in the interval $\langle 0, +\infty)$. We shall show that the class of all sets $A \subset N$ for which $R(A)$ is a dense set in $\langle 0, +\infty)$ contains every set with positive asymptotic density.

THEOREM 3. *Let the set* $A \subset N$ *satisfy the following condition: for each* a, b; $0 < a < b$, *we have*

$$\liminf_{n\to\infty} \frac{A(bn)}{A(an)} > 1.$$

Then $R(A)$ *is a dense set in* $\langle 0, +\infty)$.

Proof. It follows from the assumption of the theorem that A is an infinite set. Let $0 < a < b$. It suffices to prove that the intersection of the set $R(A)$ with the interval $(a, b\rangle$ is non-empty.

Considering the assumption of the theorem there exists a natural number n_0 such that for $n > n_0$ we have $\frac{A(bn)}{A(an)} > 1$. Because A is an infinite set, there exists a $q \in A$ such that $q > n_0$. For this number q the inequality $A(bq)-A(aq) > 0$ is true. Thus there exists a number $p \in A$ such that $aq < p \leqslant bq$ and so we have

$$a < \frac{p}{q} \leqslant b, \quad \frac{p}{q} \in R(A).$$

THEOREM 4. *If the set* $A \subset N$ *has a positive asymptotic density, then the set* $R(A)$ *is a dense set in* $\langle 0, +\infty)$.

Proof. Let

$$\delta = \delta(A) = \lim_{n\to\infty} \frac{A(n)}{n} > 0.$$

On account of the foregoing theorem it suffices to prove that for each a, b; $0 < a < b$, the following inequality

$$\liminf_{n\to\infty} \frac{A(bn)}{A(an)} > 1 \tag{4}$$

is true.

Let us choose an ε such that

$$0 < \varepsilon < \frac{\delta(b-a)}{a+b}. \tag{5}$$

Then there exists an $x_0 > 0$ such that for $x > x_0$ we have

$$(\delta-\varepsilon)x < A(x) < (\delta+\varepsilon)x. \tag{6}$$

Let us choose a n_0 such that for $n > n_0$ we have $an > x_0$. Then with the use of a simple estimation we obtain for $n > n_0$ with the aid of (5) and (6)

$$\frac{A(bn)}{A(an)} > \frac{(\delta-\varepsilon)bn}{(\delta+\varepsilon)an} = \frac{(\delta-\varepsilon)b}{(\delta+\varepsilon)a} > 1.$$

From this (4) follows immediately.

EXAMPLE. Let

$$A(x) \sim \frac{c_1 x}{\log^a x}, \quad c_1 > 0, \ a > 0.$$

Then it is easy to see that for the set A the relation

$$\lim_{n\to\infty} \frac{A(bn)}{A(an)} = \frac{b}{a} > 1 \quad (0 < a < b)$$

holds. It follows from Theorem 3 that the set $R(A)$ is dense in $\langle 0, +\infty)$. Especially it follows from this on account of the prime number theorem that $R(P)$ is dense in $\langle 0, +\infty)$, P being the set of all prime numbers (see [4], p. 155).

Further, if for the number $P_2(x)$ of prime-pairs $p, p+2$ with $p \leqslant x$ the hypothesis

$$P_2(x) \sim \frac{2c_2 x}{\log^2 x} \quad (c_2 > 0)$$

holds (see [1], p. 412) and if P^* is the union of all sets $\{p, p+2\}$, where $p, p+2$ are prime numbers, then obviously $P^* \subset P$ and simultaneously $R(P^*)$ is dense in $\langle 0, +\infty)$.

Let us remark finally that in Theorem 4 the assumption $\delta(A) > 0$ cannot be replaced by the following weaker assumption:

$$\delta_1(A) = \liminf_{n\to\infty} \frac{A(n)}{n} > 0.$$

This can be seen from the following example:

Let $A = \bigcup_{k=0}^{\infty} A_k$, where

$$A_k = \{2^{k+1}+1, 2^{k+1}+2, \ldots, 2^{k+1}+2^k\} \quad (k = 0, 1, \ldots).$$

It is easy to see that $\delta_1(A) = \frac{1}{2}$, $\delta_2(A) = \frac{3}{4}$ and it can easily be proved that $(\frac{3}{2}, \frac{4}{3}) \cap R(A) = \emptyset$.

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An effective p-adic analogue of a theorem of Thue

by

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I. Introduction. A famous theorem of Thue [11] states that the diophantine equation

$$f(x, y) = m, \tag{1}$$

where f denotes an irreducible binary form with integer coefficients and degree at least 3, and m is any integer, possesses only a finite number of solutions in integers x, y. Thue's theorem was extended by Siegel [10], both with regard to the basic result obtained by Thue on rational approximations to algebraic numbers, from which the theorem referred to above followed as a corollary, and in connexion with generalizations to integer solutions of equations in algebraic number fields. This work gave rise to many further developments; in particular Mahler [5], [6], [7], using Siegel's methods, established far-reaching p-adic analogues of the original theorems, and, in 1955, Roth [9] succeeded in establishing a profound improvement on the work of Thue-Siegel, giving a best possible approximation inequality.

All the work described above, however, is non-effective, in that although it establishes the finiteness of the number of solutions of diophantine equations of the type (1), it does not yield an effective algorithm for their explicit determination. In a recent paper [3], Baker gave the first effective proof of Thue's original theorem, obtaining thereby an explicit upper bound for the size of all integer solutions x, y of (1). The object of the present paper is to prove, by means of Baker's method, certain effective p-adic analogues of Thue's theorem, similar to those first obtained by Mahler in a non-effective form. As above, $f(x, y)$ will signify a binary form with integer coefficients and degree $n \geqslant 3$, irreducible over the rationals, and m will signify a non-zero integer. By $p_1, \ldots, p_s$ we shall denote a fixed set of s prime numbers, and we shall use $\mathfrak{m}$ to denote the largest integer, comprised solely of powers of $p_1, \ldots, p_s$, which divides m. Further, we shall suppose that $\varkappa$ is any number satisfying

$$\varkappa > n(s+1)+1. \tag{2}$$