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An improved estimate for the summatory function of the Möbius function

by

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1. Introduction. Let $\mu(n)$ be the Möbius function and let $M(x) = \sum_{1\leqslant n\leqslant x}\mu(n)$ be its summatory function (where x may assume non-integral values). Then MacLeod [4] has shown that

(1) $$|M(x)| < x/80 \quad \text{if} \quad x \geqslant 1119$$

and hence that(1)

(2) $$|M(x)| \leqslant \frac{x+1}{80} + \frac{11}{2} \quad \text{if} \quad x \geqslant 0.$$

There is also the special result of Neubauer [6] that

(3) $$|M(x)| < \tfrac{1}{2}\sqrt{x} \quad \text{if} \quad 201 \leqslant x \leqslant 10^8.$$

Inasmuch as Walfisz [9], p. 191, has shown that there exists a constant $a > 0$ such that

(4) $$M(x) = O\left(x\exp\{-a(\log x)^{3/5}(\log\log x)^{-1/5}\}\right)$$

as $x\to\infty$, it is clear that (1) is weak for large x. Moreover, (1) is not even as strong as the prime number theorem which is equivalent to $M(x) = o(x)$ as $x\to\infty$. In this paper, we prove the following stronger results:

(5) $$|M(x)| < \frac{1.2x}{(\log x)^{2/3}} \quad \text{if} \quad x > 1,$$

(6) $$|M(x)| < \frac{12x}{\log x} \quad \text{if} \quad x > 1,$$

(7) $$|M(x)| < \frac{26x}{(\log x)^{10/9}} \quad \text{if} \quad x > 1.$$

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(1) MacLeod states that $|M(x)| < x/80+5$ for all x, but this is false if $199 < x < 201$ where $M(x) = -8$.

These proofs depend on various inequalities given by Rosser and Schoenfeld [7].

In fact, it is possible to improve the coefficients in (5)-(7) by using the recently announced results of Rosser, Schoenfeld and Yohe [8] that the first 3500000 zeros of the Riemann zeta-function are all on the critical line. The above inequalities can then be replaced by:

$$|M(x)| < \frac{.47x}{(\log x)^{2/3}} \quad \text{if} \quad x \geqslant 6, \tag{5'}$$

$$|M(x)| < \frac{2.9x}{\log x} \quad \text{if} \quad x > 1, \tag{6'}$$

$$|M(x)| < \frac{5.3x}{(\log x)^{10/9}} \quad \text{if} \quad x > 1. \tag{7'}$$

Consequently,

$$|M(x)| < \frac{.55x}{(\log x)^{2/3}} \quad \text{if} \quad x > 1. \tag{5''}$$

2. Several lemmas. We begin with the following result.

LEMMA 1. *If $x \geqslant 1$, then*

$$0 < \sum_{1\leqslant n\leqslant x} \frac{|\mu(n)|}{n} - \frac{6}{\pi^2}\log x < 2. \tag{8}$$

Proof. Let $Q(x)$ and $R(x)$ be defined by

$$Q(x) = \sum_{1\leqslant n\leqslant x} |\mu(n)|, \qquad R(x) = Q(x) - 6x/\pi^2$$

so that $Q(x)$ is the number of square-free integers not exceeding x. Standard techniques, based on integration by parts, give

$$\sum_{n\leqslant x} \frac{|\mu(n)|}{n} = \frac{Q(x)}{x} + \int_1^x \frac{Q(u)}{u^2}\,du.$$

If $C > 0$ and $x \geqslant C$, then

$$\sum_{n\leqslant x} \frac{|\mu(n)|}{n} = \sum_{n\leqslant C} \frac{|\mu(n)|}{n} - \frac{Q(C)}{C} + \frac{Q(x)}{x} + \int_C^x \frac{Q(u)}{u^2}\,du$$

$$= \sum_{n\leqslant C} \frac{|\mu(n)|}{n} - \frac{Q(C)}{C} + \frac{6}{\pi^2} + \frac{R(x)}{x} + \frac{6}{\pi^2}(\log x - \log C) + \int_C^x \frac{R(u)}{u^2}\,du$$

$$= \frac{6}{\pi^2}\log x + C_1 + \frac{R(x)}{x} + \int_C^x \frac{R(u)}{u^2}\,du$$

where

$$C_1 = \sum_{n\leqslant C} \frac{|\mu(n)|}{n} - \frac{Q(C)}{C} - \frac{6}{\pi^2}(\log C - 1).$$

Now Moser and MacLeod [5] have shown, among other things, that

$$|R(x)| \leqslant \tfrac{13}{18}\sqrt{x} + \tfrac{9}{2} \quad \text{if} \quad x \geqslant 0. \tag{9}$$

Hence, if $x \geqslant C > 0$, then

$$\left|\sum_{n\leqslant x} \frac{|\mu(n)|}{n} - \frac{6}{\pi^2}\log x - C_1\right|$$

$$\leqslant \frac{13}{18\sqrt{x}} + \frac{9}{2x} + \frac{13}{9}\left(\frac{1}{\sqrt{C}} - \frac{1}{\sqrt{x}}\right) + \frac{9}{2}\left(\frac{1}{C} - \frac{1}{x}\right) < \frac{13}{9\sqrt{C}} + \frac{9}{2C}.$$

On taking $C = 10$, we see that the last expression is less than .91 and that $.95 < C_1 < .96$. This implies (8) if $x \geqslant 10$; if $1 \leqslant x < 10$, it is easily seen that (8) still holds.

Clearly, the bounds 0 and 2 in (8) are not the best obtainable. (In fact, if in place of (9) we had used $|R(x)| < \frac{1}{2}\sqrt{x}$ which Moser and MacLeod [5] state is valid for $x \geqslant 8$ and if we had defined $C = 13$, then we would have obtained the bounds .68 and 1.25, valid for $x \geqslant 2$.) However, since the main expression in (8) has the limiting value

$$\frac{6}{\pi^2}\left\{\gamma - 2\frac{\zeta'(2)}{\zeta(2)}\right\} = 1.0438945\ldots$$

as $x \to \infty$, it is clear that our bounds can not be greatly improved; in any case, they are sufficient for our application. This limiting value is obtained as follows:

$$\sum_{n\leqslant x} \frac{|\mu(n)|}{n} = \sum_{n\leqslant x} \frac{1}{n} \sum_{k^2|n} \mu(k) = \sum_{k^2\leqslant x} \mu(k) \sum_{\substack{n\leqslant x\\ k^2|n}} \frac{1}{n}$$

$$= \sum_{k\leqslant\sqrt{x}} \frac{\mu(k)}{k^2} \sum_{m\leqslant x/k^2} \frac{1}{m} = \sum_{k\leqslant\sqrt{x}} \frac{\mu(k)}{k^2}\left\{\log\frac{x}{k^2} + \gamma + O\left(\frac{k^2}{x}\right)\right\}$$

$$= \{\log x + \gamma\}\left\{\sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} + O\left(\frac{1}{\sqrt{x}}\right)\right\} -$$

$$-2\left\{\sum_{k=1}^{\infty} \frac{\mu(k)}{k^2}\log k + O\left(\frac{\log x}{\sqrt{x}}\right)\right\} + O\left(\frac{1}{\sqrt{x}}\right)$$

$$= (\log x+\gamma)\frac{1}{\zeta(2)}+2\left\{\frac{d}{ds}\sum_{k=1}^{\infty}\frac{\mu(k)}{k^s}\right\}_{s=2}+O\left(\frac{\log x}{\sqrt{x}}\right)$$

$$= \frac{6}{\pi^2}(\log x+\gamma)+2\left\{\frac{d}{ds}\frac{1}{\zeta(s)}\right\}_{s=2}+O\left(\frac{\log x}{\sqrt{x}}\right)$$

$$= \frac{6}{\pi^2}\log x+\frac{6}{\pi^2}\left\{\gamma-2\frac{\zeta'(2)}{\zeta(2)}\right\}+O\left(\frac{\log x}{\sqrt{x}}\right)$$

and the numerical value of the constant term is obtained by using the value for $\zeta'(2)/\zeta(2)$ given in Table IV of Rosser and Schoenfeld [7].

LEMMA 2. *If* $x>1$, *then*

(10) $$\sum_{1\leqslant n\leqslant x}\frac{1}{n}|\Lambda(n)-1| < 2\log x-2\log\log x.$$

Proof. Let the above sum be $L(x)$ and, in what follows, let p denote a prime. Then for $x\geqslant 2$

$$L(x) = \sum_{p^k\leqslant x}\frac{1}{p^k}|\log p-1|+\sum_{\substack{n\leqslant x\\ n\neq p^k}}\frac{1}{n}$$

$$= \sum_{p^k\leqslant x}\frac{1}{p^k}|\log p-1|-\sum_{p^k\leqslant x}\frac{1}{p^k}+\sum_{n\leqslant x}\frac{1}{n}$$

$$= \sum_{p\leqslant x}\frac{1}{p}\{|\log p-1|-1\}+\sum_{\substack{p^k\leqslant x\\ k\geqslant 2}}\frac{1}{p^k}\{|\log p-1|-1\}+\sum_{n\leqslant x}\frac{1}{n}$$

$$= \sum_{p\leqslant x}\frac{\log p-2}{p}+\sum_{n\leqslant x}\frac{1}{n}+G(x)$$

where

$$G(x) = 1-\log 2+\sum_{\substack{p^k\leqslant x\\ k\geqslant 2}}\frac{1}{p^k}\{|\log p-1|-1\}$$

$$\leqslant 1-\log 2+\sum_{\substack{k\geqslant 2\\ p\geqslant 11}}\frac{\log p-2}{p^k} < .31+\sum_{n=11}^{\infty}\frac{\log n-2}{n(n-1)}$$

$$< .31+\sum_{n=11}^{\infty}\frac{\log(n-1)}{(n-1)^2}-2\sum_{n=11}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right)$$

$$\leqslant .31+\int_9^{\infty}\frac{\log u}{u^2}\,du-\frac{2}{10} = .11+\frac{\log 9+1}{9} < .47.$$

On applying (3.23), (2.11), (3.17) and (2.10) of Rosser and Schoenfeld [7], we find that for $x\geqslant 32$

$$L(x) < \left(\log x-1.33+\frac{1}{\log x}\right)-2\left(\log\log x+.26-\frac{1}{2\log^2 x}\right)+\sum_{n\leqslant x}\frac{1}{n}+G(x)$$

$$< \log x+\sum_{n\leqslant x}1/n-2\log\log x-1.38+1/\log x+1/\log^2 x.$$

Since the second term in the last expression does not exceed $\log x+1$, we see that (10) holds if $x\geqslant 32$. The verification for $1<x<32$ is easily completed.

COROLLARY. *If* $x>\sqrt{e}$, *then*

(11) $$\sum_{1\leqslant n\leqslant x}\frac{1}{n}|\Lambda(n)-1| < 2\log x.$$

LEMMA 3. *If* $x>1$, *then*

(12) $$|\psi(x)-x| < \frac{652x}{\log^2 x}.$$

Proof. If $1<x\leqslant 10^8$, then (4.5) of Rosser and Schoenfeld [7] gives

$$\psi(x)-x\geqslant\theta(x)-x > -2.06\sqrt{x} = -2.06\frac{\log^2 x}{\sqrt{x}}\cdot\frac{x}{\log^2 x}$$

$$\geqslant -2.06\left(\frac{4}{e}\right)^2\frac{x}{\log^2 x} > -\frac{4.5x}{\log^2 x}.$$

Likewise, it follows from (4.12) and (4.5) of Rosser and Schoenfeld that if $1<x\leqslant 10^8$ then

$$\psi(x)-x < \theta(x)-x+x^{1/2}+3x^{1/3} < x^{1/2}+3x^{1/3}$$

$$\leqslant\left(\frac{4}{e}\right)^2\frac{x}{\log^2 x}+3\left(\frac{3}{e}\right)^2\frac{x}{\log^2 x} < \frac{5.9x}{\log^2 x}.$$

Moreover, Theorem 11 of Rosser and Schoenfeld states that if $x\geqslant 2$ then

(13) $$|\psi(x)-x| < x(\log x)^{1/2}\exp\{-\sqrt{(\log x)/R}\}.$$

Hence, if $x \geqslant \exp 10000$, then

$$|\psi(x)-x| < \frac{x}{\log^2 x}(10000)^{5/2}\exp\{-\sqrt{10000/R}\} < \frac{x}{\log^2 x}$$

since $R < 17.52$. Therefore, if either $1 < x \leqslant 10^8$ or $x \geqslant \exp 10000$, then

$$(14) \qquad |\psi(x)-x| < \frac{5.9x}{\log^2 x}.$$

To handle the remaining range for x, we apply Table I of Rosser and Schoenfeld. This table lists, for various values of b, appropriate values of ε such that for $x \geqslant e^b$

$$|\psi(x)-x| < \varepsilon x = \varepsilon \log^2 x \cdot \frac{x}{\log^2 x}.$$

Hence,

$$|\psi(x)-x| < \varepsilon b_1^2 \cdot \frac{x}{\log^2 x} \quad \text{if} \quad e^b \leqslant x < e^{b_1}.$$

Taking b to be, in succession, 18.4, 175, 825, 1050, 1200, 1300, 1400, 1500, 1600, 1800, 2000, 2400, 3400, 5000 and letting b_1 be the next succeeding value of b with the last value of b_1 being 10 000, this Table I shows that $\varepsilon b_1^2 < 651.93$. Together with (14) this yields

$$(15) \qquad |\psi(x)-x| < \frac{651.93x}{\log^2 x} \quad \text{if} \quad x > 1.$$

This completes the proof.

Corollary. *Letting*

$$(16) \qquad \varrho(x) = \psi(x)-[x],$$

we have

$$(17) \qquad |\varrho(x)| < \frac{652x}{\log^2 x} \quad \textit{if} \quad x > 1.$$

Proof. This follows from (15) if $x \geqslant 600$ since

$$|\varrho(x)| < |\psi(x)-x|+1.$$

And if $1 < x < 600$, then (14) implies (17).

3. Proof of the main results. The estimation of $M(x)$ is accomplished by using results of the type (17). However, it is not easy to use such results because there does not seem to be any simple formula connecting the functions $M(x)$ and $\varrho(x)$. Instead, there is a relation expressed in terms of

$$(18) \qquad N(x) = \sum_{1\leqslant n\leqslant x} \mu(n)\log n.$$

To derive this relation, we note that

$$\left\{-\frac{\zeta'(s)}{\zeta(s)} - \zeta(s)\right\}\cdot\frac{1}{\zeta(s)} = \frac{d}{ds}\left\{\frac{1}{\zeta(s)}\right\} - 1.$$

We get, on expanding into Dirichlet series and equating coefficients,

$$\sum_{kj=n}\{\Lambda(k)-1\}\mu(j) = -\mu(n)\log n - \Delta_n$$

where $\Delta_1 = 1$ and $\Delta_n = 0$ for all $n > 1$. If $x \geqslant 1$, then on summing for $1 \leqslant n \leqslant x$ we find

$$\sum_{kj\leqslant x}\{\Lambda(k)-1\}\mu(j) = -N(x)-1.$$

If $y > 0$ is arbitrary, then a standard rearrangement gives

$$-N(x)-1 = \sum_{k\leqslant y}\{\Lambda(k)-1\}\sum_{j\leqslant x/k}\mu(j) + \sum_{j\leqslant x/y}\mu(j)\sum_{k\leqslant x/j}\{\Lambda(k)-1\} - \sum_{k\leqslant y}\{\Lambda(k)-1\}\sum_{j\leqslant x/y}\mu(j).$$

Thus, if $x \geqslant 1$ and $y > 0$

$$(19) \qquad -N(x)-1 = \sum_{k\leqslant y}\{\Lambda(k)-1\}M\left(\frac{x}{k}\right) + \sum_{j\leqslant x/y}\mu(j)\varrho\left(\frac{x}{j}\right) - \varrho(y)M\left(\frac{x}{y}\right)$$

which is the relation given in (44) and (45) on page 110 of Ayoub [1].

As we show below, (19) enables us to obtain an upper bound for $|N(x)|$ from bounds on $|M(x)|$. The reverse is also true since integration by parts gives for $x > 1$

$$(20) \qquad M(x) = \sum_{2\leqslant n\leqslant x}\mu(n)\log n\cdot\frac{1}{\log n} + 1 = \frac{N(x)}{\log x} + \int_2^x \frac{N(u)}{u\log^2 u}\,du + 1.$$

Hence, if $x \geqslant C > 1$, we have

$$(21) \qquad M(x) = M(C) - \frac{N(C)}{\log C} + \frac{N(x)}{\log x} + \int_C^x \frac{N(u)}{u\log^2 u}\,du.$$

Let us now assume that

(22) $$|M(u)| \leqslant \frac{Au}{g(u)} \quad \text{if } u \geqslant \frac{x}{y}, \qquad |\varrho(v)| < \frac{Bv}{h(v)} \quad \text{if } v \geqslant y$$

where A, B are constants and $g(u)$, $h(v)$ are positive, monotone increasing functions. Then for $x \geqslant 1$ and $y > 0$, we obtain from (19)

$$|N(x)+1| \leqslant Ax\sum_{k\leqslant y}\frac{|\Lambda(k)-1|}{kg(x/k)} + Bx\sum_{j\leqslant x/y}\frac{|\mu(j)|}{jh(x/j)} + \frac{ABx}{h(y)g(x/y)}$$

$$\leqslant \frac{Ax}{g(x/y)}\sum_{k\leqslant y}\frac{1}{k}|\Lambda(k)-1| + \frac{Bx}{h(y)}\sum_{j\leqslant x/y}\frac{|\mu(j)|}{j} + \frac{ABx}{h(y)g(x/y)}$$

$$\leqslant \frac{2Ax(\log y - \log\log y)}{g(x/y)} + \frac{Bx}{h(y)}\left(\frac{6}{\pi^2}\log\frac{x}{y}+2\right) + \frac{ABx}{h(y)g(x/y)}$$

as a result of (10) and (8) provided $x \geqslant y > 1$. Hence,

(23) $$|N(x)| < 2x\left\{\frac{A\log y}{g(x/y)} + \frac{3B\log(x/y)}{\pi^2 h(y)} - \frac{A\log\log y}{g(x/y)}k_0(x,y)\right\}$$

where

(24) $$k_0(x,y) = 1 - \frac{1}{\log\log y}\left\{\frac{Bg(x/y)}{Ah(y)} + \frac{B}{2h(y)} + \frac{g(x/y)}{2Ax}\right\}.$$

We begin by sketching the rest of the argument in the particularly simple case that the functions are given by

(25) $$g(u) = (\log u)^{\alpha}, \qquad h(v) = (\log v)^{\beta}$$

where α and β are non-negative. In this case, the first two terms in the braces of (23) are

$$\frac{A\log y}{\{\log(x/y)\}^{\alpha}} + \frac{3B\log(x/y)}{\pi^2(\log y)^{\beta}}.$$

If $\log y$ is small compared to $\log x$, then this sum is minimized by specifying that

(26) $$\log y = D(\log x)^{(\alpha+1)/(\beta+1)}$$

where

(27) $$D = \left(\frac{3B\beta}{\pi^2 A}\right)^{1/(\beta+1)}.$$

Regardless of the choice of D, if we define y by (26) and if $x > y > e$, then (23) and (24) yield

(28) $$|N(x)| < \frac{2x}{(\log x)^{(\alpha\beta-1)/(\beta+1)}}\left\{\frac{AD}{(1-u)^{\alpha}} + \frac{3B}{\pi^2 D^{\beta}}(1-u) - \frac{A\log\log y}{(1-u)^{\alpha}(\log x)^{(\alpha+1)/(\beta+1)}}k(x,y)\right\}$$

where

(29) $$u = \frac{\log y}{\log x} = \frac{D}{(\log x)^{(\beta-\alpha)/(\beta+1)}},$$

(30) $$k(x,y) = 1 - \frac{1}{\log\log y}\left\{\frac{B}{AD^{\beta}(\log x)^{(\beta-\alpha)/(\beta+1)}} + \frac{B}{2D^{\beta}(\log x)^{\beta(\alpha+1)/(\beta+1)}} + \frac{(\log x)^{\alpha}}{2Ax}\right\}.$$

If x is sufficiently large and $\beta > \alpha$, then $k(x,y) > 0$ and

(31) $$N(x) = O\left(x(\log x)^{1-\alpha_1}\right)$$

where

$$\alpha_1 = \frac{\beta}{\beta+1}(\alpha+1).$$

Consequently, (20) will yield

(32) $$M(x) = O\left(\frac{x}{(\log x)^{\alpha_1}}\right).$$

Inasmuch as we assumed $M(x) = O\left(x(\log x)^{-\alpha}\right)$ in (22) and (25), the result (32) is of exactly the same nature. Moreover, (32) will be an improved estimate if $\beta > \alpha$ since this implies $\alpha_1 > \alpha$. In fact, this procedure can be iterated to yield

(33) $$M(x) = O\left(\frac{x}{(\log x)^{\alpha_n}}\right)$$

where

(34) $$\alpha_n = \beta - (\beta-\alpha)\left(\frac{\beta}{\beta+1}\right)^n.$$

As a result, we can secure an exponent α_n in (33) which will be as close to β as we wish; for our purposes, it will suffice to take $n = 2$.

The numerical details go as follows. By (17), we can take $B = 652$ and $\beta = 2$. Moreover, (1) shows that we can take $A = 1/80$ and $\alpha = 0$ provided $x \geqslant 1119y$ and $y > e$. We now specify y by (26) and (27).

If $x \geqslant e^{189}$, then it is easy to verify that $x \geqslant 1119y$, $y > e$ and $k(x, y) > 0$. As a result, (28) implies that for $x \geqslant e^{189}$

$$|N(x)| < \frac{3x}{20}\left(\frac{4890}{\pi^2}\log x\right)^{1/3} < 1.19\,x(\log x)^{1/3}. \tag{35}$$

We note that if $x \geqslant C^{10/9} > 1$ then

$$\int_C^x \frac{du}{(\log u)^{5/3}} \leqslant \frac{1}{(\log C)^{5/3}}\int_C^{x^{9/10}} du + \frac{1}{(.9\log x)^{5/3}}\int_{x^{9/10}}^x du \leqslant \frac{x^{9/10}}{(\log C)^{5/3}} + \frac{x}{(.9\log x)^{5/3}}.$$

Taking $C \geqslant e^{189}$ and $x \geqslant C^{10/9}$, we obtain from (21), (1) and (35) that

$$|M(x)| < \frac{C}{80} + \frac{1.19C}{(\log C)^{2/3}} + \frac{|N(x)|}{\log x} + 1.19\int_C^x \frac{du}{(\log u)^{5/3}}$$

$$\leqslant \left\{\frac{1}{80} + \frac{1.19}{(\log C)^{2/3}}\right\}C + \frac{|N(x)|}{\log x} + \frac{1.19\,x^{9/10}}{(\log C)^{5/3}} + \frac{1.19\,x}{(.9\log x)^{5/3}}. \tag{36}$$

Hence, (35) and the choice $C = e^{189}$ gives for $x \geqslant e^{210}$

$$|M(x)| < \frac{x}{(\log x)^{2/3}}\left\{.049C\frac{(\log x)^{2/3}}{x} + 1.19 + \frac{(\log x)^{2/3}}{5200\,x^{1/10}} + \frac{1.42}{\log x}\right\}.$$

If $x \geqslant e^{900}$, then this implies (5). And if $1 < x < e^{900}$, then (2) yields

$$|M(x)| \leqslant \frac{x}{(\log x)^{2/3}}\left\{\frac{1}{80}(\log x)^{2/3} + \frac{441(\log x)^{2/3}}{80x}\right\} \equiv \frac{x}{(\log x)^{2/3}}K(x). \tag{37}$$

Now

$$K'(x) = \frac{441}{80x^2(\log x)^{1/3}}\left\{\frac{2x}{1323} + \frac{2}{3} - \log x\right\} \equiv \frac{441}{80x^2(\log x)^{1/3}}J(x).$$

Here $J(x)$ is strictly decreasing for $0 < x < 1323/2$ and is strictly increasing for $x > 1323/2$. Moreover, $J(2) < 0 < J(6000)$. Hence, $J(x)$ has a unique zero ξ on the half-line $(2, \infty)$. Then $K'(x) < 0$ for $2 \leqslant x < \xi$ and $K'(x) > 0$ for $x > \xi$. Consequently,

$$\max_{8 \leqslant x \leqslant e^{900}} K(x) = \max\{K(8), K(e^{900})\} \leqslant 1.17 \tag{38}$$

so that (5) holds for all $x \geqslant 8$. As it clearly holds for $1 < x < 8$, the proof of (5) is complete.

Therefore, (22) and (25) hold with $A = 1.2$, $a = 2/3$ and $B = 652$, $\beta = 2$. We let y be defined by (26) where $D = 5.76$. If $x \geqslant e^{1000}$, then $x \geqslant y > e$, $k(x, y) > 0$ and

$$0 \leqslant u \leqslant 5.76/1000^{4/9} \leqslant .268. \tag{39}$$

Hence (28) gives for $x \geqslant e^{1000}$

$$|N(x)| < \frac{2x}{(\log x)^{1/9}}\left\{\frac{1.2(5.76)}{(1-u)^{2/3}} + \frac{3(652)}{\pi^2(5.76)^2}(1-u)\right\} < \frac{25.8x}{(\log x)^{1/9}} \tag{40}$$

inasmuch as the expression in braces is a concave up fonction of u and hence has a maximum either at $u = 0$ or at $u = .268$. We use this in (36) to obtain, where $C = e^{189}$ and $x \geqslant e^{1000}$,

$$|M(x)| < \frac{x}{(\log x)^{10/9}}\left\{.049C\frac{(\log x)^{10/9}}{x} + 25.8 + \frac{(\log x)^{10/9}}{5200\,x^{1/10}} + \frac{1.42}{(\log x)^{5/9}}\right\}$$

$$< \frac{25.84\,x}{(\log x)^{10/9}} < \frac{12x}{\log x}.$$

Thus (7) and (6) hold for $x \geqslant e^{1000}$. And if $1 < x < e^{1000}$, then (5) gives

$$|M(x)| < 1.2(\log x)^{1/3}\cdot\frac{x}{\log x} < \frac{12x}{\log x} < \frac{25.86\,x}{(\log x)^{10/9}}$$

so that (6) and (7) hold in this range also. This completes the proof of (5)-(7).

4. Concluding remarks. Inasmuch as we have taken $\beta = 2$ as a consequence of (17), it follows from earlier remarks that we could prove a result of the kind (33) with a_n replaced by $2-\eta$ for an arbitrary $\eta > 0$; moreover, we could replace this O-estimate by an inequality with an explicitly determined constant. As this exponent $2-\eta$ essentially results from (12), we can raise the question as to what can be proved by our methods if we were to use the stronger inequality (13). This means that we will be using (22) with

$$h(v) = (\log v)^{-1/2}\exp\sqrt{(\log v)/R}.$$

If we now let $g(u) = (\log u)^a/(\log\log u)^\delta$, then the optimal choice of y is given by

$$\sqrt{(\log y)/R} = (a+1)\log\log x - \delta\log\log\log x$$

so that we will not be able to obtain a better bound from (23) than

$$N(x) = O\left(x\frac{(\log\log x)^{\delta+2}}{(\log x)^a}\right).$$

Then (20) yields

$$M(x) = O\left(x\frac{(\log\log x)^{\delta_1}}{(\log x)^{a_1}}\right)$$

where $\delta_1 = \delta+2$ and $a_1 = a+1$. Iterating repeatedly, the exponents δ_1 and a_1 can be replaced by $\delta_n = \delta+2n$ and $a_n = a+n$. Thus, the use of (13) will not yield anything better than

$$M(x) = O\left(\frac{x}{(\log x)^F}\right)$$

with arbitrarily large F. In view of (4), this indicates that (19) is a somewhat unsatisfactory result.

The constants in (5)-(7) result from the use of the constant 652 in (12). This, in turn, depends on Table I of Rosser and Schoenfeld [7]. The construction of this table used Lehmer's [3] result that the first 25000 non-trivial zeros of $\zeta(s)$ are all on the critical line. Recently, Lehman [2] has shown that the first 250000 zeros of $\zeta(s)$ are on the line.

Further, Rosser, Schoenfeld and Yohe [8] have announced a proof that the first 3500000 zeros of $\zeta(s)$ are on the line. As a consequence, one can construct a new Table I; this enables one to replace 652 in (12) and (17) by 37.6. If we assume that $x \geqslant e^{54}$, then the previous method of proof leads to the replacement of the constant 1.19 in (35) by .459 since now $B = 37.6$. If $C = e^{54}$ and $x \geqslant C^{10/9} = e^{60}$, then (36) holds with 1.19 replaced by .459 throughout; hence

$$(41) \qquad |M(x)| < .045C + \frac{|N(x)|}{\log x} + \frac{x^{9/10}}{1680} + \frac{.548x}{(\log x)^{5/3}}.$$

If $x \geqslant e^{220}$, then our sharpened version of (35) yields

$$(42) \qquad |M(x)| < .462\,x/(\log x)^{2/3}.$$

Proceeding as before, we use (37) and, in place of (38), we have

$$\max_{29 \leqslant x \leqslant e^{220}} K(x) = \max\{K(29), K(e^{220})\} \leqslant .456$$

so that (42) holds for all $x \geqslant 29$. Hence (42) also holds for $x \geqslant 6$ so that (5′) and (5″) now follow. Continuing as previously, we can now take $A = .462$, $a = 2/3$ and $B = 37.6$, $\beta = 2$. We define y by (26) with $D = 3.06$. If $x \geqslant e^{242}$, then $x \geqslant 6y > 6e$ and (39) holds with .268 replaced by .267; also, $k(x, y) > 0$. Now (28) implies (40) with 25.8 replaced by 5.27 provided $x \geqslant e^{242}$. Consequently, we obtain from (41) that both (7′) and (6′) hold for these x. And if $6 \leqslant x < e^{242}$, then (42) implies that (6′) and (7′) hold. Since they also hold for $1 < x < 6$, the proof of these results is finished.

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