ACTA ARITHMETICA
XV (1969)

A necessary and sufficient condition for the Riemann hypothesis for zeta functions attached to eigenfunctions of the Hecke operators*

by

L. J. GOLDSTEIN (New Haven, Conn.)

Dedicated to Hans Rademacher

1. Introduction. Let Γ denote the modular group $SL_2(Z)$, and let $\sigma_k(\Gamma)$ denote the space of cusp forms of weight k associated to Γ. If $f \in \sigma_k(\Gamma)$, let

$$(1) \qquad f(\tau) = \sum_{n=1}^{\infty} c_n \exp(2\pi i n\tau), \qquad \operatorname{Im}(\tau) > 0,$$

be the Fourier expansion of f about the cusp $i\infty$. Following Hecke, we associate to f the Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$$

which converges absolutely for $\operatorname{Re}(s) > (k+1)/2$ and converges for $\operatorname{Re}(s) > k/2$ (Hecke [3], pp. 651-652). In the present paper we derive several necessary and sufficient conditions for the Riemann hypothesis for $\varphi(s)$ in case f is a simultaneous eigenfunction of the Hecke operators for Γ.

There is every reason to believe that the Riemann hypothesis is true for these zeta functions. The present theory is completely analogous to the theory for the Riemann zeta function (Titchmarsh [7], pp. 282-315). However, the interesting feature of the present result is that the Riemann hypothesis is reduced to an arithmetic question concerning the coefficients c_n, which bears a close resemblance to the conjecture of Hecke and Petersson. Using this arithmetic formulation, one can apply the

* The present work constituted a portion of the author's doctoral dissertation submitted to Princeton University. The author would like to thank Drs. Shimura, Langlands and Grosswald, as well as the referee, for their many helpful suggestions.

Selberg-Eichler trace formula to reduce the Riemann hypothesis for $\varphi(s)$ to a question about class numbers of imaginary quadratic fields.

In order to state our main results, let us first state the principal facts about Hecke operators (Hecke [3], Shimura [5], [6]). Let $f\in\sigma_k(\Gamma)$, $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(R)$ and define $f^\gamma(z)=f(\gamma z)(cz+d)^{-k}$ for z in the upper half plane H, where $\gamma z=(az+b)/(cz+d)$. If $\gamma\in\mathrm{SL}_2(Z)$ and $\Gamma\gamma\Gamma=\bigcup_{i=1}^{r}\Gamma a_i$ is a coset decomposition, define

$$f|\Gamma\gamma\Gamma=\sum_{i=1}^{r}f^{a_i}.$$

If $f\in\sigma_k(\Gamma)$, then so is $f|\Gamma\gamma\Gamma$, and is independent of the choice of the coset representatives a_i. For n a positive integer, the Hecke operator T_n is defined formally as $\sum_{\det(\gamma)=n}\Gamma\gamma\Gamma$, where the sum is over the distinct double cosets $\Gamma\gamma\Gamma$, $\gamma\in\mathrm{SL}_2(Z)$ for which $\det(\gamma)=n$. There are only a finite number of such double cosets and T_n acts as a linear operator on $\sigma_k(\Gamma)$ via

$$T_nf=\sum_{\det(\gamma)=n}f|\Gamma\gamma\Gamma.$$

$\sigma_k(\Gamma)$ can be given the structure of a finite-dimensional Hilbert space with respect to the Petersson inner product and the T_n are a family of commuting normal operators with respect to this structure. Thus, the Hecke operators can be simultaneously diagonalized and it is well-known that the simultaneous eigenfunctions of the Hecke operators, normalized so that their first Fourier coefficient is 1, have associated zeta functions $\varphi(s)$ with an Euler product of the form

$$\varphi(s)=\prod_p(1-c_pp^{-s}+p^{k-1-2s})^{-1} \tag{2}$$

where the product is extended over all primes p and is absolutely convergent for $\mathrm{Re}(s)>(k+1)/2$ (Hecke [3], I, Satz 40 and 42). Moreover, $\varphi(s)$ has an analytic continuation as an entire function of s and satisfies the functional equation (Hecke [3], p. 653):

$$R(s)=(-1)^{k/2}R(k-s), \tag{3}$$

where

$$R(s)=(2\pi)^{-s}\Gamma(s)\varphi(s).$$

In previous work we have shown:

Theorem A (Goldstein [1]). *Let $N(T)$ denote the number of zeros of $\varphi(s)$ in the rectangle $-T\leqslant\mathrm{Im}(s)\leqslant T$, $-T\leqslant\mathrm{Re}(s)\leqslant T$, then*

$$N(T)=(2/\pi)T\log T-(2/\pi)\big(1+\log(2\pi)-\pi/2\big)T+O(\log T).$$

Theorem B (Goldstein [2]). *If $M(T)$ denotes the number of zeros ϱ in the rectangle of Theorem A such that $\mathrm{Re}(\varrho)=k/2$, then there exists a positive constant A such that*

$$M(T)>AT/\log T.$$

The main goal of the present paper is to show that the Riemann hypothesis for $\varphi(s)$ is equivalent to the convergence of the series defined by

$$\prod_p(1-c_pp^{-s}+p^{k-1-2s})=\sum_{n=1}^{\infty}\mu(n:\varphi)n^{-s}$$

for $\mathrm{Re}(s)>k/2$.

2. Preliminary lemmas.

Lemma 2.1 (Hadamard Three-Circles Theorem, [8], p. 173). *Let f be analytic in the annulus $r_1\leqslant|z|\leqslant r_3$, and let $r_1<r_2<r_3$. Let $M(r)$ denote the maximum of $|f(z)|$ on the circle $|z|=r$. Then*

$$\log M(r_2)\leqslant\{\log(r_3/r_2)/\log(r_3/r_1)\}\log M(r_1)+\{\log(r_2/r_1)/\log(r_3/r_1)\}\log M(r_3).$$

Lemma 2.2 ([7], p. 54). *Let $x>0$ be unequal to an integer, $T>0$, $c>0$. Then for n an integer*

$$(1/2\pi i)\int_{c-iT}^{c+iT}(x/n)^w w^{-1}dw=\begin{cases}1+O\big((x/n)^c/T\log(x/n)\big), & n<x,\\ O\big((x/n)^c/T\log(x/n)\big), & n>x,\end{cases}$$

as $T\to\infty$, where the constants in the O-terms do not depend on x, n, or T.

Lemma 2.3 (Borel-Carathéodory, [8], p. 174). *Let $f(z)$ be analytic in the disc $|z-z_0|\leqslant R$ and for $r<R$ define*

$$M(r)=\sup_{|z-z_0|=r}f(z),\quad A(r)=\sup_{|z-z_0|=r}\mathrm{Re}\big(f(z)\big).$$

Then for $0<r<R$,

$$M(r)\leqslant\frac{2r}{R-r}A(R)+\frac{R+r}{R-r}|f(z_0)|.$$

Lemma 2.4. *Let the Dirichlet series*

$$f(s)=\sum_{n=1}^{\infty}a_nn^{-s},\quad s=\sigma+it,$$

be absolutely convergent for $\sigma\geqslant k$. Let $c>0$, $\sigma_0>0$, $T>0$, $x=$ a positive integer $+\frac{1}{2}$, $\sigma_0+c>k$, $s=\sigma_0+it$. Then

$$\sum_{n<x}a_nn^{-s}=(1/2\pi i)\int_{c-iT}^{c+iT}f(s+w)(x^w/w)\,dw+O(x^{c+2}/T),$$

where the constant in the O-term does not depend on T or x.

Proof. Multiply the equation of Lemma 1.2 by $a_n n^{-s}$ and sum over n. We may interchange summation and integration since $\sigma_0+c>k$ and for any closed subregion of $\operatorname{Re}(s)>k$ the series representation of $f(s)$ converges uniformly and absolutely. Thus,

$$\sum_{n<x} a_n n^{-s} = (1/2\pi i)\int_{c-iT}^{c+iT} f(s+w)(x^w/w)\,dw + O\Big(\sum_{n=1}^{\infty} x^c|a_n|\,n^{-\sigma_0-c}/T\log(x/n)\Big),$$

as $T\to\infty$. Let us estimate the O-term. The terms for which $n\leqslant x/3$ or $n\geqslant 3x$ are bounded by

$$O\Big((x^c/T)\sum_n |a_n|\,n^{-\sigma_0-c}\Big) = O(x^c/T), \qquad T\to\infty,$$

where the constant in the O-term does not depend on x or T. If $x\leqslant n<3x$ then

$$|\log(x/n)|\geqslant \log\big(1+(2[x]+1)^{-1}\big)\geqslant c'/x$$

for some $c'>0$ and x sufficiently large. A similar result holds for $x/3<n<x$. Thus the terms of the sum corresponding to such n are $O(x^{c+1}/T)$, and there are at most $3x$ terms so that their sum is at most $O(x^{c+2}/T)$, where the constant in the O-term does not depend on x or T.

LEMMA 2.5. *There exist positive constants a, b, c such that*

$$|\varphi(x+iy)|\leqslant b\,|y|^a, \qquad |y|>c,$$

uniformly for $k/2\leqslant x\leqslant k$.

Proof. This fact follows from the Phragmén-Lindelöf theorem and the functional equation for φ in the same way as one proves the corresponding statement for the Riemann zeta function ([4], pp. 157-159, [1], Lemma 3.3).

3. A necessary and sufficient condition for the Riemann hypothesis. Let all notation be as above.

THEOREM 3.1. *Assume that $\varphi(s)$ satisfies the Riemann hypothesis, that is, all non-trivial zeros of $\varphi(s)$ are on the line $\operatorname{Re}(s)=k/2$. Let $k/2<\sigma_0<k$. Then there exist positive constants $K(\varepsilon)$, $A(\sigma_0)$ such that $|t|>A(\sigma_0)$ implies that*

$$|\varphi(\sigma+it)|<K(\varepsilon)\,|t|^{\varepsilon},$$
$$|\varphi(\sigma+it)^{-1}|<K(\varepsilon)\,|t|^{\varepsilon}.$$

uniformly for $k/2<\sigma_0<\sigma\leqslant k$.

Proof. Let us fix σ_0 and let us consider a point $\sigma+it$, with $|t|$ large. We will later specify how large $|t|$ must be in order for all our statements to make sense, and we will see that the lower bound for $|t|$ need depend

only on σ_0. Let us consider three circles C_1, C_2, and C_3, all centered at $z_0=\log\log|t|+it$ and passing through the points

$$3k+(1/\log\log|t|)+it, \qquad \sigma+it, \qquad k/2+(1/\log\log|t|)+it,$$

respectively, where $|t|$ is so large that

$$k/2\leqslant k/2+(1/\log\log|t|)\leqslant\sigma_0<\sigma$$

a condition which can be realized by making $|t|$ larger than some bound depending only on σ_0.

The radii of these circles are, respectively, $r_1=\log\log|t|-3k-(1/\log\log|t|)$, $r_2=\log\log|t|-\sigma$, $r_3=\log\log|t|-k/2-(1/\log\log|t|)$. Since we have assumed the Riemann hypothesis, we can unambiguously define $\log\varphi(s)$ for $\operatorname{Re}(s)>k/2$. Let us choose a branch of the logarithm and fix it throughout the remainder of the discussion. Let M_i $(i=1,2,3)$ denote the maximum of $|\log\varphi(s)|$ on C_i relative to this branch. By Lemma 2.2 and the fact that $\sigma+it$ is on C_2, we have

$$|\log\varphi(\sigma+it)|\leqslant M_2\leqslant M_1^{\log(r_3/r_2)/\log(r_3/r_1)}M_3^{\log(r_2/r_1)/\log(r_3/r_1)}. \tag{4}$$

Note that

$$\log(r_3/r_2)=\log\left[\frac{\log\log|t|-k/2-(1/\log\log|t|)}{\log\log|t|-\sigma}\right] \tag{5}$$
$$=\log\big[1+(\sigma-k/2)/\log\log|t|+O\big(1/(\log\log|t|)^2\big)\big]$$
$$=(\sigma-k/2)/\log\log|t|+O\big((\log\log|t|)^{-2}\big),$$

$$\log(r_3/r_1)=\log\left[\frac{\log\log|t|-k/2-(1/\log\log|t|)}{\log\log|t|-3k-(1/\log\log|t|)}\right] \tag{6}$$
$$=\log\big[1+5k/(2\log\log|t|)+O\big((\log\log|t|)^{-2}\big)\big]$$
$$=5k/(2\log\log|t|)+O\big((\log\log|t|)^{-2}\big),$$

$$\log(r_2/r_1)=\log\left[\frac{\log\log|t|-\sigma}{\log\log|t|-3k-(1/\log\log|t|)}\right] \tag{7}$$
$$=\log\big[1+(3k-\sigma)/\log\log|t|+O\big((\log\log|t|)^{-2}\big)\big]$$
$$=(3k-\sigma)/\log\log|t|+O\big((\log\log|t|)^{-2}\big),$$

where the constants in the O-terms do not depend on σ.

Let us now estimate M_1. Since (Hecke [3], Satz 6), the Dirichlet series for $\varphi(s)$ and $1/\varphi(s)$ converge in the half-plane $\operatorname{Re}(s)\geqslant k$, it follows that $\log|\varphi(s)|$ is bounded in this half-plane. Also, $\arg\varphi(s)$ is bounded in this half-plane, for a fixed, continuous branch of the argument. Therefore, $\log\varphi(s) = \log|\varphi(s)| + i\arg\varphi(s)$ is bounded in the half-plane $\mathrm{Re}(s) \geqslant k$. Since C_1 is contained in this half-plane, we have

$$M_1 \leqslant D \tag{8}$$

for D some positive constant depending only on φ and k.

Let us now turn to the estimation of M_3. By Lemma 2.5 there exist positive constants a, b, c such that

$$|\varphi(x+iy)| \leqslant b|y|^a, \quad |y| > c,$$

uniformly for $k/2 \leqslant x \leqslant k$. The inequality also holds in the region $|y| > c$, $x \geqslant k$ since here $\varphi(s)$ is bounded. Thus, for $|t|$ sufficiently large, $s = x+iy$ on C_3 implies that for $d > a$ an appropriate constant,

$$\log|\varphi(s)| \leqslant \log b + a\log\big||t| + \log\log|t| - k/2 - (1/\log\log|t|)\big| \leqslant d\log|t|.$$

Now apply Lemma 2.3 to $f(z) = \log\varphi(z)$ in the disc $|z-z_0| = r_4 = r_3 + (1/2\log\log|t|)$ to get

$$\begin{aligned} M_3 &\leqslant 4A(r_4)(\log\log|t|)^2 + (2r_3 + 1/2\log\log|t|)(2\log\log|t|)\,|\log\varphi(z_0)| \\ &\leqslant E\log|t|(\log\log|t|)^2, \end{aligned} \tag{9}$$

where E depends only on φ and k. Applying (5)-(9) to (4), we obtain

$$\begin{aligned} \big|\log|\varphi(\sigma+it)|\big| &\leqslant M_2 \leqslant D^{(2\sigma-k)/5k+O(1/\log\log|t|)} \times \\ &\quad \times [E\log|t|(\log\log|t|)^2]^{2(3k-\sigma)/5k+O(1/\log\log t)} \\ &\leqslant F(\log|t|)^{2(3k-\sigma)/5k}(\log\log|t|)^2 \\ &\leqslant J(\varepsilon)(\log|t|)^{2(3k-\sigma)/5k+\varepsilon} \end{aligned}$$

for all $\varepsilon > 0$ and all sufficiently large $|t|$, for appropriate constants F and $J(\varepsilon)$. Since $3k-\sigma < 5k/2$, for ε sufficiently small we have

$$2(3k-\sigma)/5k + \varepsilon < 1,$$

which implies

$$\big|\log|\varphi(\sigma+it)|\big| \leqslant J(\varepsilon)(\log|t|)^\lambda, \quad 0 < \lambda < 1,$$

uniformly for $k/2 < \sigma_0 < \sigma$. Therefore

$$-J(\varepsilon)(\log|t|)^\lambda \leqslant \log|\varphi(\sigma+it)| \leqslant J(\varepsilon)(\log|t|)^\lambda$$

for all sufficiently large $|t|$. Thus, for $|t|$ sufficiently large, say $|t| > M$, given $\eta > 0$ there exists a positive number $K(\eta)$ such that

$$K(\eta)|t|^{-\eta} \leqslant \exp\big(-J(\varepsilon)(\log|t|)^\lambda\big) \leqslant |\varphi(\sigma+it)| \leqslant \exp\big(J(\varepsilon)(\log|t|)^\lambda\big) \leqslant K(\eta)|t|^\eta$$

uniformly for $\frac{1}{2} \leqslant \sigma_0 < \sigma \leqslant k$. Since $\eta > 0$ is arbitrary, the theorem is proved.

From the above proof, we derive

THEOREM 3.2. *Let $\mu(\sigma)$ denote the Phragmén-Lindelöf function for $\varphi(s)$, and assume that the Riemann hypothesis is satisfied. Then*

$$\mu(\sigma) = \begin{cases} 0, & \sigma > k/2, \\ k-2\sigma, & \sigma \leqslant k/2. \end{cases}$$

Proof. The assertion for $\sigma > k/2$ follows from Theorem 3.1. The assertion for $\sigma = k/2$ follows from the continuity of μ. For $\sigma < k/2$ the assertion follows from the functional equation for φ, in just the same way as one proves the corresponding fact for the Riemann zeta function (Titchmarsh [7], p. 81, Goldstein [1], Proposition 2.4).

For $\mathrm{Re}(s) > (k+1)/2$, we have the absolutely convergent product representation

$$\varphi(s) = \prod_p (1 - c_p p^{-s} + p^{k-1-2s})^{-1}.$$

Since $\varphi(s)$ does not vanish for $\mathrm{Re}(s) > (k+1)/2$, we may write for s in this half-plane

$$1/\varphi(s) = \prod_p (1 - c_p p^{-s} + p^{k-1-2s}).$$

Thus, $1/\varphi(s)$ can be expanded in a Dirichlet series, absolutely convergent for $\mathrm{Re}(s) > (k+1)/2$. Let

$$1/\varphi(s) = \sum_{n=1}^{\infty} \mu(n:\varphi)n^{-s}$$

be this Dirichlet series representation. The series is unique (Titchmarsh [8], p. 309), so that the quantities $\mu(n:\varphi)$ are intrinsically defined by φ. We easily see that

PROPOSITION 3.3. *Let $r \geqslant 1$ be an integer. Then, if p is prime*

$$\mu(p^r:\varphi) = \begin{cases} -c_p, & r = 1, \\ p^{k-1}, & r = 2, \\ 0, & \textit{otherwise}. \end{cases}$$

PROPOSITION 3.4. *If m, n are positive integers and relatively prime, then*

$$\mu(mn:\varphi) = \mu(m:\varphi)\mu(n:\varphi).$$

The function $\mu(n:\varphi)$ can be regarded as a generalization of the Möbius function $\mu(n)$, since if $\zeta(s)$ denotes the Riemann zeta function, then

$$1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n)n^{-s}, \quad \mathrm{Re}(s) > 1.$$

THEOREM 3.5. *A necessary and sufficient condition that the Riemann hypothesis for $\varphi(s)$ be true is that the series*

$$\sum_{n=1}^{\infty} \mu(n:\varphi) n^{-s}$$

converge for $\operatorname{Re}(s) > k/2$.

Proof. If the series converges for $\operatorname{Re}(s) > k/2$, then it converges uniformly in every half-plane $\operatorname{Re}(s) \geqslant k/2+\varepsilon$, $\varepsilon > 0$, and represents an analytic function there. We have seen that this function coincides with $1/\varphi(s)$ for $\operatorname{Re}(s) > (k+1)/2$, so that by analytic continuation, the function defined by the series coincides with $1/\varphi(s)$ for $\operatorname{Re}(s) > k/2$. In particular, $1/\varphi(s)$ is analytic for $\operatorname{Re}(s) > k/2$ so that $\varphi(s)$ does not vanish for $\operatorname{Re}(s) > k/2$. The functional equation implies that $\varphi(s)$ does not vanish for $0 < \operatorname{Re}(s) < k/2$, for if it did then there would exist a zero s for which $\operatorname{Re}(s) > k/2$, which is impossible. On the line $\operatorname{Re}(s) = 0$, a similar argument shows that no point except zero is a zero of $\varphi(s)$. Thus, all the non-trivial zeros of $\varphi(s)$ are on the critical line and $\varphi(s)$ satisfies the Riemann hypothesis.

Conversely, assume that $\varphi(s)$ satisfies the Riemann hypothesis. Let us apply Lemma 2.4 with $f(s) = 1/\varphi(s)$. If $x =$ positive integer $+\frac{1}{2}$, $c > 0$, $s = \sigma+it$, $\sigma+c > k$, $T > 0$, $\sigma > k/2$, then we have

$$\sum_{n<x} \mu(n:\varphi) n^{-s} = (1/2\pi i) \int_{c-iT}^{c+iT} \frac{x^w}{w\varphi(s+w)}\, dw + O(x^{c+2}/T).$$

By the assumed Riemann hypothesis, the integrand has a simple pole at $w = 0$ with residue $1/\varphi(s)$, and no other poles for $\operatorname{Re}(w+s) > k/2$. Thus, by Cauchy's theorem

$$(1/2\pi i) \int_{c-iT}^{c+iT} \frac{x^w}{w\varphi(s+w)}\, dw$$

$$= 1/\varphi(s) + (1/2\pi i) \left[\int_{c+iT}^{k/2+\delta-\sigma+iT} + \int_{k/2+\delta-\sigma+iT}^{k/2+\delta-\sigma-iT} + \int_{k/2+\delta-\sigma-iT}^{c-iT} \right] \frac{x^w}{w\varphi(s+w)}\, dw$$

for all sufficiently small $\delta > 0$. On the path of integration for the first and third integrals on the right, Theorem 3.1 shows that

$$|1/\varphi(s+w)| \leqslant K(\varepsilon) T^{\varepsilon}$$

for all $\varepsilon > 0$, uniformly for $k/2+\delta-\sigma \leqslant \operatorname{Re}(w) \leqslant c$, for T sufficiently large, where the lower bound for T is independent of ε. Therefore, we may estimate the first integral

$$\left| \int_{c+iT}^{k/2+\delta-\sigma+iT} \frac{x^w}{w\varphi(s+w)}\, dw \right| \leqslant K'(\varepsilon) T^{\varepsilon-1} \int_{c}^{k/2+\delta-\sigma} x^w\, dw \leqslant K''(\varepsilon, c, k, \delta, \sigma) T^{\varepsilon-1} x^c.$$

A similar estimate holds for the third integral. Let us now estimate the second integral:

$$\left| \int_{k/2+\delta-\sigma-iT}^{k/2+\delta-\sigma+iT} \frac{x^w}{w\varphi(s+w)}\, dw \right| \leqslant x^{k/2+\delta-\sigma} \int_{-T}^{T} \frac{1}{|\varphi(k/2+\delta+it)t|}\, dt$$

$$\leqslant K(\varepsilon) x^{k/2+\delta-\sigma} \int_{-T}^{T} |t|^{\varepsilon-1}\, dt \leqslant K'(\varepsilon) x^{k/2+\delta-\sigma} T^{\varepsilon}$$

for all sufficiently large T and all $\varepsilon > 0$. Therefore,

$$\sum_{n<x} \mu(n:\varphi) n^{-s} = 1/\varphi(s) + O(x^c T^{\varepsilon-1}) + O(x^{k/2+\delta-\sigma} T^{\varepsilon})$$

for all $\varepsilon > 0$, and for x and T tending independently to ∞. Set $T = x^{c+1}$, and choose ε and δ so that $c+(\varepsilon-1)(c+1) < 0$ and $\varepsilon c + k/2+\delta-\sigma < 0$. Then as x tends to ∞, the error terms tend to zero and thus, the series on the left converges for $\sigma > k/2$ to $1/\varphi(s)$ and the converse assertion is proved.

THEOREM 3.6. *For* $x > 0$ *define the function*

$$\lambda(x) = \sum_{n<x} \mu(n:\varphi).$$

Then a necessary and sufficient condition that φ *satisfy the Riemann hypothesis is that*

$$\lambda(x) = O(x^{k/2+\varepsilon}), \qquad x \to \infty$$

for all $\varepsilon > 0$, *where the constant in the O-term depends on* ε. *Moreover, if* θ *denotes the supremum of the abcissae of the zeros of* $\varphi(s)$, *then*

$$\lambda(x) = O(x^{\theta+\varepsilon})$$

for all $\varepsilon > 0$, *where the constant in the O-term depends on* ε. *The smallest possible value of* θ *is* $k/2$.

Proof. Note that $\lambda(x)$ is the coefficient sum function of the Dirichlet series for $1/\varphi(s)$. By the proof of Theorem 3.5, if $\varphi(s)$ has no zeros in the half-plane $\operatorname{Re}(s) > \theta$, then the Dirichlet series for $1/\varphi(s)$ converges for $\operatorname{Re}(s) > \theta$, and thus, by a standard theorem on Dirichlet series (Titchmarsh [8], p. 292), we get $\lambda(x) = O(x^{\theta+\varepsilon})$ for all $\varepsilon > 0$, where the constant in the O-term depends on ε. This implies the next-to-last assertion of the theorem. Moreover, it is clear that assuming the Riemann hypothesis, we get $\lambda(x) = O(x^{k/2+\varepsilon})$ for all positive ε. If θ assumed a value less than $k/2$ then $\varphi(s)$ would have no zeros on the critical line which contradicts Theorem B. Lastly, if $\lambda(x) = O(x^{k/2+\varepsilon})$ for all positive ε, then a standard argument using partial summation shows that the Dirichlet series for $1/\varphi(s)$ converges for $\mathrm{Re}(s) > k/2$, which by Theorem 3.5 implies the Riemann hypothesis.

COROLLARY 3.7. *Given $\varepsilon > 0$, there exists a sequence $x_1, x_2, \ldots$ tending to ∞ and such that*

$$\lambda(x_n)/x_n^{k/2-\varepsilon} \to \infty \quad \text{as} \quad n \to \infty.$$

Let $\{f_1, \ldots, f_r\}$ be a basis of $\sigma_k(\Gamma)$ composed of simultaneous eigenfunctions of the Hecke operators $T(n)$, and let

$$f_j(\tau) = \sum_{n=1}^{\infty} a_n^{(j)} \exp(2\pi i n\tau), \quad \mathrm{Im}(\tau) > 0, \quad a_1^{(j)} = 1, \quad 1 \leqslant j \leqslant r,$$

be their respective Fourier expansions about the cusp $i\infty$. Let

$$\varphi_j(s) = \sum_{n=1}^{\infty} a_n^{(j)} n^{-s}, \quad 1 \leqslant j \leqslant r$$

be their associated zeta functions. The using the same argument as above, we prove

THEOREM 3.8. *The Riemann hypothesis for the functions defined by $\varphi_1, \ldots, \varphi_r$ is simultaneously true if and only if*

$$\sum_{n=1}^{\infty} \psi(n:k) n^{-s} = \prod_{p,j} (1 - c_p^{(j)} p^{-s} + p^{k-1-2s})$$

converges for $\mathrm{Re}(s) > k/2$.

Based on some empirical evidence, we conjecture the following analogue of Merten's conjecture:

CONJECTURE.

$$\Big|\sum_{n<x} \mu(n:\varphi)\Big| < x^{k/2}.$$

It is clear that the conjecture implies the Riemann hypothesis.

4. Generalizations. In [5] and [6] Shimura has derived a general theory of zeta functions attached to automorphic forms associated to the unit group of a maximal order in an indefinite quaternion algebra defined over a totally real algebraic number field. The results of the present paper carry over completely to the case of zeta functions corresponding to simultaneous eigenfunctions of the Hecke operators in this setting, provided that one assumes that the field of definition is the rational numbers Q and that one only considers automorphic forms of level 1. It is likely that all of our results hold in the most general setting. However, one does not know the position of the poles in the analytic continuation

of the zeta function explicitly enough and the analogue of Theorem B is lacking in the general case. In the case of the principal congruence subgroups of $\mathrm{SL}_2(Z)$ we have proved all of our results except for the fact that the minimum value of θ is $k/2$.

References

[1] L. Goldstein, *A Riemann–von Mangoldt formula for Dirichlet series with functional equation*, to appear.

[2] — *Weak forms of the Riemann hypothesis for zeta functions associated to cusp forms*, to appear.

[3] E. Hecke, *Über Modulfunktionen und die Dirichletschen Reihen mit Eulersche Produktentwicklung*, I, II 1937, Mathematische Werke, pp. 644-671; pp. 672-707.

[4] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 2nd ed., New York 1953.

[5] G. Shimura, *On the zeta functions of the algebraic curves uniformized by certain automorphic functions*, J. Math. Soc. Japan 13 (1961), pp. 275-331.

[6] — *On Dirichlet series and Abelian varieties attached to automorphic forms*, Ann. of Math. 76 (1962), pp. 237-294.

[7] E. C. Titchmarsh, *The Theory of the Riemann Zeta Function*, Oxford 1951.

[8] — *The Theory of Functions*, Oxford 1939.

YALE UNIVERSITY
New Haven, Connecticut

Reçu par la Rédaction le 9. 12. 1967