Sums involving Farey fractions

by

J. Lehner and M. Newman (College Park, Md.)

1. Let \( F_n \) be the Farey series of order \( n \), \( n \geq 1 \). Then \( F_n \) consists of all fractions \( h/k \) arranged in ascending order, where \( 0 \leq h \leq k \leq n \), \((h, k) = 1\). For example, the Farey series of order 5 is

\[
\begin{align*}
0 & 1 & 1 & 2 & 1 & 3 & 2 & 3 & 4 & 1 \\
1 & 5 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 5 & 1
\end{align*}
\]

Let \( h/k \) be a term of \( F_n \), \( h/k < 1 \). Then the term that follows \( h/k \) in \( F_n \) will be denoted by \( h/k' \). The set of pairs \((k, k')\), where \( h/k \) runs over all terms of \( F_n \) less than 1, will be denoted by \( Q_2 \). Thus \( Q_2 \) consists of the pairs

\[(1, 5), (5, 4), (4, 3), (3, 5), (5, 2), (2, 5), (5, 3), (3, 4), (4, 5), (5, 1)\].

H. Gupta has proved in his paper [1] that

\[
\sum_{k=1}^{n} \sum_{1 \leq k' \leq n, (k', k) = 1} \frac{1}{r^2(r+k)} = \frac{3}{4}.
\]

This sum is connected with the Farey fractions. In fact we show in what follows that

\[
\frac{1}{2} \sum_{(k,k') \in Q_2} \frac{1}{kk'(k+k')} = \frac{3}{4} - \sum_{r=1}^{n} \sum_{1 \leq k' \leq n} \frac{1}{r^2(r+k')}
\]

This identity reduces a sum involving Farey fractions to one which does not. More generally we show in Section 2 that for an arbitrary function \( f \),

\[
\sum_{(k,k') \in Q_2} f(k, k') = f(1, 1) + \sum_{r=1}^{n} \sum_{1 \leq k' \leq n} \{ f(k, r) + f(r, k) - f(k, r-k) \}.
\]

In Section 3 we discuss the asymptotic behaviour of the left member of (3) as \( n \to \infty \), for certain functions \( f \). We show for example that the sum in (2) is asymptotic to \( \sigma/n \), where \( \sigma = 6\log 2/n^2 \); and that it is asym-
ptotic to \( c_{a,b}n^{a+\beta+1} \) for \( f = n^{a+b} \), where \( a, \beta \) are non-negative and \( c_{a,b} \) is a certain constant which is determined precisely. From the latter results we can determine bounds for the sum when \( f \) is any polynomial in \( k, k' \); and even its exact asymptotic behaviour, in many cases. Finally, some additional results of a similar nature are quoted without proof.

We remark parenthetically that the sum in (2) admits an interesting interpretation. Let \( |z| = \min(|x|, 1-|x|) \) represent the distance from \( x \) to its nearest integer, and consider the \( n \) functions

\[
y_r = |rz|, \quad 1 \leq r \leq n.
\]

Then these functions vanish in \([0, 1]\) if and only if \( x \) belongs to \( F_n \); and if \( h/k, h'/k' \) are consecutive terms of \( F_n \), the triangle whose sides are the lines \( y = 0, y = kx - k, \) and \( y = -k'x + k' \) has area \( 1/2kk'(k+k') \). Thus the sum in (2) represents the total area of all such triangles. The configuration is of interest in questions of diophantine approximation.

2. Assume from now on that \( n > 1 \). It is known that \( h/k, h'/k' \) are consecutive terms of \( F_n \) if and only if

\[
0 \leq h \leq k \leq n, \quad 0 \leq h' \leq k' \leq n, \quad h/k < h'/k', \quad h-k = h'-k' = 1.
\]

A further result is that \( k+k' > n \), so that

\[
n+1 \leq k+k' \leq 2n-1.
\]

(For these and other results on Farey fractions, see [2].)

Thus the conditions

\[
0 < a, b \leq n, \quad (a, b) = 1, \quad n+1 < a+b < 2n-1,
\]

are certainly necessary for \((a, b) \) to belong to \( Q_n \). But they are also sufficient. Indeed, we can find integers \( c, d \) such that

\[
0 < c < a-1, \quad 1 < d < b, \quad ad - bc = 1.
\]

This easily implies that \( c/a, d/b \) are consecutive terms of \( F_n \), so that

\[
(a, b) \in Q_n.
\]

Thus we have proved

**Lemma 1.** \( Q_n \) consists of all pairs \((a, b)\) which satisfy (4).

This lemma leads directly to the proof of the transformation equation (3).

Define

\[
S_n = \sum_{(k,k')Q_n} f(k, k'),
\]

where \( f \) is an arbitrary function. Then we have

**Theorem 1.** Let \( n > 1 \). Then

\[
S_n = f(1, 1) + \sum \sum \{f(k, r) + f(r, k) - f(k, r - k)\}.
\]

**Proof.** Observe that \( Q_n \) consists of the coprime pairs \((x, y)\) lying in or on the triangle bounded by the lines \( x+y = r+1, \) \( x = r, \) \( y = r \). Hence the \( k \) points of \( Q_n \), not in \( Q_{n-1} \), are the coprime pairs \((x, y)\) on the line segments \( x+y = r, \) \( 0 < y < r; \) \( y = r, \) \( 0 < x < r; \) and the points of \( Q_{n-1} \) not in \( Q_n \) are the coprime pairs \((x, y)\) on the line segment \( x+y = r, \) \( 0 < x, y < r \). This implies that

\[
S_n - S_{n-1} = \sum \sum \{f(k, r) + f(r, k) - f(k, r-k)\}.
\]

The result now follows by summation on \( r \).

Another result of this kind (but not as useful as the previous one) is

\[
S_n = \sum_{a+b=n+1}^{n} f(r, k),
\]

3. For our first application of Theorem 1 we turn to the problem treated by Gupta, and choose \( f(k, k') = 1/kk'(k+k') \). Since

\[
\sum_{(k,k')Q_n} \frac{1}{kk'} = \sum_{(k,k')Q_n} \left( \frac{k'}{k} - \frac{k}{k'} \right) = 1,
\]

and \( n+1 < k+k' < 2n-1 \), it follows that

\[
\frac{1}{2n-1} \leq S_n \leq \frac{1}{n+1}.
\]

Thus \( S_n = O(1/n) \). We shall prove

**Theorem 2.** Suppose that \( S_n \) is defined by (8). Then

\[
\frac{n}{2} S_n = \frac{6\log 2}{\pi^2} + O\left(\frac{\log n}{n}\right).
\]

Consequently,

\[
\lim_{n \to \infty} \frac{n}{2} S_n = \frac{6\log 2}{\pi^2}.
\]
Proof. From (6) we obtain

\[ S_n - S_{n-1} = \sum_{1 \leq k \leq n} \frac{1}{kr(k+r)} + \frac{1}{rk(r+k)} - \frac{1}{k(r-h)r} \]

\[ = \sum_{1 \leq k \leq n} \frac{1}{k} \left( \frac{1}{r+k} - \frac{1}{r-k} \right) = -2 \sum_{1 \leq k \leq n} \frac{1}{r^2(r+k)} \]

Since \( f(1,1) = \frac{1}{2} \), Theorem 1 implies that

\[ \frac{1}{2} S_n = \frac{3}{4} - \sum_{r=1}^{n} \frac{1}{r^2(r+k)} \]

Since \( S_n = O(1/n) \), this yields Gupta's result (1) when \( n \to \infty \). It follows that

\[ \frac{1}{2} S_n = \sum_{r=1}^{n} A_r, \]

where

\[ A_r = \sum_{1 \leq k \leq n} \frac{1}{k^2(r+k)}. \]

We now transform \( A_r \). Since

\[ \sum_{d|n} \mu(d) = \begin{cases} 1, & (k, r) = 1, \\ 0, & \text{otherwise}, \end{cases} \]

we have

\[ A_r = \sum_{d=1}^{n} \sum_{d|n, d|k} \frac{\mu(d)}{r^2(r+k)}. \]

Replacing \( k \) by \( kd \), we find that

\[ A_r = \sum_{d=1}^{n} \sum_{d|n} \frac{\mu(d)}{r^2(r+kd)} \]

and replacing \( r \) by \( r/d \), we find that

\[ A_r = \sum_{d=1}^{n} \sum_{d|n} \frac{d \mu(r/d)}{r^2(k+d)}. \]

Now

\[ \sum_{d=1}^{n} \frac{1}{k+d} = \log 2 + O\left( \frac{1}{n} \right). \]

Hence

\[ A_r = \sum_{d=1}^{n} \frac{d \mu(r/d)}{r^2} \left( \log 2 + O\left( \frac{1}{n} \right) \right) \leq \log 2 \cdot \frac{g(r)}{r^2} + O\left( \frac{1}{r^2} \sum_{d|r} \mu(d) \right). \]

Now

\[ \sum_{r=1}^{n} \frac{d \mu(r/d)}{r^2} = \sum_{r=1}^{n} \sum_{d|r} \mu(r) \left[ \frac{r}{r} - \frac{r-1}{r} \right] \]

\[ = \sum_{r=1}^{n} \left[ \frac{\mu(r)}{r} \frac{n}{r} \right] = O(n \log n), \]

and

\[ \sum_{r=1}^{n} g(r) = \frac{3n^2}{n^2} + O(n \log n). \]

(See (2), for example, for formula (12).)

Using these formulas it follows by partial summation that

\[ \sum_{r=1}^{n} \frac{1}{r^2} \sum_{d|r} \mu(d) = O\left( \frac{\log n}{n^2} \right), \]

and that

\[ \sum_{r=1}^{n} \frac{g(r)}{r^2} = \frac{6}{n^2} + O\left( \frac{\log n}{n^2} \right). \]

These now yield (9), and (10) is an immediate consequence of (9). This completes the proof of the theorem.

Before going on to the second application of Theorem 1, we state two more lemmas.

**Lemma 2.** Suppose that \( a \geq 0 \). Then

\[ \sum_{r=1}^{n} r^a g(r) = \frac{6}{n^2} + \frac{a+2}{n^{a+2}} + O(n^{a+1} \log n). \]

**Proof.** Lemma 2 follows directly from (12) by partial summation.

**Lemma 3.** Suppose that \( a, b \geq 0 \). Then

\[ \sum_{r=1}^{n} r^a g(r)^{b} = \frac{G(1+a)G(1+b)}{G(2+a+b) - \frac{a+2}{n^{a+2}} + O(n^{a+b+1})}. \]

**Proof.** Lemma 3 follows by comparing the sum with

\[ \int \frac{r^a g(r)^b}{r} dr \]

via the Euler-Maclaurin summation formula.
The restriction that \( a, b \geq 0 \) in these lemmas can of course be weakened.

We now go on to our second application of Theorem 1. We choose
\[
f(k, k') = k^a k'^b,
\]
where \( a, b \geq 0 \). Then (6) becomes
\[
S_n - S_{n-1} = \sum_{k \mid 2^r \cdot k'} (k^a k'^b - k^a k'^b - k^a(k - k')), \tag{15}
\]
where now
\[
S_n = \sum_{k \mid 2^r \cdot k'} k^a k'^b. \tag{16}
\]

We find as before that
\[
\sum_{k \mid 2^r \cdot k'} k^a = \frac{r^a \varphi(r)}{a+1} + r^a O\left( \sum_{d \mid r^a} |\mu(d)| \right),
\]
and
\[
\sum_{k \mid 2^r \cdot k'} k^a = \frac{r^a \varphi(r)}{\beta + 1} + r^a O\left( \sum_{d \mid r^a} |\mu(d)| \right). \tag{19}
\]

Also, Lemma 3 implies that
\[
\sum_{k \mid 2^r \cdot k'} k^a k'^b = \sum_{k \mid 2^r \cdot k'} \sum_{d \mid k} \mu(d) \left( \frac{k'}{d} - 1 \right)^b
\]
\[
= \sum_{d \mid r^a} d^{a+b} \mu(d) \left( \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)} \right) \left( \frac{r^a \varphi(r)}{d^a} \right)^b + O\left( \left( \frac{r^a \varphi(r)}{d^a} \right)^{b+1} \right)
\]
\[
= \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)} \frac{r^a \varphi(r) + r^a O\left( \sum_{d \mid r^a} |\mu(d)| \right)}{r^a \varphi(r) + r^a O\left( \sum_{d \mid r^a} |\mu(d)| \right)}.
\]

Putting these together we get that
\[
S_n - S_{n-1} = K_{a,b} r^a \varphi(r) + r^a O\left( \sum_{d \mid r^a} |\mu(d)| \right), \tag{17}
\]
where
\[
K_{a,b} = \frac{1}{1+a} + \frac{1}{1+b} - \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)}. \tag{20}
\]

Now if we sum (17) over \( r \) and use (11) and Lemma 2, we get

**Theorem 3.** Suppose that \( a, b \geq 0 \), and that \( S_n \) is defined by (16).

Then
\[
S_n = c_{a,b} \pi^{a+b+1} + O(n^{a+b+1} \log n), \tag{18}
\]
where
\[
c_{a,b} = \frac{1}{\pi^{a+b+1}} \left( \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)} \right).
\]

As an example of the above, we have
\[
\sum_{\substack{k \mid 2^r \cdot k'} \nu k^2} = \frac{32 - 3\pi}{12\pi^2} \pi^4 + O(n^{4+\log n}).
\]

Finally we mention some special results which are derivable in the same way:

\[
\sum_{\substack{k \mid 2^r \cdot k'} \nu k^2} = \frac{3}{2} \sum_{\nu k^2} \varphi(r) - \frac{1}{2} = 2n^2 + O(n \log n), \tag{19}
\]
\[
\sum_{\substack{k \mid 2^r \cdot k'} \nu k^2} = \frac{6}{\pi^2} (2\log 2 - 1)n + O((\log n)^2), \tag{20}
\]
\[
\sum_{\substack{k \mid 2^r \cdot k'} \nu k^2} = \frac{11 - 12\log 2}{3\pi^2} n^3 + O(n^{3+\log n}). \tag{21}
\]

**References**


UNIVERSITY OF MARYLAND
NATIONAL BUREAU OF STANDARDS

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