Therefore if $s \neq p-2$ by Staudt's theorem we see

$$\frac{1}{p}\sum_{a=1}^{p} a\chi^s(a) \equiv B_{sp^l+1} \pmod{p^l},$$

so that (1) is proved.

For demonstration of analogy of (1) for cyclotomic field $Q\left(\exp\frac{2\pi i}{p^n}\right)$, $n > 1$, a construction of the character with a value in algebraic completion of Q_p is required.

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A number-theoretic constant*

by

R. J. Miech (Los Angeles, Calif.)

Let k be a positive integer, χ be a character modulo k, $L(s,\chi)$ be the associated L-series, and $\mathfrak{L}(s)$ be the product of all the L-series modulo k. It is well known that there is a positive constant c that does not depend on k such that $\mathfrak{L}(s)$ has at most one zero in the region

$$1-\frac{c}{\log k(2+|t|)} \leqslant \sigma \leqslant 1, \quad t \text{ arbitrary}.$$

There have been several investigations about the size of c. Landau, for example, proved that one could take $c > 1/(18.52)$ for the zeta function; that is, $c > 1/(18.52)$ for $k = 1$. ([5], p. 320.) More recently Pan Cheng-tung stated that one could take $c > 1/200$ for all large k [6] and Chen Jing-run claimed that this could be improved to $c > 1/(104.5)$ [2]. The interest of both of these authors in the nature of c was based on the fact that its size plays an important role in the determination of the size of the smallest prime of an arithmetic progression.

The purpose of this paper is to prove that one can take $c > 1/20$ for all large k. I think that this result can be employed to prove that the smallest prime in the progression $kn+l$ is less than k^a where a is somewhere in the neighborhood of 200; this would be an improvement on Chen Jing-run's result that $a \leqslant 777$. However I have not yet carried out the details of the proof.

Formally, the main result of this paper is the

THEOREM. *Let $\mathfrak{L}(s)$ be defined as above. Then there is a constant d_1 such that for all $k \geqslant d_1$ $\mathfrak{L}(s)$ has at most one zero in the region*

$$1-\frac{1}{20\log k(1+|t|)} \leqslant \sigma \leqslant 1, \quad t \text{ arbitrary}.$$

Moreover, if $\mathfrak{L}(s)$ does have one zero β in this region then β is real and it is a zero of $L(s,\chi_1)$ where χ_1 is a real character modulo k.

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Generally speaking the proof of this theorem follows customary lines. However its details, which stem from ideas of Heilbronn and Rademacher [4], [8], are somewhat complicated so I believe there is some point in giving a complete proof. I also believe that the size of the constant, 1/20, is about the best one can do with this type of argument; i.e. it might be possible to prove that $c>1/10$, but I doubt if one could prove that $c>1/4$. A short heuristic discussion of the limitations of the proof can be found in the final section of this paper.

1. The main result of this section is:

LEMMA 1. *Suppose that χ is a non-principal character modulo k and let ϱ denote a zero of $L(s,\chi)$. Let ε_1 and ε_2 be any fixed positive numbers and let a be any number such that $\varepsilon_1 \leqslant a \leqslant 500$. Set*

$$\sigma_0 = 1+\frac{a}{\log k(1+|t_0|)}, \qquad s_0 = \sigma_0+it_0,$$

$$r = \frac{1{,}000}{\log k(1+|t_0|)} \quad \text{and} \quad \lambda = \frac{1}{\pi}+\frac{1}{4} = .56830.$$

Then there is a constant d_2, which depends only on ε_1 and ε_2, such that if $k \geqslant d_2$ then

$$\operatorname{Re}\frac{L'}{L}(s_0,\chi) \geqslant \operatorname{Re}\sum_{\substack{\varrho \\ |\varrho-s_0|\leqslant r}} \frac{1}{s_0-\varrho} - (\lambda+\varepsilon_2)\log k(1+|t_0|).$$

The proof of Lemma 1 is based on the following result of Rademacher:

LEMMA 2. *Let χ be a primitive non-principal character modulo k. Let η be any number such that $0<\eta\leqslant 1/2$. Then, for $-\eta \leqslant \sigma \leqslant 1+\eta$ we have*

$$|L(s,\chi)| \leqslant \left(\frac{k|s+1|}{2\pi}\right)^{\frac12(1-\sigma+\eta)} \zeta(1+\eta).$$

See [8] for a proof.

If χ is not a primitive character we have

LEMMA 3. *Let χ be a non-principal character modulo k and suppose that the character χ^* formed to the modulus k^* is the conductor of χ. Let η be defined as above. Then if $k>d_2$ and $1/2 \leqslant \sigma \leqslant 1+\eta$ we have*

$$|L(s,\chi)| \leqslant c_1\big(k(1+|t|)\big)^{\frac12(1-\sigma+\eta)}\zeta(1+\eta)(\log\log k)^{c_2},$$

where c_1 and c_2 are positive constants.

Incidentally, from this point on $c_1, c_2, \ldots$ will denote positive absolute constants.

Proof of the lemma. We have

$$L(s,\chi) = L(s,\chi^*)\prod_{p|k/k^*}\left(1-\frac{\chi^*(p)}{p^s}\right)$$

and we can suppose that $k/k^* > \exp(e^{10})$.

Let M denote the number of prime factors of k/k^* and p_j denote the jth prime. We then have

$$\log\prod_{p|k/k^*}\left(1+\frac{1}{p^\sigma}\right) \leqslant \sum_{j=1}^{M}\frac{1}{p_j^\sigma}+c_3.$$

Now if

$$\sigma \leqslant 1-\frac{1}{\log\log(k/k^*)}$$

then

$$\sum_{j=1}^{M}\frac{1}{p_j^\sigma} \leqslant 1+\frac{M^{1-\sigma}}{1-\sigma} \leqslant 1+[\log\log(k/k^*)]\left[\frac{\log(k/k^*)}{\log 2}\right]^{1-\sigma}$$

$$\leqslant \frac{(1-\sigma+\eta)}{2}\log(k/k^*)+1.$$

Consequently,

$$\left|\prod_{p|k/k^*}\left(1-\frac{\chi^*(p)}{p^s}\right)\right| \leqslant c_4\left(\frac{k}{k^*}\right)^{\frac12(1-\sigma-\eta)}$$

and this inequality, combined with Lemma 2 yields the desired result if σ is not too close to 1.

If

$$1-\frac{1}{\log\log(k/k^*)} \leqslant \sigma \leqslant 1$$

then

$$\sum_{j=1}^{M}\frac{1}{p_j^\sigma} \leqslant M^{1-\sigma}\sum_{j=1}^{M}\frac{1}{p_j} \leqslant \left[\frac{\log(k/k^*)}{\log 2}\right]^{1-\sigma} c_5\log\log\left[\frac{\log(k/k^*)}{\log 2}\right]$$

$$\leqslant \frac{1-\sigma+\eta}{2}\log(k/k^*)+c_6\log\log\log k.$$

This implies that

$$\left|\prod_{p|k/k^*}\left(1-\frac{\chi^*(p)}{p^s}\right)\right| \leqslant (\log\log k)^{c_7}\left(\frac{k}{k^*}\right)^{\frac12(1-\sigma+\eta)}.$$

This result completes the proof of Lemma 3.

LEMMA 4. *Let χ, ϱ, a, s_0, r and λ be defined as in Lemma 1. Let R be any number such that $1/2 \leqslant R\log\log k \leqslant 2$ and $L(s,\chi) \neq 0$ for $|s-s_0| = R$. Then there is a positive number d_1 such that for $k \geqslant d_1$ we have*

$$\left|\frac{L'}{L}(s_0,\chi) - \sum_{\substack{\varrho \\ |\varrho - s_0| \leqslant R}} \left(\frac{1}{s_0-\varrho} + \frac{\overline{\varrho - s_0}}{R^2}\right)\right| \leqslant \left(\lambda + \frac{\varepsilon_2}{2}\right)\log k(1+|t_0|).$$

Proof. We begin with a change of variables. First, set

$$s' = \frac{s-s_0}{R} \quad \text{and} \quad \varrho' = \frac{\varrho - s_0}{R}.$$

Then let

$$\frac{L(s,\chi)}{L(s_0,\chi)} = f\left(\frac{s-s_0}{r}\right) = f(s').$$

Finally, define $g(s')$ by

$$g(s') = f(s')\prod_{\varrho}\frac{1-\bar{\varrho}'s'}{s'-\varrho'}$$

where ϱ runs through the zeros of $L(s,\chi)$ inside the circle $|s-s_0| = R$. (The form of the factors of the above product, which is important in this proof, is due to Heilbronn [4].)

We then have:

$$g(s') \text{ is analytic for } |s'| < 1,$$

$$g(s') \neq 0 \quad \text{for} \quad |s'| \leqslant 1,$$

$$|g(s')| = |f(s')| \quad \text{for} \quad |s'| = 1,$$

$$|g(0)| = |f(0)|\prod_{\varrho}\left|-\frac{1}{\bar{\varrho}'}\right| \geqslant 1.$$

These four conditions imply that there is a function $h(s')$ such that:

$$h(s') \text{ is analytic for } |s'| \leqslant 1,$$

$$g(s') = e^{h(s')} \quad \text{for} \quad |s'| \leqslant 1,$$

$$\operatorname{Re} h(s') = \log|f(s')| \quad \text{for} \quad |s'| = 1,$$

$$\operatorname{Re} h(0) \geqslant 0.$$

The last two conditions follow directly from the equations:

$$|g(s')| = |e^{h(s')}| = \exp\left(\operatorname{Re} h(s')\right),$$

$$|g(s')| = |f(s')| = \exp\log|f(s')| \quad \text{for} \quad |s'| = 1.$$

To continue, let

$$h(s') = a_0 + a_1 s' + \ldots + a_n (s')^n + \ldots$$

where

$$a_n = |a_n|e^{i\alpha_n} \quad \text{and} \quad |s'| = \vartheta e^{i\varphi},\ 0 \leqslant \vartheta \leqslant 1.$$

Integrating term-by-term gives us

$$\int_0^{2\pi} \operatorname{Re}[h(e^{i\varphi})]\,d\varphi = \sum_{m=0}^{\infty} |a_m| \int_0^{2\pi} \cos(\alpha_m + m\varphi)\,d\varphi = 2\pi \operatorname{Re}(a_0).$$

Moreover, for $n \geqslant 1$,

$$\int_0^{2\pi} \operatorname{Re}[h(e^{i\varphi})]\cos(\alpha_n + n\varphi)\,d\varphi = \pi|a_n|.$$

Thus, since

$$\operatorname{Re}(a_0) = \operatorname{Re}[h(0)] \geqslant 0,$$

we have for $n \geqslant 1$

$$\pi|a_n| \leqslant \pi|a_n| + 2\pi\operatorname{Re}(a_0) = \int_0^{2\pi} \operatorname{Re}[h(e^{i\varphi})]\left(1+\cos(\alpha_n + n\varphi)\right)d\varphi.$$

Recall that we have

$$\operatorname{Re}[h(e^{i\varphi})] = \operatorname{Re}[h(s')] = \log|f(s')| = \log\left|\frac{L(s,\chi)}{L(s_0,\chi)}\right|$$

where

$$s' = e^{i\varphi} \text{ if and only if } s = s_0 + Re^{i\varphi}.$$

Thus if $0 \leqslant \varphi \leqslant \pi/2$ then, since

$$|L(s,\chi)| \leqslant \sum_{n=1}^{\infty}\frac{1}{n^{\sigma_0}} = \zeta(\sigma_0) \quad \text{and} \quad \frac{1}{|L(s_0,\chi)|} \leqslant \sum_{n=1}^{\infty}\frac{1}{n^{\sigma_0}} = \zeta(\sigma_0),$$

it follows that

$$\int_0^{\pi/2} \operatorname{Re}[h(e^{i\varphi})]\left(1+\cos(\alpha_n+n\varphi)\right)d\varphi \leqslant \left(2\log\zeta(\sigma_0)\right)(2)(\pi/2).$$

Since a similar result holds if $(3/2)\pi \leqslant \varphi \leqslant 2\pi$ we have

$$\pi|a_n| \leqslant \int_{\pi/2}^{3\pi/2} \operatorname{Re}[h(e^{i\varphi})]\left(1+\cos(\alpha_n+n\varphi)\right)d\varphi + 4\pi\log\zeta(\sigma_0).$$

If $\pi/2 \leqslant \varphi \leqslant 3\pi/2$ and $s' = e^{i\varphi}$ we then have $s = s_0 + Re^{i\varphi}$. That is

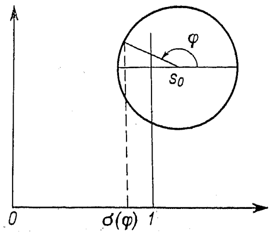

Fig. 1

$$-\cos\varphi = \cos(\pi-\varphi) = \frac{\sigma_0-\sigma(\varphi)}{R} \quad\text{or}\quad -\sigma(\varphi)+\sigma_0 = -R\cos\varphi.$$

Thus if we apply Lemma 3, with $\eta = \sigma_0-1$, to the function

$$\operatorname{Re}[h(e^{i\varphi})] = \log\left|\frac{L(s,\chi)}{L(s_0,\chi)}\right| = \log\left|\frac{L(s_0+Re^{i\varphi},\chi)}{L(s_0,\chi)}\right|$$

we get

$$\operatorname{Re}[h(e^{i\varphi})] \leqslant \frac{1-\sigma(\varphi)+\sigma_0-1}{2}\log k(1+|t_0|)+2\log\zeta(\sigma_0)+c_8\log_3 k$$
$$= \frac{-R\cos\varphi}{2}L+2\log\zeta(\sigma_0)+c_8\log_3 k$$

where $L = \log k(1+|t_0|)$. This, in turn implies that

$$\pi|a_n| \leqslant L\frac{R}{2}\int_{\pi/2}^{3\pi/2}(-1)\cos\varphi\big(1+\cos(a_n+n\varphi)\big)\,d\varphi+c_9\big(\log\zeta(\sigma_0)+\log_3 k\big),$$

or

$$|h'(0)| = |a_1| \leqslant \frac{R}{\pi}\left(1-\frac{\pi}{4}\cos a_1\right)L+c_{10}\big(\log\zeta(\sigma_0)+\log_3 k\big).$$

It is not difficult to see, from the definitions at the beginning of this proof, that

$$\frac{L'}{L}(s,\chi)-\sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{\overline{\varrho-s_0}}{R^2-(\overline{\varrho-s_0})(s-s_0)}\right) = \frac{|h'(s')|}{R}$$

for $|s-s_0| < R$. Consequently, for $s = s_0$

$$\left|\frac{L'}{L}(s_0,\chi)-\sum_{\varrho}\left(\frac{1}{s_0-\varrho}+\frac{\overline{\varrho-s_0}}{R^2}\right)\right| \leqslant \frac{h'(0)}{R}$$
$$\leqslant \frac{1}{\pi}\left(1-\frac{\pi}{4}\cos a_1\right)L+\frac{c_{10}}{R}\big(\log\zeta(\sigma_0)+\log_3 k\big) = B.$$

Since $\sigma_0 = 1+(a/\log k)$ and $\zeta(\sigma_0) \leqslant c_{11}\log k$ we have $\log\zeta(\sigma_0) \leqslant c_{12}\log_2 k$. Thus, since $L = \log k(1+|t_0|)$ and $1/R \leqslant 2\log\log k$, we have

$$B \leqslant \left(\frac{1}{\pi}+\frac{1}{4}+\frac{\varepsilon_2}{2}\right)\log k(1+|t_0|)$$

for sufficiently large k. This completes the proof of Lemma 4.

Lemma 1 follows directly from Lemma 4. Note first of all that since

$$\operatorname{Re}\left(\frac{1}{s_0-\varrho}+\frac{\overline{\varrho-s_0}}{R^2}\right) = \operatorname{Re}\left[\frac{1}{s_0-\varrho}\left(1-\frac{|s_0-\varrho|^2}{R^2}\right)\right] \geqslant 0$$

we have

$$\operatorname{Re}\frac{L'}{L}(s_0,\chi) \geqslant \operatorname{Re}\sum_{\substack{\varrho\\ |\varrho-s_0|\leqslant r}}\left(\frac{1}{s_0-\varrho}+\frac{\overline{\varrho-s_0}}{R^2}\right)-\left(\lambda+\frac{\varepsilon_2}{2}\right)\log k(1+|t_0|)$$

where $r = 1{,}000/\log k(1+|t_0|)$. Secondly, if $|\overline{\varrho-s_0}| \leqslant r$ then

$$\frac{\overline{\varrho-s_0}}{R^2} \leqslant c_{13}\frac{(\log\log k)^2}{\log k(1+|t_0|)}.$$

Thirdly, since the rectangle $|t-t_0| \leqslant 1$, $1/2 \leqslant \sigma \leqslant 1$ contains at most $c_{14}\log k(1+|t_0|)$ zeros of $L(s,\chi)$ we have

$$\sum_{\substack{\varrho\\ |\varrho-s_0|\leqslant r}}\left|\frac{\overline{s-\varrho_0}}{R^2}\right| \leqslant c_{15}(\log\log k)^2.$$

If these results are brought together we have Lemma 1.

The next lemma contains the information we need about the principal character.

Lemma 5. *If χ_0 is the principal character modulo k and $s = \sigma+it$ where $\sigma > 1$ then*

$$\operatorname{Re}\frac{L'}{L}(s,\chi_0) \geqslant -\operatorname{Re}\frac{1}{s-1}-c_{16}\frac{\log k(1+|t|)}{\log\log k}$$

for all real t.

Proof. Since χ_0 is principal

$$\frac{L'}{L}(s,\chi_0) = \frac{\zeta'}{\zeta}(s)+\sum_{p|k}\frac{\log p}{p^s-1}.$$

If M is the number of prime factors of k and p_j is the jth prime then

$$\Big|\sum_{p|k}\frac{\log p}{p^s-1}\Big| \leqslant 2\sum_{j=1}^{M}\frac{\log p}{p} \leqslant c_{17}\log\log k.$$

Suppose now that $|t| > \log k$. Then it is known that

$$\Big|\frac{\zeta'}{\zeta}(s)\Big| \leqslant c_{18}\frac{\log t}{\log\log t}.$$

See [9], p. 52, p. 98 for a proof. Thus

$$\Big|\frac{L'}{L}(s,\chi_0)\Big| \leqslant c_{17}\log\log k + c_{18}\frac{\log t}{\log\log t} \leqslant c_{19}\frac{\log k(1+|t|)}{\log\log k}$$

for $|t| > \log k$.

On the other hand if $|t| \leqslant \log k$, we can apply the known representation ([7], p. 218)

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} - \frac{1}{s} + b - \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big) + \sum_{\varrho}\Big(\frac{\varrho}{s-\varrho}+\frac{1}{\varrho}\Big),$$

where b is a constant. Since

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + O\Big(\frac{1}{|s|^2}\Big)$$

for $\sigma > 1/2$ we have

$$\Big|\mathrm{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)\Big| \leqslant c_{20}\log\log k$$

for $|t| \leqslant \log k$. Moreover, since

$$\mathrm{Re}\Big(\frac{1}{s-\varrho}+\frac{1}{\varrho}\Big) > 0$$

it follows that

$$\mathrm{Re}\frac{\zeta'}{\zeta}(s) \geqslant -\mathrm{Re}\Big(\frac{1}{s-1}\Big) - c_{21}\log\log k \geqslant -\mathrm{Re}\Big(\frac{1}{s-1}\Big) - c_{22}\frac{\log k(1+|t|)}{\log\log k}.$$

This completes the proof of Lemma 5.

LEMMA 6. *Suppose that*

$$0 \geqslant -\frac{Q}{a} + \frac{R}{a+b} - S$$

where $R > Q > 0$ *and* $S > 0$. *Then if we set*

$$a = \frac{-Q+\sqrt{RQ}}{S}$$

we have

$$b \geqslant \frac{(\sqrt{Q}-\sqrt{R})^2}{S}.$$

LEMMA 7. *If* φ *is real then*

$$47+80\cos\varphi+49\cos 2\varphi+20\cos 3\varphi+4\cos 4\varphi \geqslant 0,$$
$$5+8\cos\varphi+4\cos 2\varphi+\cos 3\varphi \geqslant 0,$$
$$17+24\cos\varphi+8\cos 2\varphi \geqslant 0.$$

The first inequality is due to Heilbronn [4]; the last two are due to Landau [5].

2. We are now in a position to prove the Theorem of this paper. Generally speaking the proof is quite simple if we are not close to the real axis; however a number of special arguments are needed when we approach this line.

Our first result is

LEMMA 8. *Suppose that* χ *is a non-principal character modulo* k. *Then there is a positive constant* d_3 *such that if* $k \geqslant d_3$ *then* $L(s,\chi) \neq 0$ *in the region*

$$1-\frac{1}{20\log k(1+|t|)} \leqslant \sigma \leqslant 1, \quad t \text{ arbitrary},$$

provided that $\chi^j \neq \chi_0$ *for* $j = 2, 3,$ *and* 4. *If* $\chi^j = \chi_0$ *for* $j = 2, 3,$ *or* 4 *then* $L(s,\chi) \neq 0$ *in the region*

$$1-\frac{1}{20\log k(1+|t|)} \leqslant \sigma \leqslant 1, \quad |t| \geqslant \frac{\sqrt{.32}}{j\log k}.$$

Before turning to the proof of this lemma I would like to point out several notational devices that will be used for the balance of this section. First of all, the symbol ε will represent a positive number that can be taken as small as desired; its value will usually be different each time it appears. Secondly, it will be assumed, without further comment that k is large enough for the purpose at hand.

Proof of Lemma 8. Let $\varrho = \beta+i\gamma$ be a zero of $L(s,\chi)$. Set

$$t_0 = \gamma, \quad \sigma_0 = 1+\frac{a}{\log k(1+|t_0|)} \quad \text{and} \quad \beta = 1-\frac{b}{\log k(1+|t_0|)}.$$

By the first inequality of Lemma 7 we have

$$0 \geqslant \operatorname{Re}\Big[47\frac{L'}{L}(\sigma_0,\chi_0)+80\frac{L'}{L}(\sigma_0+it_0,\chi)+49\frac{L'}{L}(\sigma_0+i2t_0,\chi^2)+ +20\frac{L'}{L}(\sigma_0+i3t_0,\chi^3)+4\frac{L'}{L}(\sigma_0+i4t_0,\chi^4)\Big].$$

By Lemma 6

$$\operatorname{Re}\frac{L'}{L}(\sigma_0,\chi_0) \geqslant -\frac{1}{\sigma_0-1}-\frac{c_{16}\log k(1+|t_0|)}{\log\log k}.$$

By Lemma 1

$$\operatorname{Re}\frac{L'}{L}(\sigma_0+it_0,\chi) \geqslant \frac{1}{\sigma_0-\beta}-(\lambda+\varepsilon)\log k(1+|t_0|).$$

In addition if $\chi^j \neq \chi_0$ for $j = 2, 3,$ and 4 then

$$\operatorname{Re}\frac{L'}{L}(\sigma_0+ijt_0,\chi^j) \geqslant -(\lambda+\varepsilon)\log k(1+j|t_0|) \geqslant -(\lambda+\varepsilon)\log k(1+|t_0|).$$

If these results are brought together we have

$$0 \geqslant \frac{-47}{\sigma_0-1}+\frac{80}{\sigma_0-\beta}-153(\lambda+\varepsilon)\log k(1+|t_0|),$$

or

$$0 \geqslant \frac{-47}{a}+\frac{80}{a+b}-153(\lambda+\varepsilon).$$

According to Lemma 6, if

$$a = \frac{-47+\sqrt{(80)(47)}}{153(\lambda+\varepsilon)} \approx .1646$$

then

$$b \geqslant \frac{(\sqrt{80}-\sqrt{47})^2}{153(\lambda+\varepsilon)} \geqslant .05016 \geqslant \frac{1}{20}.$$

This completes the proof of the first part of Lemma 8.

If $\chi^j = \chi_0$ for $j = 2, 3,$ or 4 then, by Lemma 6,

$$\operatorname{Re}\frac{L'}{L}(\sigma_0+ijt_0) \geqslant -\operatorname{Re}\left(\frac{1}{\sigma_0+ijt_0-1}\right)-c_{16}\frac{\log k(1+|t_0|)}{\log\log k}.$$

Thus, if we put a condition on t, say $|t| \geqslant A/\log k$, that will insure that

$$\operatorname{Re}\left[\frac{1}{\sigma_0+ijt_0-1}\right] \leqslant (\lambda+\varepsilon)\log k(1+|t_0|)$$

we can use the argument employed above, for we will have the same lower bound at σ_0+ijt_0.

Set $L = \log k(1+|t_0|)$. Then, supposing that $|t| > A/\log k$,

$$\operatorname{Re}\left(\frac{1}{\sigma_0-1+ijt_0}\right) = \frac{a}{L}\left[\left(\frac{a}{L}\right)^2+(jt_0)^2\right]^{-1} \leqslant \frac{a}{a^2+j^2A^2}L.$$

Since $\lambda+\varepsilon \geqslant .56$ we will have what we want if

$$\frac{a}{a^2+j^2A^2} \leqslant \frac{1}{2} \quad\text{or}\quad 2a-a^2 \leqslant j^2A^2.$$

Since $a \leqslant .17$ the last inequality will hold if we set $A = \sqrt{.32}/j$.

Once we have done so we have what is needed to prove the second part of Lemma 8.

Lemma 9. *If χ is a complex character such that $\chi^4 = \chi_0$ and $L(s,\chi)$ has a zero $\varrho = \beta+i\gamma$ with*

$$|\gamma| \leqslant \frac{\sqrt{.32}}{4}\cdot\frac{1}{\log k} = \frac{\sqrt{.02}}{\log k}$$

then

$$\beta \leqslant 1-\frac{1}{20\log k(1+|\gamma|)}.$$

Proof. Let

$$\gamma = t_0,\quad \beta = 1-\frac{b}{\log k(1+|t_0|)}\quad\text{and}\quad \sigma_0 = 1+\frac{a}{\log k(1+|t_0|)}$$

where $a = .17928$.

The second inequality of Lemma 7 implies that

$$0 \geqslant \operatorname{Re}\Big[5\frac{L'}{L}(\sigma_0,\chi_0)+8\frac{L'}{L}(\sigma_0+it_0,\chi)+4\frac{L'}{L}(\sigma_0+i2t_0,\chi^2)+ +\frac{L'}{L}(\sigma_0+i3t_0,\chi^3)\Big].$$

Now, the first three terms of this inequality can be treated in the usual fashion. We have, by Lemmas 5 and 1,

$$\operatorname{Re}\frac{L'}{L}(\sigma_0,\chi_0) \geqslant -\frac{1}{\sigma_0-1}-c_{16}\frac{\log k}{\log\log k},$$

$$\operatorname{Re}\frac{L'}{L}(\sigma_0+it_0,\chi) \geqslant \frac{1}{\sigma_0-\beta}-(\lambda+\varepsilon)\log k(1+|t_0|)$$

and

$$\operatorname{Re}\frac{L'}{L}(\sigma_0+i2t_0,\chi^2) \geqslant -(\lambda+\varepsilon)\log k(1+|t_0|).$$

The term associated with χ^3 requires special attention. Since $\chi^4 = \chi_0$ we have $\chi^3 = \bar{\chi}$, i.e. $\bar{\varrho} = \beta - i\gamma$ is a zero of $L(s, \chi^3)$. Moreover $\bar{\varrho}$ will appear in the sum of the inequality

$$\operatorname{Re}\frac{L'}{L}(\sigma_0+i3t_0, \chi^3) \geqslant \sum_{\substack{\varrho \\ |\varrho-(\sigma_0+i3t_0)|\leqslant r}} \frac{1}{(\sigma_0+i3t_0-\varrho)} - (\lambda+\varepsilon)\log k(1+|t_0|)$$

for $\sigma_0 = 1+\big(a/\log k(1+|t_0|)\big)$, $a = .17928$, $r = 1{,}000/\log k$, and $|t_0| \leqslant \sqrt{.02}/\log k$. Thus it follows that

$$\operatorname{Re}\frac{L'}{L}(\sigma_0+i3t_0, \chi^3) \geqslant \left[\frac{a+b}{(a+b)^2+.33} - (\lambda+\varepsilon)\right]\log k(1+|t_0|).$$

If we bring these results together we have

$$0 \geqslant -\frac{5}{a} + \frac{8}{a+b} + \frac{a+b}{(a+b)^2+.33} - 13\lambda_1$$

where $\lambda_1 = \lambda+\varepsilon$. This inequality is equivalent to

$$b \geqslant -a + \frac{8a}{5+13\lambda_1 a} + \frac{(a+b)^2}{\big((a+b)^2+.33\big)}\cdot\frac{a}{(5+13\lambda_1 a)}.$$

If we set

$$a = \frac{-5+\sqrt{40}}{13\lambda_1} = .17928$$

we have

$$-a + \frac{8a}{(5+13\lambda_1 a)} \geqslant .04749 \quad\text{and}\quad \frac{a}{5+13\lambda_1 a} \geqslant .02834.$$

That is,

$$b \geqslant .04749 + \frac{(.17928+b)^2}{(.17928+b)^2+.33}(.02834),$$

or

$$b^3 + (.28273)b^2 + (.33495)b - (.01810) \geqslant 0.$$

This last inequality completes the proof of Lemma 9, for it implies that $b > .05$.

LEMMA 10. *Let χ be a complex character modulo k such that $\chi^3 = \chi_0$. Suppose that $\varrho = \beta+i\gamma$ is a zero of $L(s, \chi)$ with*

$$|\gamma| \leqslant \sqrt{.32}/(3\log k).$$

Then

$$\beta \leqslant 1 - \frac{1}{20\log k(1+|\gamma|)}.$$

Proof. There are two parts to the proof. Suppose first that

$$\frac{.12}{\log k} \leqslant |\gamma| \leqslant \frac{\sqrt{.32}}{3\log k}.$$

The argument for this case will be based on the inequality

$$0 \geqslant \operatorname{Re}\Big[5\frac{L'}{L}(\sigma_0+it_0, \chi_0) + 8\frac{L'}{L}(\sigma_0+it_0, \chi) + 4\frac{L'}{L}(\sigma_0+i2t_0, \chi^2) + \frac{L'}{L}(\sigma_0+i3t_0, \chi^3)\Big]$$

where $t_0 = \gamma$, $\beta = 1-\big(b/\log k(1+|t_0|)\big)$, $\sigma_0 = 1+\big(a/\log k(1+|t_0|)\big)$ and $a = .19405$. This is the same inequality as the one employed in the proof of Lemma 9 and the argument is similar to that one: Since $\chi^3 = \chi_0$ we have $\chi^2 = \bar{\chi}$. Hence $\bar{\varrho} = \beta - i\gamma$ is a zero of $L(s, \chi^2)$ and, as before,

$$\operatorname{Re}\frac{L'}{L}(\sigma_0+i2t_0, \chi^2) \geqslant \left[\frac{a+b}{(a+b)^2+.33} - (\lambda+\varepsilon)\right]\log k(1+|t_0|).$$

Moreover, since $\chi^3 = \chi_0$ and $|\gamma| = |t_0| > .12/\log k$,

$$\operatorname{Re}\frac{L'}{L}(\sigma_0+i3t_0, \chi^3) \geqslant \left[\frac{-a}{a^2+.1296} - \frac{c_{16}}{\log\log k}\right]\log k(1+3|t_0|).$$

Utilizing the customary estimates at σ_0 and σ_0+it_0 we get

$$0 \geqslant -\frac{5}{a} + \frac{8}{a+b} - 12(\lambda+\varepsilon) + \frac{4(a+b)}{(a+b)^2+.33} - \frac{a}{a^2+.1296}.$$

Since

$$\frac{4(a+b)}{(a+b)^2+.33} - \frac{a}{a^2+.1296} = \frac{3a^3+.1884a+2a^2b+b(.5184-ab)}{\big((a+b)^2+.33\big)(a^2+.1296)} > 0,$$

for we can assume that $b < 1$ and $a < 1/2$, we have

$$0 \geqslant -\frac{5}{a} + \frac{8}{a+b} - 12(\lambda+\varepsilon).$$

If we set

$$a = \frac{-5+\sqrt{40}}{12(\lambda+\varepsilon)} = .19405$$

then

$$b \geqslant \frac{(\sqrt{8}-\sqrt{5})^2}{12(\lambda+\varepsilon)} \geqslant .051,$$

the desired inequality.

If $L(s,\chi)$ has a zero $\varrho = \beta+i\gamma$ with

$$0 \leqslant |\gamma| \leqslant .12/\log k$$

we shall base our argument on the inequality

$$0 \geqslant \operatorname{Re}\left[17\frac{L'}{L}(\sigma_0,\chi_0)+24\frac{L'}{L}(\sigma_0,\chi)+8\frac{L'}{L}(\sigma_0,\chi^2)\right].$$

If we set $c = .12$, $\sigma_0 = 1+(a/\log k)$, $\beta = 1-(b/\log k)$ we have

$$\operatorname{Re}\frac{L'}{L}(\sigma_0,\chi) \geqslant \left[\frac{a+b}{(a+b)^2+c^2}-(\lambda+\varepsilon)\right]\log k.$$

Similarly, since $\bar{\varrho} = \beta-i\gamma$ is a zero of $L(s,\chi^2)$

$$\operatorname{Re}\frac{L'}{L}(\sigma_0,\chi^2) \geqslant \left[\frac{a+b}{(a+b)^2+c^2}-(\lambda+\varepsilon)\right]\log k.$$

Consequently,

$$0 \geqslant -\frac{17}{a}+\frac{32(a+b)}{(a+b)^2+c^2}-32(\lambda+\varepsilon)$$

or, with $\lambda_1 = \lambda+\varepsilon$ and $x = (32a/[17+32a\lambda_1])$

$$b \geqslant -a+\tfrac{1}{2}x\left(1+\sqrt{1-(2c/x)^2}\right).$$

If we set

$$a = \frac{-17+\sqrt{(17)(32)}}{32\lambda_1} \leqslant .34774$$

then

$$x \geqslant .475 \quad \text{and} \quad 1/x \leqslant 2.11.$$

That is, since $c = .12$,

$$b \geqslant -.348+(.475)(.93) \geqslant .10.$$

This completes the proof of Lemma 10.

Lemma 11. *Suppose that χ is a non-principal real character modulo k and $\varrho = \beta+i\gamma$ is a zero of $L(s,\chi)$ with*

$$0 < |\gamma| \leqslant \sqrt{.32}/(2\log k).$$

Then

$$\beta \leqslant 1-\frac{.15}{\log k(1+|\gamma|)}.$$

The proof of this result is based on the inequality

$$0 \geqslant \operatorname{Re}\frac{L'}{L}(\sigma_0,\chi_0)+\operatorname{Re}\frac{L'}{L}(\sigma_0,\chi)$$

and the fact that $\bar{\varrho} = \beta-i\gamma$ is also a zero of $L(s,\chi)$. If we set $\sigma_0 = 1+(a/\log k)$ where $a = 1/2$, $\beta = 1-(b/\log k)$ and $c = \sqrt{.32}/2$ we have

$$\operatorname{Re}\frac{L'}{L}(\sigma_0,\chi) \geqslant \left[\frac{2(a+b)}{(a+b)^2+c^2}-(\lambda+\varepsilon)\right]\log k.$$

Arguing as before we get

$$0 \geqslant -\frac{1}{a}+\frac{2(a+b)}{(a+b)^2+c^2}-(\lambda+\varepsilon)$$

or

$$b \geqslant -a+\frac{a}{1+a\lambda_1}\left(1+\left[1-\left(\frac{c(1+a\lambda_1)}{a}\right)^2\right]^{1/2}\right)$$

where $\lambda_1 = \lambda+\varepsilon$. If we set $a = 1/2$ then

$$\frac{a}{1+a\lambda_1} \geqslant .389, \qquad \frac{1+a\lambda_1}{a} \leqslant 2.57$$

and

$$\left[1-\left(\frac{c(1+a\lambda_1)}{a}\right)^2\right]^{1/2} \geqslant \sqrt{1-.53} \geqslant .68.$$

That is

$$b \geqslant -.500+(.389)(1.68) \geqslant .153.$$

Lemma 12. *If χ is a non-principal real character modulo k then $L(s,\chi)$ has at most one zero on the line*

$$1-\frac{.28}{\log k} \leqslant \sigma \leqslant 1, \quad t = 0.$$

Proof. Suppose that

$$\beta_1 = 1-\frac{b_1}{\log k} \quad \text{and} \quad \beta_2 = 1-\frac{b_2}{\log k},$$

where $0 < b_1 \leqslant b_2$, are two real zeros of $L(s,\chi)$. Then the inequality

$$0 \geqslant \operatorname{Re}\frac{L'}{L}(\sigma_0,\chi_0)+\operatorname{Re}\frac{L'}{L}(\sigma_0,\chi)$$

implies that

$$0 \geqslant -\frac{1}{a}+\frac{1}{a+b_1}+\frac{1}{a+b_2}-(\lambda+\varepsilon) \geqslant -\frac{1}{a}+\frac{2}{a+b_2}-(\lambda+\varepsilon).$$

That is,

$$b_2 \geqslant \frac{(\sqrt{2}-1)^2}{(\lambda+\varepsilon)} \geqslant .28.$$

At this stage we have proved, with one qualification, that if χ is any non-principal character modulo k then $L(s,\chi) \neq 0$ in the region

$$1-\frac{1}{20\log k(1+|t|)} \leqslant \sigma \leqslant 1, \quad t \text{ arbitrary}.$$

The qualification arises if χ is real; then $L(s,\chi)$ may have one real zero β in this region.

Let E denote this set of exceptional real zeros for the modulus k and suppose that β_1, a zero of $L(s,\chi_1)$, is the largest element in E. We then have

$$\beta_1 = 1-\frac{b_1}{\log k} \quad \text{where} \quad b_1 < 1/20.$$

Next, let χ_2 be any non-principal character modulo k that is not equal to χ_1 and let

$$\beta_2 = 1-\frac{b_2}{\log k}$$

be a real zero of $L(s,\chi_2)$. Since

$$0 \geqslant \operatorname{Re}\left[\frac{L'}{L}(\sigma_0,\chi)+\frac{L'}{L}(\sigma_0,\chi_1)+\frac{L'}{L}(\sigma_0,\chi_2)+\frac{L'}{L}(\sigma_0,\chi_1\chi_2)\right]$$

we have

$$0 \geqslant -\frac{1}{a}+\frac{1}{a+b_1}+\frac{1}{a+b_2}-3(\lambda+\varepsilon) \geqslant -\frac{1}{a}+\frac{2}{a+b_2}-3(\lambda+\varepsilon)$$

or

$$b_2 \geqslant \frac{(\sqrt{2}-1)^2}{3(\lambda+\varepsilon)} \geqslant \frac{.05709}{.56831} \geqslant \frac{1}{10}.$$

This shows that at most one L-series has a zero on the real line

$$1-\frac{1}{10\log k} \leqslant \sigma \leqslant 1,$$

and it completes the proof of our theorem for all non-principal characters modulo k.

If $L(s,\chi_0)$ is the principal character series modulo k then since the zeros of $L(s,\chi_0)$ coincide with those of $\zeta(s)$, since there are constants c_1, c_2, and c_3 such that $\zeta(s) \neq 0$ in the region

$$\sigma \geqslant 1-\frac{c_1}{\log(2+|t|)}, \quad |t| \leqslant c_2,$$

$$\sigma \geqslant 1-c_3\frac{\log\log t}{\log t}, \quad |t| \geqslant c_2,$$

and since this region contains the region $\sigma \geqslant 1-20\log k(1+|t|)^{-1}$, for sufficiently large k, we will have $L(s,\chi_0) \neq 0$ for $\sigma \geqslant 1-20\log k(1+|t|)^{-1}$.

3. The limitations of the proof given here are based on two considerations. First of all, let A be a number such that

$$\left|\frac{L'}{L}(s_0,\chi)-\sum_{\substack{\varrho \\ |\varrho-s_0|\leqslant r}}\frac{1}{s_0-\varrho}\right| \leqslant A\log k(1+|t_0|)$$

where $s_0 = \sigma_0+it_0$, $\sigma_0 = 1+\big(a/\log k(1+|t_0|)\big)$ and r is large enough so that any possible zero $\varrho = \beta+it_0$ is included in the circle $|s-s_0| \leqslant r$. Secondly, let $a_0, a_1, \ldots, a_n$ be a set of real numbers such that

$$a_1 > a_0 > 0$$

and

$$a_0+a_1\cos\theta+\ldots+a_n\cos n\theta \geqslant 0$$

for all real θ. Let

$$B = \frac{(\sqrt{a_1}-\sqrt{a_0})^2}{a_1+|a_2|+\ldots+|a_n|}.$$

An examination of the proof of Lemma 8 reveals that if $\varrho = \beta+it_0$ is a zero of $L(s,\chi)$ that is not too close to the real axis, say $|t_0| > 1/\log k$, and $\beta = 1-\big(b/\log k(1+|t_0|)\big)$ then

$$b \geqslant (B/A)-\varepsilon.$$

Thus the problem of finding and upper bound for β is that of minimizing A and maximizing B.

The minimal value A might assume is rather obscure. On one hand there are results that might lead one to believe that it is very small. Vinogradov's method, for example, shows that A approaches 0 for values of t_0 that are large relative to k ([7], Ch. 8); an argument of Linnik, Barban, and Tschudakov shows that A approaches 0 as k increases for values of t_0 that are not to large relative to k, provided that k is a power of a prime [1]. On the other hand, the techniques employed in the proof of Lemma 1 seem to be the only known ones which work for all values of t and all large moduli k. An examination of the formula

$$\frac{L'}{L}(s,\chi)=\frac{1}{2}\log\pi-\frac{1}{2}\log k-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}(s+a)\right)+a_1+\sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right)$$

where a and a_1 are constants that depend on χ and χ is a primitive character modulo k, suggests that any argument that would produce a value of A less than 1/2 would be extremely deep.

It is fairly easy to show that we must have $B \leqslant .58$. We begin by dividing the numbers $a_0, a_1, \ldots, a_n$ to get 1 as a constant term. Our problem then becomes that of finding a set of numbers $a_1, a_2, \ldots, a_n$ with $a_1 > 1$ such that

(1) $$1+a_1\cos\theta+\ldots+a_n\cos n\theta \geqslant 0 \quad \text{for all real } \theta,$$

(2) $$B=\frac{(\sqrt{a_1}-1)^2}{a_1+|a_2|+\ldots+|a_n|} \quad \text{is as large as possible.}$$

If we replace θ by π in (1) we get

$$|a_2|+|a_3|+\ldots+|a_n| \geqslant a_2-a_3+\ldots+(-1)^n a_n \geqslant a_1-1.$$

That is,

$$B \leqslant \frac{(\sqrt{a_1}-1)^2}{2a_1-1}.$$

It is also known [3] that if (1) is true then

$$|a_1| \leqslant 2\cos\big(\pi/(n+2)\big) < 2.$$

Since we must have $a_1 > 1$ it follows that

$$B < \frac{(\sqrt{2}-1)^2}{3} < .058.$$

In short, if we cannot find a value of A that is less than 1/2 then, since $B \leqslant .058$ we cannot expect anything better than $b = .116 < 1/8$.

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UNIVERSITY OF CALIFORNIA
Los Angeles

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