Here

\[
\frac{1}{|G(a)|} \leq J_s(Q) = \prod_{\mu=1}^{\nu} (M_s^\mu(Q))^{b_\mu}
\]

\[
|R(z)G(z)|_{|z|=R} \leq r_1 \ldots r_\nu |J_1(Q)|^{b_1-1} J_\nu(R) \leq |J_s(Q)|^{b_1-1-q} J_\nu(R),
\]

where

\[
J_s(R) = \prod_{\mu=1}^{\nu} (M_s^\mu(R))^{b_\mu},
\]

\[
\prod_{\nu \neq \mu} \left( \frac{a_\nu - a_\mu}{z - a_\mu} \right) \leq \left( \frac{2D(Q)}{R - D(Q)} \right)^{N(Q-1)} \leq \left( \frac{4D(Q)}{R} \right)^{N(Q-1)},
\]

\[
1 < \frac{1}{|z - a_\mu|} \leq \frac{1}{R - D(Q)} \leq \frac{2}{R}
\]

and so

\[
|y| \leq J_s(Q) \frac{1}{2\pi} \cdot \frac{2\pi R |J_1(Q)|^{b_1-1} J_\nu(R)}{R} \left( \frac{4D(Q)}{R} \right)^{N(Q-1)}
\]

\[
\leq |J_s(Q)|^{b_1-1-q} J_\nu(R) \frac{8D(Q)}{R} \left( \frac{4D(Q)}{R} \right)^{N(Q-1)}.
\]

Combining all our estimates for \(\gamma\), we get

\[
1 \leq |J_s(Q)|^{b_1-1-q} J_\nu(R) \frac{8D(Q)}{R} \left( \frac{4D(Q)}{R} \right)^{N(Q-1)}
\]

and since the exponent of \(J_s(Q)\) on the right does not exceed \(b_1 + b_3 \log (1 + \epsilon) = 8b_1 + 8\), for \(\epsilon = 1\), this proves the validity of the inequality of the Main Theorem for \(R \geq 2D(Q)\) and the inequality is trivial for \(R \leq 8D(Q)\). This completes the proof of the Main Theorem.

**Contributions to the theory of transcendental numbers (II)**

by

K. Ramachandra (Bombay)

To the 60-th birthday of A. O. Gelfond

§1. A special case of the Main Theorem. In this section we apply a special case of the Main Theorem of the earlier paper I ([9], p. 69) to deduce Theorem 1 below which will be the only theorem to which we shall refer in the later sections of this paper. We begin with some definitions (incidentally we also recall the notation). From now on we deal only with meromorphic functions which are quotients of entire functions of finite order. Given \(s (\geq 2)\) algebraically independent meromorphic functions \(F_1(z), \ldots, F_s(z)\) we introduce with respect to these

DEFINITION 1. A weighted sequence \(S\) (often we write \(\{a_\mu\}_0^\infty\) for \(S\)) is an infinite sequence \(\{a_\mu\}_0^\infty\) of distinct complex numbers together with an infinite subsequence \(\{a_\mu\}_0^\infty\) \(\mu = 1, 2, 3, \ldots\) which may be the same as \(\{a_\mu\}_0^\infty\) and an infinite sequence \(\{a_\mu\}_0^\infty\) \(\mu = 1, 2, 3, \ldots\) of natural numbers not necessarily distinct satisfying the following conditions.

(i) The sequence \(\{a_\mu\}_0^\infty\) is non-decreasing.

(ii) For each \(q = 1, 2, 3, \ldots\) there are only finitely many \(\{a_\mu\}_0^\infty\) for which \(a_\mu \neq q\). We denote this number by \(N(q)\). It follows that there are only finitely many numbers \(a_\mu\) for which \(a_\mu \neq q\) and this number \(N_q\) does not exceed \(N(q)\).

(iii) The limits

\[
\delta = \lim_{Q \to \infty} \frac{\log N(Q)}{\log Q} \quad \text{and} \quad \delta_1 = \lim_{Q \to \infty} \frac{\log N_1(Q)}{\log Q}
\]

exist and are finite.

(iv) The upper limit

\[
\limsup_{Q \to \infty} \frac{\max |a_\mu|}{Q}
\]

is finite.

(v) Whenever a polynomial in \(F_1(z), \ldots, F_s(z)\) with complex coefficients vanishes for all values \(z = a_\mu\) with \(a_\mu \leq Q\), it also vanishes for all values \(z = a_\mu\) with \(a_\mu \leq Q\).
(vi) The numbers \( F_t(s_\mu) \) \( (t = 1, \ldots, s; \mu = 1, 2, 3, \ldots) \) are all algebraic numbers lying in some fixed algebraic number field of degree \( h \) (the number field and its degree may be different for different \( \mu \)).

**Definition 2.** The numbers \( \delta, \delta_1 \) in (iii) are called the major and minor densities of the weighted sequence \( S \) and \( \delta - \delta_1 \) (which we know to be non-negative) is called the deviation of \( S \).

**Definition 3.** Next we write \( F_t(z) = H_t(z) G_t(z) \) \( (t = 1, \ldots, s) \) where \( H_t(z) \) and \( G_t(z) \) are entire functions without common zeros and the maximum of the orders of \( H_t(z) \) and \( G_t(z) \) is least possible. We define this number \( g_t \) as the order of the function \( F_t(z) \).

**Definition 4.** A weighted sequence \( S \) is said to be special if it satisfies the following hypotheses:

\[
\limsup_{Q \to \infty} \frac{\log|\log(\max_{n \leq Q} F_t(s_\mu))|}{\log Q} < g_t
\]

and

\[
\limsup_{Q \to \infty} \frac{\log|\log(\max_{n \leq Q} G_t(s_\mu))|}{\log Q} < g_t.
\]

(Since an algebraic number \( a \) is as usual \( d(a) + n \) where \( d(a) \) is the least natural number for which \( ad(a) \) is an algebraic integer.)

**Remark.** We can work with weaker hypotheses where \( g_t \) is replaced by larger constants \( g_t \) and the results obtained will then be rough. However we are lucky to have the truth of the hypotheses in applications.

**Theorem 1.** Let \( F_1(z), \ldots, F_s(z) \) \( (s > 1) \) be \( s \) algebraically independent meromorphic functions of orders \( q_1, \ldots, q_s \), and let \( \delta \) be the maximum and \( q_1 \) the minimum of these orders. Then for any weighted sequence \( S \) which is also special with major and minor densities \( \delta, \delta_1 \) we have necessarily

\[
\delta < \frac{q_1 - \sum \delta \delta q - (\delta - \delta_1)}{\delta - 1},
\]

provided the second term on the right is non-negative.

**Remark.** We will have occasion to apply this result only when \( \sum \delta \delta q \) is zero, and further \( \delta \) and \( q \) are non-negative rational integers.

**Proof.** If \( \delta \leq q \) there is nothing to prove. Let now \( \delta > q \). In this case

\[
\delta + \sum q = q_1 - \sum (q - q) - (\delta - \delta_1) + \delta + (e - 1) \delta > \delta_0.
\]

Putting \( R = O(Q) \) for a big constant \( C \) and taking logarithms twice in the inequality of the Main Theorem we get easily the inequality

\[
\log N(Q-1) \leq \sum r_t Q^{\delta t} + O(1) \quad (e > 0; \text{ fixed})
\]

valid for infinitely many \( Q \). The condition on the natural numbers \( r_t \) is that their product shall be asymptotic to \( h(k+1) N_1(q) \) where \( h \) is the degree of the number field occurring in the definition of a weighted sequence and \( q \) is a natural number less than \( Q \) and related to it in a certain way (both \( q \) and \( Q \) will be arbitrarily large). We set \( r_t \) to be the smallest natural number which exceeds

\[
q^{-\eta - e} (k(k+1) N_1(q) q^{2(k+1) \delta e})
\]

In view of the inequality \( \delta + \sum q > \delta_0 \) it is easy to verify the asymptotic condition on \( r_t \) \( (t = 1, \ldots, s) \), provided \( e \) is small enough. Also

\[
r_t = O(q^{-\eta - e} (k(k+1) N_1(q) q^{2(k+1) \delta e}))
\]

and so we are led to

\[
\log N(Q-1) \leq \frac{1}{\delta} \log (N_1(Q) Q^{2(k+1) \delta e}) + O(1)
\]

Dividing this by \( \log Q \) and passing to the limit \( Q \to \infty \) we get

\[
\delta \leq \frac{1}{\delta} (\delta_1 + \sum q)
\]

since \( e \) is arbitrary. This is precisely the desired inequality and completes the proof of Theorem 1.

**Remark.** Here we have applied the Main Theorem to deduce Theorem 1. We can also make other deductions. One can prove that if \( F(z) \) is a single valued (analytic except at singularities) transcendental function for which \( F'(1/a) \) are algebraic numbers, lying in a fixed algebraic number field, for all sufficiently large \( a \) and \( \log |F'(1/a)| = O(n/\log n) \) then \( F(z) \) must have an essential singularity in a finite part of the plane.

**§ 2. Some preparations.** In this section we set out some preparations which will enable us to verify the necessary hypotheses in applications of Theorem 1. These preparations are rather lengthy and spread over quite a few lemmas. We have also to specify the conditions of algebraic independence of certain meromorphic functions and the conditions we give are not quite satisfactory in some cases. We begin with

**Lemma 1.** Let \( a \) and \( \beta \) be two algebraic numbers of degree not exceeding \( h \). Then size \( (a+\beta) \) and size \( (a\beta) \) do not exceed size \( (a) \) size \( \beta \); for natural numbers \( n \), size \( (a^n) \) lies between \( 2^n \cdot \text{size} (a)^n \) and \( \text{size} (a)^n ) \); and finally size \( (1/a) \) does not exceed \( 2 \cdot \text{size} (a) \).
Remark. It can be proved that size $a$ does not exceed $2^{n^2}[H(a)]^b$ and that $H(a)$ does not exceed $2^n(\text{size }a)^3$, where $H(a)$ denotes the major height of $a$.

Proof. It is clear that $d(a+b)$ and $d(ab)$ do not exceed $d(a)d(b)$. (We denote by $d(a)$ the least natural number for which $ad(a)$ is an algebraic integer.) Also $|a+\beta|$ and $|ab|$ do not exceed $|a|+|\beta|$ and $|a||\beta|$ and hence the assertions regarding size $(a+b)$ and size $(ab)$ follow. Now $\text{size }a \leq \delta_0(a)d(a)$ and the inequality $(a+b)^\beta \geq a^\beta + b^\beta \geq 2^{n^2}(a+b)$ valid for all real positive $a$, $b$ and natural numbers $n$, proves the second statement. Again multiplying the numerator and denominator in $1/a$ by
\begin{align*}
\alpha^{-1}N(ad(a)) = d(a)[ad(a)]^{-1}N(ad(a)),
\end{align*}

it is easily seen that the denominator becomes a rational integer and the numerator an algebraic integer. It follows that $d(a)$ does not exceed
\begin{align*}
[N(ad(a))] \leq \left|\alpha\right|^{-1}(\text{size }a)^2 \leq (\text{size }a)^3.
\end{align*}

The result mentioned in the remark can be proved in the following way. Let $\alpha$ be different from zero and of degree $n$ not exceeding $\beta$. Suppose $a$ satisfies $a_0^2 + a_1^2 + \cdots + a_n = 0$ where $a_0, a_1, \ldots, a_n$ are rational integers with g.c.d. 1. $a_0 > 0$ and $a_1, \ldots, a_n$ different from zero. If $a_0$ is any conjugate of $a$, which does not exceed $\gamma$ in absolute value it follows that $1 \leq |a_0| \leq |a_0| + \cdots + a_{n-1} = 2 - |a| = H(a)$. Now since $H(a) > 0$, $|a|$, $H(a) > |a|^{1/\gamma}(c)^{\gamma}$, $\exp$, where $r$ runs over conjugates of $a$, whose absolute values do not exceed $\gamma$ and those with absolute value exceeding $\gamma$, both with the exception of one conjugate for which the absolute value is maximal. Now $\prod |a|^\gamma(c)^{\gamma}$, $\exp$, where $r$, $s$ are non-negative rational integers with sum less than $n$ and $\gamma \geq \gamma_0$, where $\gamma_0$ is a function of $n$. Hence $\alpha = a_0\alpha_0$ is an algebraic integer, $\text{size }\alpha \leq \alpha_0 < H(a)$. These two together prove the upper estimate for size $\alpha$. Again since $ad(a)$ is an algebraic integer $a_0$ divides $a_d$, $a_1d^2$, $\ldots$, $aNd^n$, where $d = d(a)$, which itself divides $b_0^\gamma(a_0, \ldots, a_n)$. But $a_0$, being prime to the second factor, $a_0$, divides $d^\gamma$ and so $a_0 \leq d^\gamma$. Now $\pm a_0H(a)$ is the jth elementary symmetric function of $a$ and its conjugates (if $j > 0$). It follows that $a_0H(a)$ does not exceed 1 if $j = 0$, $2^n|a|^3$ if $j > 0$. Now since $a_0 < \gamma_0 < (a_0)^3$, it follows that $H(a)$ does not exceed $2^3\text{size }a_0^3$.

The following lemma is an improvement of a certain statement implicitly contained in Schneider's work ([11], II) (the result of Schneider here referred to is size $\gamma(n)$ $\leq A^\delta n$ in the notation of the lemma that follows). Using Mahler's result [8] and our remark below Lemma 1 we can show that our result is the best possible in the sense that if $\gamma(\beta)$, $\gamma_1$, $\beta_1$, $a_0$, $\beta_1$ are rational, size $\gamma(n)$ exceeds, for infinitely many $n$, $A^\delta n$ for a suitable constant $A_\delta > 1$ which depends on $\beta_1$, $\gamma_1$, $\alpha_0$.

**Lemma 1.** Let $\gamma(n) = \gamma(\varepsilon_1, \varepsilon_2, \varepsilon_3, a_0)$ be the Weierstrass elliptic function with periods $\varepsilon_1, \varepsilon_2$ whose invariants $\gamma_1, \gamma_2$ which appear in its differential equation of the first order, are algebraic numbers. Suppose $\gamma(\beta)$ is an algebraic number. Then the numbers $\gamma(n\beta)$ (with $n = 1, 2, \ldots$) are algebraic numbers belonging to a fixed algebraic number field and size $\gamma(n\beta)$ does not exceed $A^n$ where $A$ is a positive constant independent of $n$.

We have the following Schrodinger ([11], II) the following formulae ([11]):

\begin{align}
\varphi(\lambda u) &= \varphi(u - \frac{\varphi_{2,3}(u)}{\varphi_{3}(u)^2})^\lambda, \quad \lambda = 2, 3, \ldots \nonumber
\end{align}

where
\begin{align}
\varphi_1(u) &= 1, \quad \varphi_2(u) = -\varphi_1(u), \nonumber
\end{align}

\begin{align}
\varphi_3(u) &= 3\varphi_1(u) - \frac{1}{2} g_2\varphi_1(u)^2 - 3g_3\varphi_1(u) - \frac{1}{2} g_4, \nonumber
\end{align}

\begin{align}
\varphi_4(u) &= \varphi_2(u) \times (3\varphi_1(u) - \frac{1}{2} g_2\varphi_1(u)^2 - 10g_3\varphi_1(u) - \frac{1}{2} g_4) - \frac{1}{2} g_2\varphi_1(u) + \frac{1}{2} g_4
\end{align}

and further

\begin{align}
\varphi_{2,3}(u) &= \varphi_3(u) - \varphi_4(u) - \varphi_1(u) - \varphi_2(u), \quad \lambda \geq 2, 
\varphi_{3,4}(u) = \varphi_3(u) - \varphi_4(u) - \varphi_1(u) - \varphi_2(u), \quad \lambda \geq 3.
\end{align}

First we remark that since $\beta_1, \alpha_1, \beta_1$ are linearly independent over the rationals $\gamma(\lambda\beta_1) \neq \gamma(\beta_1)$ if $\lambda = 2, 3, \ldots$ Now $\gamma(\beta_1)$ is different from zero. From these two it follows (on using (2.1)) that $\gamma(\lambda\beta_1) (\lambda = 2, 3, \ldots)$ are all finite and further they are algebraic numbers different from zero lying in a fixed algebraic number field. We make the following assertions which we verify by induction. We write $\Psi$ for $\gamma(\beta_1)$.

(i) $|\Psi(\lambda\beta_1)| \leq A_{\max}^{\lambda\beta_1} \delta^n \beta_1^{-\lambda}$, where $A_{\lambda}$ is a constant positive independent of $\lambda$ ($\lambda = 1, 2, 3, \ldots$).

Suppose this is true for all $\lambda < u$. Then to verify the truth of the statement for $\lambda = u + 1$, we have to verify

(a) If $u = 2m$,
\begin{align}
A_{\max}^{\lambda\beta_1} &= \alpha_0^{u+1}\beta_1^{u+1},
\end{align}

(b) If $u = 2m - 1$,
\begin{align}
A_{\max}^{\lambda\beta_1} &= \alpha_0^{u+1}\beta_1^{u+1}.
\end{align}

For all $n > 2$, $1/2, 3, \ldots$ does not exceed $\alpha_0^{u+1}\beta_1^{u+1}$ for all $m > 2$.

2. $1/2$, $3, \ldots$ does not exceed $\alpha_0^{u+1}\beta_1^{u+1}$ for all $m > 3$. 

\begin{align}
A_{\max}^{\lambda\beta_1} &= \alpha_0^{u+1}\beta_1^{u+1}.
\end{align}
We rewrite (a) and (b) as
\[
2A_1 \max(m^3 + m^2 + m + 1; \quad m^3 + m^2 + m + 1) \leq A_1^{m^3 + m^2}, \quad m \geq 2,
\]
\[
2 \frac{1}{\Psi_2} A_1^{m^3 + m^2 + m + 1} \leq A_1^{m^3 + m^2}, \quad m \geq 3.
\]
The first inequality is satisfied for all \( m \geq 2 \) if \( 2A_1 < A_1^m \), i.e. if \( A_1^2 > 2 \).
The second inequality is satisfied for all \( m \geq 1 \) if \( 2 \frac{1}{\Psi_2} A_1 < A_1^m \), i.e.
\[
\frac{1}{\Psi_2} A_1^2 < A_1^m.
\] We choose \( A_1 \) big enough to satisfy these and also
\[
\frac{1}{\Psi_2} A_1^2 \leq A_1^{-1}\text{ for } \lambda = 1, 2, \ldots, 7.
\] This secures (i) by induction for all \( \lambda \).
Next we note that in the second equality of (2.3), \( \Psi_2(w) \) divides the right hand side. We can thus regard \( \Psi_2(w) \) as a polynomial in \( \phi(w) \), \( \phi'(w) \), \( \phi''(w) \), \( \phi'''(w) \), with rational numbers as coefficients. We now assert
(ii) The denominators of the rational number coefficients of \( \Psi_2(w) \) divide \( 2^{\lambda-1} \).
We verify this for \( \lambda = 1, 2, 3, 4 \). Then we have only to verify
\[
\max \{3(m^3 - m) + (m^2 + 1)^2; \quad 3(m^3 - m) + (m^2 + 1)^2; \quad 3(m - 1)^3 - (m - 1)\} \leq (2m + 1)^2 - (2m + 1), \quad m \geq 2
\]
and
\[
\max \{3(m - 1)^3 - (m - 1)\} + \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad...
Proof. We first consider the case \( a = 0 \). Let \( L \) be a natural number and divide \([0,1]\) into \( L \) subintervals of length \( 1/L \) each. Let \( \lambda_1 \) denote the maximum number of these subintervals contained wholly in \( I \) and \( \lambda_2 \) the minimum number of these subintervals which contain \( I \). Then it is easy to verify the inequalities

\[(i) \quad \lambda_1 \frac{1}{L} + \frac{2}{L} \geq |I| \geq \lambda_2 \frac{1}{L} - \frac{2}{L},\]

\[(ii) \quad \lambda_2 \left( \frac{1}{L} + \eta \right) \leq \varepsilon^{-1} N_\theta \leq \lambda_1 \left( \frac{1}{L} - \eta \right) \quad \text{for all} \quad x \geq x_\theta(\beta, L, \eta_i).\]

The second of these follows from Weyl’s result ([13], see also [4] for other references and also for a simple proof of the result using only the simplest property of continued fractions). These inequalities are valid even if one or both of \( \lambda_1, \lambda_2 \) are zero.

Now,

\[\lambda_2 \left( \frac{1}{L} - \eta \right) < \left( |I| - \frac{2}{L} \right) (1 - L \eta_i) \leq |I| - \frac{2}{L} + L \eta_i + 2 \eta_i\]

and

\[\lambda_1 \left( \frac{1}{L} + \eta \right) \geq \left( |I| + \frac{2}{L} \right) (1 + L \eta_i) \geq |I| + \frac{2}{L} - L \eta_i - 2 \eta_i.\]

Hence

\[|\varepsilon^{-1} N_\theta - |I|| \leq \frac{2}{L} + L \eta_i + 2 \eta_i \quad \text{for all} \quad x \geq x_\theta(\beta, L, \eta_i).\]

We first fix \( L \) large enough and then a small enough \( \eta \), to get the result.

Now if \( a \) is different from zero, consider the interval \( I' = I + 1 - (a) \) with an obvious meaning. Now the points of \([0,1]\) which differ from those of \( I \) by a rational integer form at most two (and at least one) intervals of total length \( |I| \). These intervals say \( I_1, I_2 \) are further disjoint and have the property that \((a)\eta \) lies in \( I \) if and only if \((a)\eta \) lies in \( I_1 \) or \( I_2 \). The general case now follows on applying the case \( a = 0 \) to the intervals \( I_1, I_2 \).

Corollary. Let \( a_1, \ldots, a_p (p \geq 1) \) be real numbers not all rational. Then the number of \( p \)-tuples \((m_1, \ldots, m_p)\) of natural numbers none of which exceed a given natural number \( Q \), for which the fractional part \((m_1 a_1 + \cdots + m_p a_p)\) lies in a given interval \( I \) contained in \([0, 1]\) is asymptotic to \( Q^p |I| \) where \(|I|\) is as usual the length of \( I \), as \( Q \to \infty \).

Proof. Without loss of generality we may assume \( a_1 \) to be irrational. In Lemma 4 we write \( \beta = a_1, \alpha = m_2 a_1 + \cdots + m_p a_p \). Then for all \( x \geq x_\theta(\beta, \eta) \) the number \( X_\theta \), lies between \( x(1 + \eta) \) and \( x(1 - \eta) \).

Let now \( Q \geq x_\theta(\beta, \eta) \) and note that \( x_\theta = \eta \) is independent of \( a \). We consider for the \((p-1)\)-tupes \((m_3, \ldots, m_p)\) any one of the \( Q^{p-1} \) possibilities. Summing up the bounds for \( X_\theta \), we see that the total number of solutions required by the corollary lies between \( Q^p (|I| - \eta) \) and \( Q^p (|I| + \eta) \). Since \( \eta \) is arbitrary this proves the corollary. However our proof also gives the following result. The total number of solutions of \((m_1 a_1 + \cdots + m_p a_p) \in I \) where \( m_i \) runs through all natural numbers not exceeding \( x_\theta(\beta, \eta) \) and \( m_1, \ldots, m_p \), \( p \) distinct natural numbers, is asymptotic to \( x_\theta(\beta, \eta) \) uniformly in \( x_\theta, \ldots, x_\theta(p) \) (with an obvious meaning) as \( x_\theta \) tends to infinity. In this result we have to assume that \( a_1 \) is irrational.

Definition. Let \( g_i(x) = g_i(x) a_1 i \) \( (i = 1, \ldots, u) \) be \( u \) Weierstrass functions (not necessarily distinct), for the purpose of this definition and the three lemmas to follow we do not even need the restriction that the invariants \( g_1, g_2, \ldots \) \( (i = 1, \ldots, u) \) be algebraic whose period groups are \( \mathcal{A}_i \) \( (i = 1, \ldots, u) \), respectively. Let \( b_1, \ldots, b_n \) be \( u \) nonzero complex numbers linearly independent over the field of rationals. With every complex number of the type \( a = m_1 b_1 + \cdots + m_p b_p \) \( (m_1, \ldots, m_p \) are natural numbers) we associate the natural number \( \mathcal{A}(a) = \max(m_1, \ldots, m_p) \).

Let \( W \) be any set of distinct complex numbers. We say that a number \( a \) is \( W \)-admissible if \( W \) contains a representative of the coset \( a + \mathcal{A}(a) \mod \mathcal{A}(a) \) for each \( i = 1, \ldots, u \).

Lemma 5. There exists in the complex plane a compact set \( W \) free from the poles of each of the functions \( g_i(x) \) \( (i = 1, \ldots, u) \) with the property that the number of \( W \)-admissible numbers \( a \) (see the definition above) with \( \mathcal{A}(a) \) not exceeding \( Q \) is at least \( 4 Q^p \) for all \( Q \) exceeding \( Q_0 \). Here \( W \) and \( Q_0 \) depend on the period groups \( \mathcal{A}_i \) \( (i = 1, \ldots, u) \) and the numbers \( b_1, \ldots, b_p \).

Proof. Consider the union of \( u \) closed parallelograms \((0, a_1 i + \cdots + a_p i) \) \( (i = 1, \ldots, u) \) corresponding to the period groups \( \mathcal{A}_i \). The union contains only finitely many points of the \( u \) groups \( \mathcal{A}_1, \ldots, \mathcal{A}_u \). We exclude from the union all those points which are at a distance less than \( \varepsilon \) (\( \varepsilon \) a fixed positive quantity) from these points and denote the resulting set by \( W \).

The set \( W \) is compact and we show that, if \( \varepsilon \) is a fixed number sufficiently small, this set has the property required. We fix \( \varepsilon \) in such a way that the sum of the projections of the excluded regions on the four sides of the parallelogram corresponding to \( \mathcal{A}_i \) is less than \( 1/3u \) times the minimum of the length of its sides (here we have used the word projection not as orthogonal projection, but as oblique projection parallel to the sides of the parallelogram).

We write \( b_1 b_2 = a_1 a_2 i + a_2 a_1 i \) \( (i = 1, \ldots, u, j = 1, \ldots, p) \) where \( a_1 \) and \( b_1 \) are real. For each fixed \( i \) \( (i = 1, \ldots, u) \) it is clear that one at least of the \( 2p \) numbers \( a_1 b_1, b_1 \) is irrational, since otherwise it would lead to the linear dependence of \( \beta_1, \ldots, \beta_p \) over the field of rationals. Now \( b_1 a_1 = a_1 i \sum a_1 b_1 + a_2 i \sum a_1 b_2 \) the sums being from \( j = 1, \ldots, p \). Since we...
are interested in a representative of $b_n \mod \rho_0$, we may reduce the coefficients of $a_0^{(i)}$ and $a_0^{(j)}$ modulo 1, to lie in the interval $[0,1)$. Suppose for definiteness that one of the coefficients $a_0^{(i)}$ is irrational. Then the total number of numbers $a$ with $a(a)$ not exceeding $Q$ for which the coefficient of $a_0^{(i)}$ in $b_n$ obtained modulo 1 is minimal is at most 2. Repeating this for each $i = 1, \ldots, n$, we find that the total number of numbers $a$ with $a(a)$ not exceeding $Q$, for which one at least of the coefficients of $a_0^{(i)}$, $a_0^{(j)}$ in $b_n$ does not lie modulo 1, in the projections of the excluded regions on $(0,1)$ (resp. $(0, q_0^{(i)})$ for each $i = 1, \ldots, n$, is at least $\frac{3}{4}Q$ for all $Q$ exceeding a certain number $Q_0$. The number $Q_0$ and the set $\mathcal{W}$ depend clearly on the parameters mentioned in the lemma. This proves Lemma 5 completely.

**Lemma 6.** Let $a_0^{(i)}, a_0^{(j)}, \varphi_0(z), b_i (i = 1, \ldots, n), \beta_1, \ldots, \beta_n$ be as in the definition preceding Lemma 5, a, the points $a = a_0 + \beta_1 b_1 + \cdots + \beta_n b_n$ which occur in connection with Lemma 5 are ordered in the manner of $n_0 = n(a_0)$ non-decreasing, and $a_0$ a zero of $\varphi_0(z)$. Let $f_1(z)$ and $f_2(z)$ be respectively $\frac{1}{2}a(z-a)(a(x-a)^2 + z) + a(z-a)^{2}$ and $\varphi_0(z)$ where $\varphi_0(z)$ is the Weierstrass zeta function corresponding to $\varphi_0(z)$ ($f_1(z)$ and $f_2(z)$ are two entire functions without common zeros and $\varphi_0(z) = f_1(z)(f_2(z))^{-1}$). Then we have

$$\max_{i=1, \ldots, n} \left| \frac{f_1(b_i z)}{f_2(b_i z)} \right| \leq B_0^2$$

and

$$\max_{i=1, \ldots, n} \left( \frac{\max |f_1(b_i z) + 1|}{\max |f_2(b_i z) + 1|} \right) \leq B_0^2 \max_{i=1, \ldots, n} \left| \frac{b_i}{B_0} \right|$$

where $B_0$ is a positive constant independent of $Q$ and $R$.

**Proof.** Denoting by $\zeta(z)$ the Weierstrass zeta function corresponding to $\varphi_0(z)$, we have the identity (14), page 448; for the result $\varphi_0(z) = f_1(z)(f_2(z))^{-1}$ see example 1 on page 461,

$$\sigma_1(x + n_0 a_0^{(i)} + n_0 a_0^{(j)}) = (-1)^{n_0 + n_0} \exp \left[ \frac{a_0^{(i)}(a_0^{(i)} - 1)}{2} (2n_0 x + 2n_0 n_0 a_0^{(i)} + n_0^2 a_0^{(i)}) + \frac{\zeta_0(a_0^{(i)})}{2} (2n_0 x + n_0^2 a_0^{(i)}) \right] \sigma_1(z),$$

where $n_0, n_0$ are arbitrary non-negative rational integers and it is easy to modify this when $n_0, n_0$ are arbitrary rational integers. To deduce the first we take for some fixed $i, x = b_i a_0^{(i)}$, and select $n_0, n_0$ such that $x + n_0 a_0^{(i)} + n_0 a_0^{(j)}$ lies in the parallelogram $0, a_0^{(i)}, a_0^{(j)}$, below $a_0^{(i)}$. We observe that $|n_0(b_i a_0^{(i)} + n_0 a_0^{(i)} + n_0 a_0^{(j)})|$ is greater than a fixed positive constant independent of $m_1, \ldots, m_0$, and also that $|m_1|$ and $|m_0|$ do not exceed a certain constant multiple of $Q$. This leads to the first inequality. The second one also follows by a similar argument.

**Lemma 7.** Let $\varphi_1(z) = \varphi(z; a_0^{(i)}, a_0^{(j)}), \varphi_2(z) = \varphi(z; a_0^{(i)}, a_0^{(j)}), \ldots$ be finitely many Weierstrass elliptic functions with periods indicated. Then $\varphi_1(z)$ and $\varphi_2(z)$ are algebraically independent (unless otherwise specified we mean over the field of complex numbers) if and only if there exists a rational $2 \times 2$ matrix $M$ such that $(a_0^{(i)}, a_0^{(j)}) = (a_0^{(i)}, a_0^{(j)}) M$. In this case we say that the periods of $\varphi_1(z)$ and $\varphi_2(z)$ are commensurable. The necessary and sufficient condition that $\varphi_1(z)$ and $\varphi_2(z)$ be algebraically independent is that $\varphi_1(z)$ and $\varphi_2(z)$ be algebraically independent. Finally suppose that $a_0^{(i)}, a_0^{(j)}, a_0^{(j)}, a_0^{(j)}$ be linearly independent over the field of rationals and the ratio of every two of them is real. Then the functions $\varphi_1(z), \varphi_2(z), \ldots$ are algebraically independent.

**Remark.** Let $\varphi(z) = \varphi(z; a_0^{(i)}, a_0^{(j)})$ with usual notation and $b_1, b_2, b_3, \ldots$ be linearly independent complex numbers linearly independent over the rationals. Then $\varphi(b_1 z)$ and $\varphi(b_2 z)$ are algebraically dependent if and only if $a_0^{(i)}, a_0^{(j)}$ is an algebraic number of degree 2 and $b_1 b_2^{-1}, a_0^{(i)}, a_0^{(j)}$ are linearly dependent over the field of rationals. Let further that one of the numbers $a_0^{(i)}, a_0^{(j)}$ be either real or purely imaginary and $b_1, b_2, \ldots$ have the property that the ratio of every two of them is real. Then the functions $\varphi(b_1 z), \varphi(b_2 z), \ldots$ are algebraically dependent.

**Proof.** The first statement of the lemma is standard. To prove the final statement we assume that $\varphi_1(z), \varphi_2(z), \ldots$ have an algebraic relation of the form

$$\varphi_1(z) P_1(z) + \varphi_2(z) P_2(z) + \cdots = 0 \quad (k \geq 1)$$

where $P_1, P_2, \ldots$ are polynomials in $\varphi_1(z), \varphi_2(z), \ldots$ (with complex number coefficients) and that $P_k$ is not identically zero. This relation is an identity and hence we replace $z$ by $z + n_0 a_0^{(j)}$, where $n_0$ is an arbitrary rational number. In each function $\varphi(z)$ remains unaltered and in the other functions we may further change $z + n_0 a_0^{(j)}$ by the addition of a suitable periodic function. For instance $\varphi_2(z + n_0 a_0^{(j)}) = \varphi_2(z + n_0 a_0^{(j)} + n_0 a_0^{(j)})$ where $n_0$ is a suitable rational integer and similarly for the other functions. Now let $z$ be a complex number different from zero and close to zero such that $z^k a_0^{(j)}$ is real. Since the fractional parts of the numbers $n_0 a_0^{(j)} + n_0 a_0^{(j)}$ are bounded by a constant independent of $x, n_0, n_0, n_0, \ldots$ and that the first term has its absolute value bounded below by a positive constant independent of $x, n_0, n_0, n_0, \ldots$ and also that $n_0$ and $n_0$ do not exceed a certain constant multiple of $Q$. This leads to the first inequality. The second one also follows by a similar argument.
constant independent of $z, m, n, \ldots$. Since $p_1(z) \sim z^{-1}$ as $z$ approaches zero the assertion of the lemma follows.

We now prove the second assertion of the lemma. The necessity of the condition is clear. Assuming the period lattices $e_0^0, e_1^0$ and $e_0^0, e_1^0$ to be incommensurable is equivalent to the algebraic independence of $p_1(z)$ and $p_2(z)$. We have plenty of periods $\Omega = m_1 e_0^0 + n_1 e_1^0$ ($m, n$ natural numbers) which are not poles of $p_2(z)$. Suppose that there exists a polynomial relation between the three functions. If for instance $p_1(z)$ does not enter this relation a contradiction is immediate since $e'$ is not doubly periodic. Consider now this polynomial as a polynomial in $p_1(z)$ and suppose that $P(e', p_1(z))$ is the coefficient of the highest degree term. Since $p_1(z) \sim z^{-1}$ as $z$ tends to zero we have on replacing $z$ by $z + \Omega$ and letting $z \to 0$, $P(e', p_1(z)) = 0$ for all periods $\Omega$ of $p_1(z)$ which are not poles of $p_2(z)$. We now examine such periods $\Omega$. Writing $e_0^0 = \alpha_0^0 + \alpha_1^0, e_1^0 = \alpha_0^0 + \alpha_1^0$ where the $\alpha_i$ are real we observe that one at least of the $\alpha_i$ is irrational. Consequently in the expression $\Omega = (m_1 + n_1) e_0^0 + (m_2 + n_2) e_1^0$ one at least of the coefficients can be restricted to lie modulo 1, in any fixed closed sub-interval of $(0,1)$ with non-empty interior, by Kronecker’s Theorem [7]. The natural numbers $m$ and $n$ are still arbitrary. If for instance $\alpha_1$ is irrational we can fix $n$ as large as we please and restrict $m$ in such a way that the periods $\Omega$ are congruent mod $(e_0^0, e_1^0)$ to points of a fixed compact set just described. Since $m$ is arbitrary there are plenty of periods which are incongruent mod $(e_0^0, e_1^0)$. If now $P$ is independent of $e'$, the elliptic function $P(p_1(z))$ has plenty of incongruent zeros and is therefore a constant. On the other hand if $P(e', p_1(z))$ is the coefficient of the highest degree term in $P$ regarded as a polynomial in $e'$ (we can assume without loss of generality that the real parts of $e_0^0$ and $e_1^0$ are non-negative), it follows that $P_1(p_1(z))$ as a function of the natural numbers $m, n$ tends to zero. But we can further restrict the points $\Omega$ (since the zeros of $P(p_1(z))$ are isolated) to lie mod $(e_0^0, e_1^0)$, in a compact set where $P_1(p_1(z))$ never vanishes. It follows that $P_1(p_1(z))$ is bounded below, in absolute value, by a positive constant independent of $m, n$. This contradiction completes the proof of the lemma.

§ 3. Principal results. We are now in a position to deduce our principal results. In view of the lemma that follows it is convenient to introduce

**Definition.** A complex number $\beta$ is said to be a pseudo-algebraic point of a meromorphic function $f(z)$ if either $\beta$ is a pole of $f(z)$ or $f(\beta)$ is an algebraic number.

**Lemma 8.** Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with an algebraic relation $P(f(z), g(z)) = 0$, where $P$ is a polynomial in $f(z)$, and $g(z)$ with algebraic numbers as coefficients, not all zero. Then the set of pseudo-algebraic points of $f(z)$ is identical with the set of pseudo-algebraic points of $g(z)$.

**Proof.** Let $\beta$ be a pseudo-algebraic point of $f(z)$ and let $g(\beta)$ be transcendental. By considering the reciprocal of $f(z)$ if necessary we can assume that $f(\beta)$ is algebraic. Then considering $f(z) - f(\beta)$ in place of $f(z)$ we can assume that $f(\beta) = 0$. This would mean that $P(f(z), g(z)) = 0$ for all $z$, i.e. $f(z)$ divides $P(f(z), g(z))$, but we can divide $P(f(z), g(z))$ by a suitable power of $f(z)$ if necessary and assume without loss of generality that $f(0, g(z))$ is not identically zero. This proves the lemma.

**Remark.** Lemma 8 shows that if $f(z)$ is a nonconstant elliptic function with periods $\alpha_1, \alpha_2$ with algebraic invariants $g_1, g_2$ and the expression of $f(z)$ in terms of $g(z), g'(z)$ as a rational function involves only algebraic number coefficients, then the pseudo-algebraic points of $f(z)$ and $g(z)$ are the same. Thus it suffices to consider the pseudo-algebraic points of $g(z)$. Addition theorem for $g(z)$ shows that these points form a group under the usual addition of complex numbers. Also since $g(z)$ and $g'(z)$ (natural number) are algebraically dependent over the field of all algebraic numbers, these points have actually a vector space structure over the field of rationals, i.e. if $\beta_1, \beta_2$ are pseudo-algebraic points of $g(z)$ so is $\beta_1 + \beta_2$ for any two rational numbers $r, s$. If $\tau = \alpha_1/\alpha_2$ is an imaginary quadratic number then it is easy to verify that $g(z, \tau)$ has the same pseudo-algebraic points as $g(z)$, and so in this case these points have a vector space structure over the imaginary quadratic field obtained by adjoining $\tau$ to the field of rationals.

**Remark 2.** Similar remarks apply to the function $e'$ and non-constant meromorphic functions which depend algebraically on $e'$ with algebraic number coefficients. Needless to say that such remarks apply also to $e'$ and $\tau$ functions of $z$ with algebraic coefficients.

It will be convenient to introduce

**Definition.** A meromorphic function $\Phi(z)$ is said possess an algebraic addition theorem if there exists a polynomial $P(z_1, z_2, z_3)$ in three variables, with complex number coefficients not all zero such that $P(z_0 + z_1, \Phi(z_0), \Phi(z_1)) = 0$ for all pairs $z_0, z_1$ of complex numbers for which $\Phi(z_0 + z_1, \Phi(z_0), \Phi(z_1))$ and $\Phi(z_1)$ are finite. If further the coefficients of $P$ are algebraic numbers it is said to be algebraically additive.

By a well known Theorem of Weierstrass (see [1], p. 363) every meromorphic function $\Phi(z)$ with an algebraic addition theorem has the property that for a suitable nonzero complex number $\delta$ depending on the function, $\Phi(z\delta)$ is a rational function of $\Phi(z), \Phi'(z)$ with complex number coefficients where $\Phi(z)$ is either $\pi, e'$ or $\phi(z)$ with invariants.
Lemma 5. \( X(Q) \) lies between \( Q_0 \) and \( Q' \) and it follows that \( X'_1(Q) \) lies between two positive constant multiples of \( Q' \). This secures (iii). The condition \((v)\) is easily satisfied. The major and minor densities are 3 and 2 and the deviation is 1. By Lemmas 3 and 6 this weighted sequence is special. Hence by Theorem 1, it follows that \( s \leq 2 + (2 - 1) \) a contradiction. This proves our principal result.

§ 4. Concluding remarks. A number of curious corollaries follow from Theorem 2, on using Lemma 6. Some of these have already been stated in the introduction. We mention one more corollary.

**Corollary.** If \( a \) and \( b \) are real positive algebraic numbers different from 1 for which \( \log a \) and \( \log b \) are irrational and \( a < b < a^{-1} \), then one of the two numbers

\[
\begin{align*}
&\left( \frac{1}{240} \sum_{n=1}^{\infty} \frac{n^3a^n}{1-a^n} \right) \prod_{n=1}^{\infty} (1-a^n)^{-n}, \\
&\left( \frac{1}{(b^{1/3}-b^{-1/3})^2} - \frac{1}{(b^{1/3}-b^{-1/3})^2} \right) \prod_{n=1}^{\infty} \left( \frac{1-a^n}{1-a^n} \right)^{-n}
\end{align*}
\]

is transcendental.

**Proof.** It is easy to verify that the functions \( e^{2\pi i a} \) and \( g(z) = \left( \left[ a \right] e^{2\pi i z} \right) \) \( (a \in \mathbb{Q}) \) are algebraically additive under the only assumption \( j(a_1, a_2) \) is algebraic and have the common period \( a_1 \). Hence \( \text{dim}(e^{2\pi i a}) \) \( g(z) \) is \( \leq 2 \). We set \( a_1 = \log a \), \( a_2 = -2\pi i \) and consider the values of both the functions at \( z = 2\pi i, \log a, \log b, \text{which are clearly linearly independent over the field of rationals.} \) It follows that one of the two numbers

\[
j \left( \frac{\log a}{2\pi i} \right)^2, \quad \left( \frac{\log b}{2\pi i} \right)^2, \quad \left( \frac{\log g}{2\pi i} \right)^2
\]

is transcendental. The Fourier expansions for \( j, g, a, b \) functions (which can be found for instance in Deuring [2] or Hasse [3]) lead to the result stated.

It may be possible to improve the bound for the dimension given by Theorem 2 probably to 1 in all cases. But even a slight improvement in the bound such as \( \leq g^{-1} (g - 1)/s \) appears to be very difficult. One may ask another question. Let \( a, b, c \) be three non-zero complex numbers such that \( \log a, \log b, \log c \) are linearly independent over the field of rationals. 

\[
\text{Let } a_1, a_2, \ldots, a_n \text{ be any complex numbers linearly independent over the field of rationals. Then if } n \geq 2 \text{ Theorem 2 shows that the 3n numbers}
\]

\[
a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3; \ldots; a_n, b_n, c_n
\]

\[
cannot be all algebraic (in fact \( n-1 \) of them must be transcendental).
\]

It is natural to ask whether among these 3n numbers there exist \( n-1 \)