Here

$$\frac{1}{|G(a_j)|} \leqslant J_2(Q) = \prod_{t=1}^{s} \big(M_2^{(t)}(Q)\big)^{r_t},$$

$$|R(z)G(z)|_{|z|=R} \leqslant r_1 \ldots r_s \big(J_1(q)\big)^{3/(1-\varepsilon)} J_3(R) \leqslant \big(J_1(Q)\big)^{1+3/(1-\varepsilon)} J_3(R),$$

where

$$J_3(R) = \prod_{t=1}^{s} \big(M^{(t)}(R)\big)^{r_t},$$

$$\Big|\prod_{n_l<Q} \Big(\frac{a_j-a_l}{z-a_l}\Big)\Big| \leqslant \Big(\frac{2D(Q)}{R-D(Q)}\Big)^{N(Q-1)} \leqslant \Big(\frac{4D(Q)}{R}\Big)^{N(Q-1)},$$

$$\frac{1}{|z-a_j|} \leqslant \frac{1}{R-D(Q)} \leqslant \frac{2}{R}$$

and so

$$|\gamma| \leqslant J_2(Q)\cdot\frac{1}{2\pi}\cdot 2\pi R \big(J_1(Q)\big)^{1+3/(1-\varepsilon)} J_3(R)\cdot\frac{2}{R}\Big(\frac{4D(Q)}{R}\Big)^{N(Q-1)}$$

$$\leqslant \big(J_1(Q)\big)^{1+3/(1-\varepsilon)} J_2(Q) J_3(R) \Big(\frac{8D(Q)}{R}\Big)^{N(Q-1)}.$$

Combining all our estimates for $\gamma$, we get

$$1 \leqslant \big(J_1(Q)\big)^{h_2+(h_2-1)[2+3/(1-\varepsilon)]+1+3/(1-\varepsilon)} J_2(Q) J_3(R) \Big(\frac{8D(Q)}{R}\Big)^{N(Q-1)}$$

and since the exponent of $J_1(Q)$ on the right does not exceed $h_2[3+ +3/(1-\varepsilon)] = 8h_2 = 8h(Q)$, for $\varepsilon = \frac{2}{5}$, this proves the validity of the inequality of the Main Theorem for $R \geqslant 2D(Q)$ and the inequality is trivial for $R \leqslant 8D(Q)$. This completes the proof of the Main Theorem.

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# Contributions to the theory of transcendental numbers (II)

by

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*To the 60-th birthday of A. O. Gelfond*

**§ 1. A special case of the Main Theorem.** In this section we apply a special case of the Main Theorem of the earlier paper I ([9], p. 69) to deduce Theorem 1 below which will be the only theorem to which we shall refer in the later sections of this paper. We begin with some definitions (incidentally we also recall the notation). From now on we deal only with meromorphic functions which are quotients of entire functions of finite order. Given $s$ ($\geqslant 2$) algebraically independent meromorphic functions $F_1(z), \ldots, F_s(z)$ we introduce with respect to these

Definition 1. A *weighted sequence* $S$ (often we write $\{a_\mu\}$ for $S$) is an infinite sequence $\{a_\mu\}$ ($\mu = 1, 2, \ldots$) of distinct complex numbers together with an infinite subsequence $\{a_{\mu_r}\}$ ($r = 1, 2, 3, \ldots$) (which may be the same as $\{a_\mu\}$) and an infinite sequence $\{n_\mu\}$ ($\mu = 1, 2, 3, \ldots$) of natural numbers not necessarily distinct satisfying the following conditions.

(i) The sequence $\{n_\mu\}$ is non-decreasing.

(ii) For each $Q = 1, 2, 3, \ldots$ there are only finitely many $\{a_\mu\}$ for which $n_\mu$ does not exceed $Q$. We denote this number by $N(Q)$. It follows that there are only finitely many numbers $a_{\mu_r}$ for which $n_{\mu_r}$ does not exceed $Q$ and this number $N_1(Q)$ does not exceed $N(Q)$.

(iii) The limits

$$\delta = \lim_{Q\to\infty} \frac{\log N(Q)}{\log Q} \quad \text{and} \quad \delta_1 = \lim_{Q\to\infty} \frac{\log N_1(Q)}{\log Q}$$

exist and are finite.

(iv) The upper limit

$$\limsup_{Q\to\infty} \Big(\frac{1}{Q}\max_{n_\mu\leqslant Q}|a_\mu|\Big)$$

is finite.

(v) Whenever a polynomial in $F_1(z), \ldots, F_s(z)$ with complex coefficients vanishes for all values $z = a_{\mu_r}$ with $n_{\mu_r} \leqslant Q$, it also vanishes for all values $z = a_\mu$ with $n_\mu \leqslant Q$.

(vi) The numbers $F_t(a_\mu)$ $(t = 1, \ldots, s;\ \mu = 1, 2, 3, \ldots)$ are all algebraic numbers lying in some fixed algebraic number field of degree $h$ (the number field and its degree may be different for different $S$).

DEFINITION 2. The numbers $\delta$, $\delta_1$ in (iii) are called the *major and minor densities* of the weighted sequence $S$ and $\delta - \delta_1$ (which we know to be non-negative) is called the *deviation* of $S$.

DEFINITION 3. Next we write $F_t(z) = H_t(z)/G_t(z)$ $(t = 1, \ldots, s)$ where $H_t(z)$ and $G_t(z)$ are entire functions without common zeros and the maximum of the orders of $H_t(z)$ and $G_t(z)$ is least possible. We define this number $\varrho_t$ as the *order of the function* $F_t(z)$.

DEFINITION 4. A weighted sequence $S$ is said to be *special* if it satisfies the following hypotheses:

$$\limsup_{Q\to\infty} \{(\log Q)^{-1} \log\log (\max_{n_\mu \leqslant Q} \operatorname{size} F_t(a_\mu))\} \leqslant \varrho_t$$

and

$$\limsup_{Q\to\infty} \{(\log Q)^{-1} \log\log (\max_{n_\mu \leqslant Q} |G_t(a_\mu)|^{-1})\} \leqslant \varrho_t.$$

(Size of an algebraic number $\alpha$ is as usual $d(\alpha) + \overline{|\alpha|}$ where $d(\alpha)$ is the least natural number for which $\alpha d(\alpha)$ is an algebraic integer.)

Remark. We can work with weaker hypotheses where $\varrho_t$ is replaced by bigger constants $\varrho_t'$ and the results obtained will then be rough. However we are lucky to have the truth of the hypotheses in applications.

THEOREM 1. *Let $F_1(z), \ldots, F_s(z)$ $(s \geqslant 2)$ be $s$ algebraically independent meromorphic functions of orders $\varrho_1, \ldots, \varrho_s$ and let $\varrho$ be the maximum and $\varrho_1$ the minimum of these orders. Then for any weighted sequence $S$ which is also special with major and minor densities $\delta$, $\delta_1$ we have necessarily*

$$\delta \leqslant \varrho + \frac{\varrho_1 - \sum_{t=2}^{s} (\varrho - \varrho_t) - (\delta - \delta_1)}{s-1}$$

*provided the second term on the right is non-negative.*

Remark. We will have occasion to apply this result only when $\sum_{t=2}^{s} (\varrho - \varrho_t)$ is zero, and further $\delta$ and $\varrho$ are non-negative rational integers.

Proof. If $\delta \leqslant \varrho$ there is nothing to prove. Let now $\delta > \varrho$. In this case

$$\delta_1 + \sum_{t=1}^{s} \varrho_t = \varrho_1 - \sum_{t=2}^{s} (\varrho - \varrho_t) - (\delta - \delta_1) + \delta + (s-1)\varrho > s\varrho.$$



Putting $R = CQ$ for a big constant $C$ and taking logarithms twice in the inequality of the Main Theorem we get easily the inequality

$$\log N(Q-1) \leqslant \log \sum_{t=1}^{s} r_t Q^{\varrho_t + \varepsilon} + O(1) \qquad (\varepsilon > 0;\ \text{fixed})$$

valid for infinitely many $Q$. The condition on the natural numbers $r_t$ is that their product shall be asymptotic to $h(h+1)N_1(q)$ where $h$ is the degree of the number field occurring in the definition of a weighted sequence and $q$ is a natural number less than $Q$ and related to it in a certain way (both $q$ and $Q$ will be arbitrarily large). We set $r_t$ to be the smallest natural number which exceeds

$$q^{-\varrho_t - \varepsilon} \{h(h+1) N_1(q) q^{\sum_i (\varrho_i + \varepsilon)}\}^{1/s}.$$

In view of the inequality $\delta_1 + \sum_{t=1}^{s} \varrho_t > s\varrho$ it is easy to verify the asymptotic condition on $r_t$ $(t = 1, \ldots, s)$, provided $\varepsilon$ is small enough. Also

$$r_t = O\left(Q^{-\varrho_t - \varepsilon} \{h(h+1) N_1(Q) Q^{\sum_i (\varrho_i + \varepsilon)}\}^{1/s}\right)$$

and so we are led to

$$\log N(Q-1) \leqslant \frac{1}{s} \log \{N_1(Q) Q^{\sum_t (\varrho_t + \varepsilon)}\} + O(1).$$

Dividing this by $\log Q$ and passing to the limit $Q \to \infty$ we get $\delta \leqslant \frac{1}{s}\{\delta_1 + \sum_t \varrho_t\}$ since $\varepsilon$ is arbitrary. This is precisely the desired inequality and completes the proof of Theorem 1.

Remark. Here we have applied the Main Theorem to deduce Theorem 1. We can also make other deductions. One can prove that if $F(z)$ is a single valued (analytic except at singularities) transcendental function for which $F(1/n)$ are algebraic numbers, lying in a fixed algebraic number field, for all sufficiently large $n$ and $\log \operatorname{size}(F(1/n)) = O(n/\log n)$ then $F(z)$ must have an essential singularity in a finite part of the plane.

**§ 2. Some preparations.** In this section we set out some preparations which will enable us to verify the necessary hypotheses in applications of Theorem 1. These preparations are rather lengthy and spread over quite a few lemmas. We have also to specify the conditions of algebraic independence of certain meromorphic functions and the conditions we give are not quite satisfactory in some cases. We begin with

LEMMA 1. *Let $\alpha$ and $\beta$ be two algebraic numbers of degree not exceeding $h$. Then* $\operatorname{size}(\alpha+\beta)$ *and* $\operatorname{size}(\alpha\beta)$ *do not exceed* $\operatorname{size}\alpha \operatorname{size}\beta$; *for natural numbers $n$,* $\operatorname{size}(\alpha^n)$ *lies between* $2^{1-n}(\operatorname{size}\alpha)^n$ *and* $(\operatorname{size}\alpha)^n$; *and finally* $\operatorname{size}(1/\alpha)$ *does not exceed* $2(\operatorname{size}\alpha)^{2h}$.

Remark. It can be proved that $\operatorname{size}\alpha$ does not exceed $2^h(H(\alpha))^h$ and that $H(\alpha)$ does not exceed $2^h(\operatorname{size}\alpha)^{2h}$, where $H(\alpha)$ denotes the familiar height of $\alpha$.

Proof. It is clear that $d(\alpha+\beta)$ and $d(\alpha\beta)$ do not exceed $d(\alpha)d(\beta)$. (We have denoted by $d(\alpha)$ the least natural number for which $\alpha d(\alpha)$ is an algebraic integer.) Also $\overline{|\alpha+\beta|}$ and $\overline{|\alpha\beta|}$ do not exceed $\overline{|\alpha|}+\overline{|\beta|}$ and $\overline{|\alpha|}\,\overline{|\beta|}$ and hence the assertions regarding $\operatorname{size}(\alpha+\beta)$ and $\operatorname{size}(\alpha\beta)$ follow. Now $\operatorname{size}\alpha^n = (d(\alpha))^n+\overline{|\alpha|}^n$ and the inequality $(a+b)^n \geqslant a^n+b^n \geqslant 2^{1-n}(a+b)^n$ valid for all real positive $a, b$ and natural numbers $n$, proves the second statement. Again multiplying the numerator and denominator in $1/\alpha$ by

$$\alpha^{-1}N(\alpha d(\alpha)) = d(\alpha)(\alpha d(\alpha))^{-1}N(\alpha d(\alpha)),$$

it is easily seen that the denominator becomes a rational integer and the numerator an algebraic integer. It follows that $d(\alpha^{-1})$ does not exceed

$$|N(\alpha d(\alpha))| \leqslant \overline{|\alpha|}^h(d(\alpha))^h \leqslant (\operatorname{size}\alpha)^{2h}.$$

The result mentioned in the remark can be proved in the following way. Let $\alpha$ be different from zero and of degree $n$ not exceeding $h$. Suppose $\alpha$ satisfies $a_0x^n+a_1x^{n-1}+\ldots+a_n = 0$ where $a_0, \ldots, a_n$ are rational integers with g.c.d. 1, $a_0 > 0$ and $a_n$ different from zero. If $\alpha'$ is any conjugate of $\alpha$, which does not exceed $\frac{1}{2}$ in absolute value it follows that $1 \leqslant |a_n| \leqslant |a_0\alpha'^n+\ldots+a_{n-1}\alpha'| \leqslant 2|\alpha'|H(\alpha)$. Now since $H(\alpha) \geqslant |a_0N(\alpha)| \geqslant |N(\alpha)|$, $H(\alpha) \geqslant \overline{|\alpha|}\prod|\alpha'|\prod|\alpha''|$ where $\alpha'$ runs over conjugates of $\alpha$, whose absolute values do not exceed $\frac{1}{2}$ and $\alpha''$ those with absolute value exceeding $\frac{1}{2}$, both with the exception of one conjugate for which the absolute value is maximal. Now $\prod|\alpha'|\prod|\alpha''| \geqslant (2H(\alpha))^{-r}2^{-s}$ (where $r$, $s$ are non-negative rational integers with sum less than $n$) $\geqslant 2^{-(h-1)}(H(\alpha))^{-h}$ and so $\overline{|\alpha|} \leqslant 2^{h-1}(H(\alpha))^h$. Also since $a_0\alpha$ is an algebraic integer, $d(\alpha) \leqslant a_0 \leqslant H(\alpha)$. These two together prove the upper estimate for size $\alpha$. Again since $\alpha d(\alpha)$ is an algebraic integer $a_0$ divides $(a_1d, a_2d^2, \ldots, a_nd^n)$, where $d = d(\alpha)$, which itself divides $d^n(a_1, \ldots, a_n)$. But $a_0$ being prime to the second factor, $a_0$ divides $d^n$ and so $a_0 \leqslant d^h$. Now $\pm a_0^{-1}H(\alpha)$ is the $j$th elementary symmetric function of $\alpha$ and its conjugates ($j \geqslant 0$). It follows that $a_0^{-1}H(\alpha)$ does not exceed 1 if $j = 0$, $2^h\overline{|\alpha|}^h$ if $j > 0$. Now since $a_0 \leqslant d^h \leqslant (\operatorname{size}\alpha)^h$, it follows that $H(\alpha)$ does not exceed $2^h(\operatorname{size}\alpha)^{2h}$.

The following lemma is an improvement of a certain statement implicitly contained in Schneider's work ([11], II) (the result of Schneider here referred to is $\operatorname{size}\wp(n\beta) \leqslant A^{n^3}$ in the notation of the lemma that follows). Using Mahler's result [8] and our remark below Lemma 1 we can show that our result is the best possible in the sense that if $\wp(\beta)$, $g_2, g_3$ are rational, $\operatorname{size}\wp(n\beta)$ exceeds, for infinitely many $n$, $A_0^{n^2}$ for a suitable constant $A_0 > 1$ which depends on $\beta, \omega_1, \omega_2$.



LEMMA 2. *Let $\wp(z) = \wp(z; \omega_1, \omega_2)$ be the Weierstrass elliptic function with periods $\omega_1, \omega_2$ whose invariants $g_2, g_3$ which appear in its differential equation of the first order, are algebraic numbers. Let $\beta$ be a complex number such that $\beta, \omega_1, \omega_2$ are linearly independent over the field of rationals. Suppose $\wp(\beta)$ is an algebraic number. Then all the numbers $\wp(n\beta)$ ($n = 1, 2, \ldots$) are algebraic numbers belonging to a fixed algebraic number field and $\operatorname{size}\wp(n\beta)$ does not exceed $A^{n^2}$ where $A$ is a positive constant independent of $n$.*

Proof. We have following Schneider ([11], II) the following formulae ([6]):

$$\wp(\lambda u) = \wp(u) - \frac{\Psi_{\lambda-1}(u)\Psi_{\lambda+1}(u)}{(\Psi_\lambda(u))^2}, \quad \lambda = 2, 3, \ldots \tag{2.1}$$

where

$$\Psi_1(u) = 1, \quad \Psi_2(u) = -\wp'(u),$$

$$\Psi_3(u) = 3\wp^4(u) - \tfrac{3}{2}g_2\wp^2(u) - 3g_3\wp(u) - \tfrac{1}{16}g_2^2,$$

$$\Psi_4(u) = \wp'(u) \times \tag{2.2}$$

$$\times\{2\wp^6(u) - \tfrac{5}{2}g_2\wp^4(u) - 10g_3\wp^3(u) - \tfrac{5}{8}g_2^2\wp^2(u) - \tfrac{1}{2}g_2g_3\wp(u) + \tfrac{1}{32}g_2^3 - g_3^2\}$$

and further

$$\Psi_{2\lambda+1}(u) = \Psi_{\lambda+2}(u)\Psi_\lambda^3(u) - \Psi_{\lambda-1}(u)\Psi_{\lambda+1}^3(u), \quad \lambda \geqslant 2, \tag{2.3}$$

$$\Psi_{2\lambda}(u)\Psi_2(u) = \Psi_\lambda(u)\{\Psi_{\lambda+2}(u)\Psi_{\lambda-1}^2(u) - \Psi_{\lambda-2}(u)\Psi_{\lambda+1}^2(u)\}, \quad \lambda \geqslant 3.$$

First we remark that since $\beta, \omega_1, \omega_2$ are linearly independent over the rationals $\wp(\lambda\beta) \neq \wp(\beta)$ if $\lambda = 2, 3, \ldots$ Now $\Psi_2(\beta)$ is different from zero. From these two it follows (on using (2.1)) that $\Psi_\lambda(\beta)$ ($\lambda = 1, 2, 3, \ldots$) are all finite and further they are algebraic numbers different from zero lying in a fixed algebraic number field. We make the following assertions which we verify by induction. We write $\Psi_\lambda$ for $\Psi_\lambda(\beta)$.

(i) $\overline{|\Psi_\lambda|} \leqslant A_1^{\lambda^2-\lambda}$, where $A_1$ is a positive constant independent of $\lambda$ ($\lambda = 1, 2, 3, \ldots$).

Suppose this is true for all $\lambda \leqslant n$. Then to verify the truth of the statement for $\lambda = n+1$, we have to verify

(a) If $n = 2m$,

$$2A_1^{\max[3(m^2-m)+(m+2)^2-(m+2);3((m+1)^2-(m+1))+(m-1)^2-(m-1)]}$$

does not exceed $A_1^{(2m+1)^2-(2m+1)}$ for all $m \geqslant 2$.

(b) If $n = 2m-1$,

$$2\,\overline{\left|\frac{1}{\Psi_2}\right|}\,A_1^{\max[2((m-1)^2-(m-1))+(m+2)^2-(m+2)+m^2-m;\,2((m+1)^2-(m+1))+(m-2)\;-(m-2)+m^2-m]}$$

does not exceed $A_1^{(2m)^2-2m}$ for all $m \geqslant 3$.

We rewrite (a) and (b) as

$$2A_1^{\max[4m^2+2;\,4m^2+2]} \leqslant A_1^{4m^2+2m}, \qquad m \geqslant 2,$$

$$2\left|\frac{1}{\Psi_2}\right| A_1^{\max[4m^2-4m+6;\,4m^2-4m+6]} \leqslant A_1^{4m^2-2m}, \qquad m \geqslant 3.$$

The first inequality is satisfied for all $m \geqslant 2$ if $2A_1^2 \leqslant A_1^{2m}$, i.e. if $A_1^2 \geqslant 2$. The second inequality is satisfied for all $m \geqslant 4$ if $2\left|\frac{1}{\Psi_2}\right| A_1^6 \leqslant A_1^{2m}$, i.e. if $A_1^2 \geqslant 2\left|\frac{1}{\Psi_2}\right|$. We choose $A_1$ big enough to satisfy these and also $\left|\Psi_\lambda\right| \leqslant A_1^{\lambda^2-\lambda}$ for $\lambda = 1, 2, \ldots, 7$. This secures (i) by induction for all $\lambda$.

Next we note that in the second equality of (2.3), $\Psi_2(u)$ divides the right hand side. We can thus regard $\Psi_\lambda(u)$ as a polynomial in $\wp(u)$, $\wp'(u), g_2, g_3$ with rational numbers as coefficients. We now assert

(ii) The denominators of the rational number coefficients of $\Psi_\lambda(u)$ divide $2^{\lambda^2-\lambda}$.

We verify this for $\lambda = 1, 2, 3, 4$. Then we have only to verify

$$\max\big[3(m^2-m)+(m+2)^2-(m+2);\ 3\big((m+1)^2-(m+1)\big)+ +(m-1)^2-(m-1)\big] \leqslant (2m+1)^2-(2m+1), \qquad \text{for} \quad m \geqslant 2$$

and

$$\max\big[2\big((m-1)^2-(m-1)\big)+ +(m+2)^2-(m+2)+m^2-m;\ 2\big((m+1)^2-(m+1)\big)+ +(m-2)^2-(m-2)+m^2-m\big] \leqslant (2m)^2-2m \qquad \text{for} \quad m \geqslant 3.$$

These have already been evident.

By the same argument it is also evident that

(iii) The degree of $\Psi_\lambda(u)$ in any of the variables $\wp(u), \wp'(u), g_2, g_3$ does not exceed $\lambda^2-\lambda$.

Combining these facts it is now evident that $d(\Psi_\lambda)$ does not exceed $A_2^{\lambda^2-\lambda}$ (for some positive constant $A_2$ independent of $\lambda$) and so size $\Psi_\lambda$ does not exceed $2A_2^{\lambda^2-\lambda}$. The truth of Lemma 2 now follows in view of (2.1), by a simple application of Lemma 1.

LEMMA 3. *Let $\wp(z)$ be as in Lemma 2, $\beta_1, \ldots, \beta_p$ $(p \geqslant 1)$ complex numbers linearly independent over the field of rational numbers. Suppose that $\wp(\beta_1), \ldots, \wp(\beta_p)$ are algebraic numbers. Let us write for brevity $a = m_1\beta_1+\ldots+m_p\beta_p$, where $m_1, \ldots, m_p$ are natural numbers. Then as $m_1, \ldots, m_p$ run independently through all natural numbers for which $a$ is not a pole of $\wp(z)$, the numbers $\wp(a)$ all lie in a fixed algebraic number field and* size $\wp(a) \leqslant B^{n^2}$, *where $n = \max(m_1, \ldots, m_p)$ and $B$ is a positive constant independent of $m_1, \ldots, m_p$ (but $B$ may depend on $p$).*



Proof. First consider the case when for each $i$ $(i = 1, \ldots, p)$ it is true that $\beta_i, \omega_1, \omega_2$ are linearly independent (hereafter we shall mean over the field of rationals unless otherwise specified). The lemma is true for $p = 1$. Suppose it is true for $p-1$ $(p \geqslant 2)$. We write $b = m_1\beta_1+\ldots+ +m_{p-1}\beta_{p-1}$, and we have either $b$ is a pole of $\wp(z)$ in which case it is a period, or size $\wp(b)$ does not exceed $B_1^{n^2}$, for some positive constant $B_1$ independent of $m_1, \ldots, m_{p-1}$. In the first case $\wp(a) = \wp(m_p\beta_p)$ and the lemma follows by Lemma 2. In the second case we apply the theorem

$$\wp(u+v) = -\wp(u)-\wp(v)+\frac{1}{4}\left\{\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right\}^2.$$

Putting $u = b$, $v = m_p\beta_p$ (and hence $u+v = a$), we apply Lemma 1 as follows. Now as a function of $u$, $\wp(u)-\wp(v)$ has a zero of order at most 2 at $u = b$ and therefore we have to apply L'Hospital's rule at most twice to the function in the curly brackets. Thus one of the expressions $\big(\wp'(b)-\wp'(a-b)\big)\big(\wp(b)-\wp(a-b)\big)^{-1}$, $\wp''(b)\big(\wp'(b)\big)^{-1}$, $\wp'''(b)\times \times\big(\wp''(b)\big)^{-1}$ is determinate. Now $\big(\wp'(z)\big)^2 = 4\wp^3(z)-g_2\wp(z)-g_3$, $\wp''(z) = 6\wp^2(z)-\frac{1}{2}g_2$, $\wp'''(z) = 12\wp(z)\wp'(z)$ and so by Lemma 1, the sizes of $\wp'(b)$, $\wp'(a-b)$, $\wp''(b)$, $\wp''(a-b)$, $\wp'''(b)$, $\wp'''(a-b)$ do not exceed respectively constant (independent of $a, b$) multiples of $\big(\text{size}\ \wp(b)\big)^2$, $\big(\text{size}\ \wp(a-b)\big)^2$, $\big(\text{size}\ \wp(b)\big)^2$, $\big(\text{size}\ \wp(a-b)\big)^2$, $\big(\text{size}\ \wp(b)\big)^3$ and $\big(\text{size}\ \wp(a-b)\big)^3$. It is now easy to complete the proof using Lemma 1.

Now consider the case when exactly one of the $\beta_1, \ldots, \beta_p$ say $\beta_p$ depends linearly on $\omega_1, \omega_2$. Under this condition the point $m_p\beta_p$ is either a period or congruent modulo the periods of $\wp(z)$ to one of a fixed number of points which are not poles. In the first case the estimate trivially follows from the case $p-1$ already considered and in the second size $\wp(m_p\beta_p)$ does not exceed a bound independent of $m_p$, and we can complete the proof as before. When only two of the $\beta_i$ have the above property the result follows similarly.

Finally if $\beta_i, \omega_1, \omega_2$ are linearly dependent for $i = p-2, p-1$ and $p$, $\beta_{p-2}, \beta_{p-1}, \beta_p$ are themselves linearly dependent contrary to the hypothesis of the lemma. This completes the proof of Lemma 3.

LEMMA 4. *Let $\alpha$ and $\beta$ be real numbers and $\beta$ irrational. Then the number $N_\alpha = N_{\alpha,I}$ of natural numbers $n$ not exceeding $x$ for which the fractional part $(n\beta+\alpha)$ of $n\beta+\alpha$ belongs to a given interval $I$ (may be open or closed or consist of a single point) contained in $[0, 1)$ of length $|I|$ has the following property. Given any positive number $\eta$ however small, there exists a positive number $x_0 = x_0(\beta, \eta)$ depending only on the parameters indicated (in particular independent of $\alpha$ and $I$) such that $x^{-1}N_\alpha$ lies between $|I|-\eta$ and $|I|+\eta$ for all $x$ exceeding $x_0$.*

Proof. We first consider the case $a = 0$. Let $L$ be a natural number and divide $[0, 1]$ into $L$ subintervals of length $1/L$ each. Let $\lambda_1$ denote the maximum number of these sub-intervals contained wholly in $I$ and $\lambda_2$ the minimum number of these sub-intervals which contain $I$. Then it is easy to verify the inequalities

$$\text{(i)}\quad \lambda_1\frac{1}{L}+\frac{2}{L}\geqslant |I|\geqslant \lambda_2\frac{1}{L}-\frac{2}{L},$$

$$\text{(ii)}\quad \lambda_2\left(\frac{1}{L}+\eta_1\right)\geqslant x^{-1}N_0\geqslant \lambda_1\left(\frac{1}{L}-\eta_1\right)\quad \text{for all } x\geqslant x_0(\beta, L, \eta_1).$$

The second of these follows from Weyl's result ([13], see also [4] for other references and also for a simple proof of the result using only the simplest property of continued fractions). These inequalities are valid even if one or both of $\lambda_1, \lambda_2$ are zero.

Now,

$$\lambda_2\left(\frac{1}{L}+\eta_1\right)\leqslant\left(|I|+\frac{2}{L}\right)(1+L\eta_1)\leqslant |I|+\frac{2}{L}+L\eta_1+2\eta_1$$

and

$$\lambda_1\left(\frac{1}{L}-\eta_1\right)\geqslant\left(|I|-\frac{2}{L}\right)(1-L\eta_1)\geqslant |I|-\frac{2}{L}-L\eta_1-2\eta_1.$$

Hence

$$|x^{-1}N_0-I|\leqslant\frac{2}{L}+L\eta_1+2\eta_1\quad \text{for all } x\geqslant x_0(\beta, L, \eta_1).$$

We first fix $L$ large enough and then a small enough $\eta_1$ to get the result.

Now if $a$ is different from zero, consider the interval $I' = I+1-(a)$ with an obvious meaning. Now the points of $[0, 1)$ which differ from those of $I'$ by a rational integer form at most two (and at least one) intervals of total length $|I|$. These intervals say $I_1, I_2$ are further disjoint and have the property that $(n\beta+a)$ lies in $I$ if and only if $(n\beta)$ lies in $I_1$ or $I_2$. The general case now follows on applying the case $a = 0$ to the intervals $I_1$ and $I_2$.

COROLLARY. *Let $a_1, \ldots, a_p$ $(p\geqslant 1)$ be real numbers not all rational. Then the number of p-tuples $(m_1, \ldots, m_p)$ of natural numbers none of which exceed a given natural number $Q$, for which the fractional part $(m_1a_1+\ldots+m_pa_p)$ lies in a given interval $I$ contained in $[0, 1)$ is asymptotic to $Q^P|I|$, where $|I|$ is as usual the length of $I$, as $Q$ tends to infinity.*

Proof. Without loss of generality we may assume $a_1$ to be irrational. In Lemma 4 we write $\beta = a_1$, $a = m_2a_2+\ldots+m_pa_p$. Then for all $x\geqslant x_0 = x_0(\beta, \eta)$ the number $N_a$ lies between $x(|I|+\eta)$ and $x(|I|-\eta)$. Let now $Q\geqslant x_0(\beta, \eta)$ and note that $x_0$ is independent of $a$. We consider for the $(p-1)$-tuple $(m_2, \ldots, m_p)$ any one of the $Q^{p-1}$ possibilities. Summing up the bounds for $N_a$ for these possibilities we see that the total number of solutions required by the corollary lies between $Q^P(|I|-\eta)$ and $Q^p(|I|+\eta)$. Since $\eta$ is arbitrary this proves the corollary. However our proof also gives the following result. The total number of solutions of $(m_1a_1+\ldots+m_pa_p)\in I$, where $m_1$ runs through all natural numbers not exceeding $x_1$ and $m_2, \ldots, m_p$ through any $x_2, \ldots, x_p$ distinct natural numbers, is asymptotic to $x_1\ldots x_p|I|$ uniformly in $x_2, \ldots, x_p$ (with an obvious meaning) as $x_1$ tends to infinity. In this result we have to assume that $a_1$ is irrational.



DEFINITION. Let $\wp_i(z) = \wp_i(z; \omega_1^{(i)}, \omega_2^{(i)})$ $(i = 1, \ldots, u)$ be $u$ Weierstrass functions (not necessarily distinct; for the purpose of this definition and the three lemmas to follow we do not even need the restriction that the invariants $g_2^{(i)}, g_3^{(i)}$, $i = 1, \ldots, u$, be algebraic) whose period groups are $\mathscr{P}_i$ $(i = 1, \ldots, u)$, respectively. Let $b_1, \ldots, b_u$ be $u$ nonzero complex numbers linearly independent over the field of rationals. With every complex number of the type $a = m_1\beta_1+\ldots+m_p\beta_p$ (where $m_1, \ldots, m_p$ are natural numbers) we associate *the natural number* $n(a) = \max(m_1, \ldots, m_p)$. Let $W$ be any set of distinct complex numbers. *We say that a number $a$ is $W$-admissible if $W$ contains a representative of the coset $b_ia$ $\mathrm{mod}\,\mathscr{P}_i$ for each $i = 1, \ldots, u$.*

LEMMA 5. *There exists in the complex plane a compact set $W$ free from the poles of each of the functions $\wp_i(z)$ $(i = 1, \ldots, u)$ with the property that the number of $W$-admissible numbers $a$ (see the definition above) with $n(a)$ not exceeding $Q$ is at least $\frac{1}{2}Q^p$ for all $Q$ exceeding $Q_0$. Here $W$ and $Q_0$ depend on the period groups $\mathscr{P}_i$ $(i = 1, \ldots, u)$ and the numbers $\beta_1, \ldots, \beta_p$.*

Proof. Consider the union of $u$ closed parallelograms $(0, \omega_1^{(i)}, \omega_1^{(i)}+\omega_2^{(i)}, \omega_2^{(i)})$ corresponding to the period groups $\mathscr{P}_i$. The union contains only finitely many points of the $u$ groups $\mathscr{P}_1, \ldots, \mathscr{P}_u$. We exclude from the union all those points which are at a distance less than $\varepsilon$ (a fixed positive quantity) from these points and denote the resulting set by $W$. The set $W$ is compact and we show that, if $\varepsilon$ is a fixed number sufficiently small, this set has the property required. We fix $\varepsilon$ in such a way that the sum of the projections of the excluded regions on the four sides of the parallelogram corresponding to $\mathscr{P}_i$ is less than $1/3u$ times the minimum of the length of its sides (here we have used the word projection not as orthogonal projection, but as oblique projection parallel to the sides of the parallelogram).

We write $b_i\beta_j = a_{ij}\omega_1^{(i)}+\beta_{ij}\omega_2^{(i)}$ $(i = 1, \ldots, u, j = 1, \ldots, p)$ where $a_{ij}$ and $\beta_{ij}$ are real. For each fixed $i$ $(i = 1, \ldots, u)$ it is clear that one at least of the $2p$ numbers $a_{ij}, \beta_{ij}$ is irrational, since otherwise it would lead to the linear dependence of $\beta_1, \ldots, \beta_p$ over the field of rationals. Now $b_ia = \omega_1^{(i)}\sum m_ja_{ij}+\omega_2^{(i)}\sum m_j\beta_{ij}$, the sums being from $j = 1, \ldots, p$. Since we are interested in a representative of $b_i a \bmod \mathscr{P}_i$ we may reduce the coefficients of $\omega_1^{(i)}$ and $\omega_2^{(i)}$ modulo 1, to lie in the interval $[0, 1)$. Suppose for definiteness that one of the coefficients $a_{ij}$ is irrational. Then the total number of numbers $a$ with $n(a)$ not exceeding $Q$ for which the coefficient of $\omega_1^{(i)}$, in $b_i a$, reduced modulo 1 lies in the projection (parallel to the side $(0, \omega_2^{(i)})$) of the excluded regions on the side $(0, \omega_1^{(i)})$ does not exceed $Q^p/2u$ for $Q \geqslant Q_0(a_{i1}, \ldots, a_{ip})$ by the Corollary to Lemma 4. If all the $a_{ij}$ are rational we consider the coefficient of $\omega_2^{(i)}$ and obtain a similar result with $\beta_{ij}$ in place of $a_{ij}$. Repeating this for each $i = 1, \ldots, u$, we find that the total number of numbers $a$ with $n(a)$ not exceeding $Q$, for which one at least of the coefficients of $\omega_1^{(i)}, \omega_2^{(i)}$ in $b_i a$ does not lie modulo 1, in the projections of the excluded regions on $(0, \omega_1^{(i)})$ (resp. $(0, \omega_2^{(i)})$) for each $i = 1, \ldots, u$, is at least $\frac{1}{2}Q^p$ for all $Q$ exceeding a certain number $Q_0$. The number $Q_0$ and the set $W$ depend clearly on the parameters mentioned in the lemma. This proves Lemma 5 completely.

LEMMA 6. *Let $\omega_1^{(i)}, \omega_2^{(i)}, \wp_i(z), b_i$ $(i = 1, \ldots, u), \beta_1, \ldots, \beta_p$ be as in the definition preceding Lemma 5, $a_\mu$ the points $a = m_1\beta_1 + \ldots + m_p\beta_p$ which occur in connection with Lemma 5 arranged in the order of $n_\mu = n(a_\mu)$ non-decreasing, and $a_i$ a zero of $\wp_i(z)$. Let $f_{1,i}(z)$ and $f_{2,i}(z)$ be respectively $-\sigma_i(z - a_i)\sigma_i(z + a_i)$ and $\sigma_i^2(z)\sigma_i(a_i)$ where $\sigma_i(z)$ is the Weierstrass sigma function corresponding to $\wp_i(z)$ ($f_{1,i}(z)$ and $f_{2,i}(z)$ are two entire functions without common zeros and $\wp_i(z) = f_{1,i}(z)\big(f_{2,i}(z)\big)^{-1}$). Then we have*

$$\operatorname*{Max}_{\substack{i=1,\ldots,u \\ n_\mu \leqslant Q}} \frac{1}{|f_{2,i}(b_i a_\mu)|} \leqslant B_2^{Q^2}$$

*and*

$$\operatorname*{Max}_{i=1,\ldots,u} \Big\{\big(\max_{|z|=R} |f_{1,i}(b_i z)| + 1\big)\big(\max_{|z|=R} |f_{2,i}(b_i z)| + 1\big)\Big\} \leqslant B_2^{R^2}$$

*where $B_2$ is a positive constant independent of $Q$ and $R$.*

Proof. Denoting by $\zeta_i(z)$ the Weierstrass zeta function corresponding to $\wp_i(z)$, we have the identity ([14], page 448; for the result $\wp_i(z) = f_{1,i}(z)\big(f_{2,i}(z)\big)^{-1}$ see example 1 on page 451),

$$\sigma_i(z + n_1\omega_1^{(i)} + n_2\omega_2^{(i)}) = (-1)^{n_1+n_2}\operatorname{Exp}\left[\zeta_i\left(\frac{\omega_1^{(i)}}{2}\right)\{2n_1 z + 2n_1 n_2\omega_2^{(i)} + n_1^2\omega_1^{(i)}\} + \right.$$
$$\left. + \zeta_i\left(\frac{\omega_2^{(i)}}{2}\right)\{2n_2 z + n_2^2\omega_2^{(i)}\}\right]\sigma_i(z),$$

where $n_1, n_2$ are arbitrary non-negative rational integers and it is easy to modify this when $n_1, n_2$ are arbitrary rational integers. To deduce the first we take for some fixed $i$, $z = b_i a_\mu$ and select $n_1, n_2$ such that $z + n_1\omega_1^{(i)} + n_2\omega_2^{(i)}$ lies in the parallelogram $0, \omega_1^{(i)}, \omega_1^{(i)} + \omega_2^{(i)}, \omega_2^{(i)}$. We observe that $|\sigma_i(b_i a_\mu + n_1\omega_1^{(i)} + n_2\omega_2^{(i)})|$ is greater than a fixed positive



constant independent of $m_1, \ldots, m_p$ and also that $|n_1|$ and $|n_2|$ do not exceed a certain constant multiple of $Q$. This leads to the first inequality. The second one also follows by a similar argument.

LEMMA 7. *Let $\wp_1(z) = \wp(z; \omega_1^{(1)}, \omega_2^{(1)})$, $\wp_2(z) = \wp(z; \omega_1^{(2)}, \omega_2^{(2)}), \ldots$ be finitely many Weierstrass elliptic functions with periods indicated. Then $\wp_1(z)$ and $\wp_2(z)$ are algebraically independent (unless otherwise specified we mean over the field of complex numbers) if and only if there exists a rational $2\times 2$ matrix $M$ such that $(\omega_1^{(2)}, \omega_2^{(2)}) = (\omega_1^{(1)}, \omega_2^{(1)})M$, in this case we say that the periods of $\wp_1(z)$ and $\wp_2(z)$ are commensurable. The necessary and sufficient condition that $e^z, \wp_1(z), \wp_2(z)$ be algebraically independent is that $\wp_1(z)$ and $\wp_2(z)$ be algebraically independent. Finally suppose that $\omega_2^{(1)^{-1}}, \omega_2^{(2)^{-1}}, \omega_2^{(3)^{-1}}, \ldots$ are linearly independent over the field of rationals and the ratio of every two of them is real. Then the functions $\wp_1(z), \wp_2(z), \ldots$ are algebraically independent.*

Remark. Let $\wp(z) = \wp(z; \omega_1, \omega_2)$ with usual notation and $b_1, b_2, b_3, \ldots$ be finitely many complex numbers linearly independent over the rationals. Then $\wp(b_1 z)$ and $\wp(b_2 z)$ are algebraically dependent if and only if $\omega_2/\omega_1$ is an algebraic number of degree 2 and $b_1 b_2^{-1}, 1, \omega_2/\omega_1$ are linearly dependent over the field of rationals. Let further that one of the numbers $\omega_1, \omega_2$ be either real or pure imaginary and $b_1, b_2, \ldots$ have the property that the ratio of every two of them is real. Then the functions $\wp(b_1 z), \wp(b_2 z), \ldots$ are algebraically independent.

Proof. The first statement of the lemma is standard. To prove the final statement we assume that $\wp_1(z), \wp_2(z), \ldots$ have an algebraic relation of the form

$$\wp_1^k(z)P_1\big(\wp_2(z), \ldots\big) + \wp_1^{k-1}(z)P_2\big(\wp_2(z), \ldots\big) + \ldots = 0 \qquad (k \geqslant 1)$$

where $P_1, P_2, \ldots$ are polynomials in $\wp_2(z), \ldots$ (with complex number coefficients) and that $P_1$ is not identically zero. This relation is an identity and hence we replace $z$ by $z + n_1\omega_2^{(1)}$, where $n_1$ is an arbitrary natural number. The function $\wp_1(z)$ remains unaltered and in other functions we may further change $z + n_1\omega_2^{(1)}$ by the addition of a suitable period. For instance $\wp_2(z + n_1\omega_2^{(1)}) = \wp_2(z + n_1\omega_2^{(1)} + n_2\omega_2^{(2)})$ where $n_2$ is a suitable rational integer and similarly for the other functions. Now let $z$ be a complex number different from zero and close to zero such that $z^{-1}\omega_2^{(1)}$ is real. Since the fractional parts of the numbers $n_1\omega_2^{(1)}\omega_2^{(2)^{-1}}, n_1\omega_2^{(1)}\omega_2^{(3)^{-1}}, \ldots$ (say $l$ numbers) regarded as points of the Euclidean $R^{(l)}$ are dense in the unit cube there by Kronecker's Theorem [7], we may for certain arbitrarily large values of $n_1$ choose $n_2, n_3, \ldots$ in such a way that

$$P_1\big(\wp_2(z + n_1\omega_2^{(1)} + n_2\omega_2^{(2)}), \ldots\big), \qquad P_2\big(\wp_2(z + n_1\omega_2^{(1)} + n_2\omega_2^{(2)}), \ldots\big)$$

are bounded above by a constant independent of $z, n_1, n_2, n_3, \ldots$ and that the first term has its absolute value bounded below by a positive constant independent of $z, n_1, n_2, \ldots$ Since $\wp_1(z) \sim z^{-2}$ as $z$ approaches zero the assertion of the lemma follows.

We now prove the second assertion of the lemma. The necessity of the condition is clear. Assuming the period lattices $\omega_1^{(1)}, \omega_2^{(1)}$ and $\omega_1^{(2)}, \omega_2^{(2)}$ to be incommensurable, which is equivalent to the algebraic independence of $\wp_1(z)$ and $\wp_2(z)$, we have plenty of periods $\Omega = m\omega_1^{(1)} + n\omega_2^{(1)}$ ($m, n$ natural numbers) which are not poles of $\wp_2(z)$. Suppose that there exists a polynomial relation between the three functions. If for instance $\wp_1(z)$ does not enter this relation a contradiction is immediate since $e^z$ is not doubly periodic. Consider now this polynomial as a polynomial in $\wp_1(z)$ and suppose that $P(e^z, \wp_2(z))$ is the coefficient of the highest degree term. Since $\wp_1(z) \sim z^{-2}$ as $z$ tends to zero we have on replacing $z$ by $z+\Omega$ and letting $z \to 0$, $P(e^{\Omega}, \wp_2(\Omega)) = 0$ for all periods $\Omega$ of $\wp_1(z)$ which are not poles of $\wp_2(z)$. We now examine such periods $\Omega$. Writing $\omega_1^{(1)} = a_1\omega_1^{(2)} + a_2\omega_2^{(2)}$, $\omega_2^{(1)} = a_3\omega_1^{(2)} + a_4\omega_2^{(2)}$ where the $a_i$ are real we observe that one at least of the $a_i$ is irrational. Consequently in the expression $\Omega = (ma_1+na_3)\omega_1^{(2)} + (ma_2+na_4)\omega_2^{(2)}$ one at least of the coefficients can be restricted to lie modulo 1, in any fixed closed sub-interval of $(0, 1)$ with non-empty interior, by Kronecker's Theorem [7]. The natural numbers $m$ and $n$ are still arbitrary. If for instance $a_2$ is irrational we can fix $n$ as large as we please and restrict $m$ in such a way that the points $\Omega$ are congruent $\bmod(\omega_1^{(2)}, \omega_2^{(2)})$ to points of a fixed compact set just described. Since $m$ is arbitrary there are plenty of points which are incongruent $\bmod(\omega_1^{(2)}, \omega_2^{(2)})$. If now $P$ is independent of $e^z$, the elliptic function $P(\wp_2(z))$ has plenty of incongruent zeros and is therefore a constant. On the other hand if $P_1(\wp_2(z))$ is the coefficient of the highest degree term in $P$ regarded as a polynomial in $e^z$ (we can assume without loss of generality that the real parts of $\omega_1^{(1)}$ and $\omega_2^{(1)}$ are non-negative), it follows that $P_1(\wp_2(\Omega))$ as a function of the natural numbers $m, n$ tends to zero. But we can further restrict the points $\Omega$ (since the zeros of $P_1(\wp_2(z))$ are isolated) to lie mod $\omega_1^{(2)}, \omega_2^{(2)}$, in a compact set where $P_1(\wp_2(z))$ never vanishes. It follows that $P_1(\wp_2(\Omega))$ is bounded below, in absolute value, by a positive constant independent of $m, n$. This contradiction completes the proof of the lemma.

**§ 3. Principal results.** We are now in a position to deduce our principal results. In view of the lemma that follows it is convenient to introduce

Definition. A complex number $\beta$ is said to be a *pseudo-algebraic point* of a meromorphic function $f(z)$ if either $\beta$ is a pole of $f(z)$ or $f(\beta)$ is an algebraic number.

Lemma 8. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with an algebraic relation $P_1(f(z), g(z)) = 0$, where $P_1$ is a polynomial in $f(z)$, and $g(z)$ with algebraic numbers as coefficients, not all zero. Then the set of pseudo-algebraic points of $f(z)$ is identical with the set of pseudo-algebraic points of $g(z)$.*



Proof. Let $\beta$ be a pseudo-algebraic point of $f(z)$ and let if possible $g(\beta)$ be transcendental. By considering the reciprocal of $f(z)$ if necessary we can assume that $f(\beta)$ is algebraic. Then considering $f(z)-f(\beta)$ in place of $f(z)$ we can assume that $f(\beta) = 0$. This would mean that $P_1(0, g(z))$ is identically zero in $z$, i.e. $f(z)$ divides $P_1(f(z), g(z))$; but we can divide $P_1(f(z), g(z))$ by a suitable power of $f(z)$ if necessary and assume without loss of generality that $P_1(0, g(z))$ is not identically zero. This proves the lemma.

Remark 1. Lemma 8 shows that if $f(z)$ is a nonconstant elliptic function with periods $\omega_1, \omega_2$ with algebraic invariants $g_2, g_3$ and the expression of $f(z)$ in terms of $\wp(z)$, $\wp'(z)$ as a rational function involves only algebraic number coefficients, then the pseudo-algebraic points of $f(z)$ and $\wp(z)$ are the same. Thus it suffices to consider the pseudo-algebraic points of $\wp(z)$. Addition theorem for $\wp(z)$ shows that these points form a group under the usual addition of complex numbers. Also since $\wp(z)$ and $\wp(nz)$ ($n$ natural number) are algebraically dependent over the field of all algebraic numbers, these points have actually a vector space structure over the field of rationals, i.e. if $\beta_1, \beta_2$ are pseudo-algebraic points of $\wp(z)$ so is $r\beta_1 + s\beta_2$ for any two rational numbers $r, s$. If $\tau = \omega_2/\omega_1$ is an imaginary quadratic number then it is easy to verify that $\wp(z\tau)$ has the same pseudo-algebraic points as $\wp(z)$, and so in this case these points have a vector space structure over the imaginary quadratic field obtained by adjoining $\tau$ to the field of rationals.

Remark 2. Similar remarks apply to the function $e^z$ and nonconstant meromorphic functions which depend algebraically on $e^z$ with algebraic coefficients. Needless to say that such remarks apply also to $z$ and rational functions of $z$ with algebraic coefficients.

It will be convenient to introduce

Definition. A meromorphic function $\Phi(z)$ is said possess an *algebraic addition theorem if there exists a polynomial* $P(\zeta_1, \zeta_2, \zeta_3)$ in three variables, with complex number coefficients not all zero such that $P(\Phi(z_1+z_2), \Phi(z_1), \Phi(z_2))$ is zero for all pairs $z_1, z_2$ of complex numbers for which $\Phi(z_1+z_2)$, $\Phi(z_1)$ and $\Phi(z_2)$ are finite. If further the coefficients of $P$ are algebraic numbers it is said to be *algebraically additive*.

By a well known Theorem of Weierstrass (see [1], p. 363) every meromorphic function $\Phi(z)$ with an algebraic addition theorem has the property that for a suitable nonzero complex number $b$ depending on the function, $\Phi(bz)$ is a rational function of $\Psi(z)$, $\Psi'(z)$ with complex number coefficients where $\Psi(z)$ is either $z$, $e^z$ or $\wp(z)$ with invariants

$g_2, g_3$ not necessarily algebraic. It can be shown that if $\Phi(z)$ is not a constant and is *algebraically additive* then the coefficients in the rational expression for $\Phi(bz)$ in terms of $\Psi(z)$, $\Psi'(z)$ has *algebraic number coefficients* and also that if $\Psi(z)$ happens to be $\wp(z)$ *the invariants* $g_2, g_3$ are *algebraic numbers*. It is easy to see that the order of a non-constant meromorphic function $\Phi(z)$ with an algebraic addition theorem is 0, 1 or 2 according as the corresponding function $\Psi(z)$ is $z$, $e^z$ or $\wp(z)$. In view of Lemma 8 and the Remarks 1 and 2 below the lemma we can make

Remark 3. The pseudo algebraic points of an algebraically additive meromorphic function form a vector space over the field of rational numbers.

Lemmas 3, 5 and 6 now enable us to deduce from Theorem 1,

THEOREM 2 (Principal result). *Let $\Phi_1(z), \ldots, \Phi_s(z)$ be $s \geqslant 2$ algebraically independent (over the field of complex numbers) meromorphic functions each of which is algebraically additive. Let $\varrho^*$ and $\varrho_*$ denote the maximum and the minimum of the orders of these functions. Then $\dim(\Phi_1(z), \ldots, \Phi_s(z))$ denoting the dimension of the vector space (over the rational number field) of common pseudo algebraic points of these functions, we have*

$$\dim(\Phi_1(z), \ldots, \Phi_s(z)) \leqslant \varrho^* + (\varrho_* - \theta)/(s-1)$$

*where $\theta$ is 1 if the functions have a common period and 0 otherwise.*

Proof. Clearly it suffices to confine to $s \leqslant 4$ and further to the case where $s-1$ of the functions have order $\varrho^*$. By Lemma 8 we can make a further reduction replacing $\Phi_t(z)$ $(t = 1, \ldots, s)$ by $\Psi_t(b_t z)$ where $b_t$ are nonzero complex numbers and $\Psi_t(z) = z$, $e^z$ or $\wp(z)$ with algebraic invariants. The proof in the various cases are all similar and by way of illustration it suffices to prove for instance that $\dim(\wp_1(\omega_1^{(1)}z), \wp_2(\omega_1^{(2)}z), \wp_3(\omega_1^{(3)}z))$ does not exceed 2 (from now on we use $\wp_i(z)$, $i = 1, 2, \ldots$ to denote Weierstrass functions with fundamental periods $\omega_1^{(i)}$, $\omega_2^{(i)}$ and algebraic invariants $g_2^{(i)}, g_3^{(i)}$). Let if possible the dimension be greater than 2. Then there exist three complex numbers $\beta_1, \beta_2, \beta_3$ ($\beta_1$ rational and $0 < \beta_1 < 1$) linearly independent over the field of rationals for which the nine numbers $\wp_i(\omega_1^{(i)}\beta_j)$ $(i = 1, 2, 3, j = 1, 2, 3)$ are algebraic numbers. In Theorem 1 we take $F_i(z) = \wp_i(\omega_1^{(i)}z)$ $(i = 1, 2, 3)$ and we can define a weighted sequence $\{a_\mu\}$ as follows. We set $a_\mu = m_1\beta_1 + m_2\beta_2 + m_3\beta_3$ ($m_i$ natural numbers), $n_\mu = \max(m_1, m_2, m_3)$ and arrange $a_\mu$ in the order of $n_\mu$ non-decreasing. We confine only to those $a_\mu$ for which $a_\mu$ is $W$-admissible, where $W$ is the compact set determined by Lemma 5 with respect to $b_i = \omega_1^{(i)}$, $i = 1, 2, 3$ (see Lemma 5 and the definition preceding it). The set $\{a_\mu\}$ contains with $a_\mu$ also the point $a_\mu + 1$ and so the subsequence $\{a_{\mu_r}\}$ may be chosen to be the subsequence obtained by taking only those points $a_\mu$ for which $m_1\beta_1$ does not exceed 1. Since $F_i(z)$ $(i = 1, 2, 3)$ have period 1 condition (v) is satisfied. By



Lemma 5 $N(Q)$ lies between $\frac{1}{2}Q^3$ and $Q^3$ and it follows that $N_1(Q)$ lies between two positive constant multiples of $Q^2$. This secures (iii). The condition (vi) is easily satisfied. The major and minor densities are 3 and 2 and the deviation is 1. By Lemmas 3 and 6 this weighted sequence is special. Hence by Theorem 1, it follows that $3 \leqslant 2 + (2-1)/2$ a contradiction. This proves our principal result.

**§ 4. Concluding remarks.** A number of curious corollaries follow from Theorem 2, on using Lemma 7. Some of these have already been stated in the introduction. We mention one more corollary.

COROLLARY. *If $a$ and $b$ are real positive algebraic numbers different from 1 for which $\log a/\log b$ is irrational and $a < b < a^{-1}$, then one at least of the two numbers*

$$\Big(\frac{1}{240} + \sum_{n=1}^{\infty} \frac{n^3 a^n}{1-a^n}\Big) \prod_{n=1}^{\infty} (1-a^n)^{-8},$$

$$\Big\{\frac{6}{(b^{1/2}-b^{-1/2})^4} - \frac{1}{(b^{1/2}-b^{-1/2})^2} - \sum_{n=1}^{\infty} \frac{n^3 a^n (b^n + b^{-n})}{1-a^n}\Big\} \prod_{n=1}^{\infty} (1-a^n)^{-8}$$

*is transcendental.*

Proof. It is easy to verify that the functions $e^{2\pi i z \omega_1^{-1}}$ and $\mathfrak{g}(z) = (\Delta(\omega_1, \omega_2))^{-1} (\wp''(z; \omega_1, \omega_2))^3$ are algebraically additive under the only assumption $j(\omega_2/\omega_1)$ is algebraic and have the common period $\omega_1$. Hence $\dim(e^{2\pi i z/\omega_1}, \mathfrak{g}(z)) \leqslant 2$. We set $\omega_2 = \log a$, $\omega_1 = -2\pi i$ and consider the values of both the functions at $z = 2\pi i$, $\log a$, $\log b$ which are clearly linearly independent over the field of rationals. It follows that one at least of the two numbers

$$j\Big(\frac{\log a}{2\pi i}\Big), \qquad (\Delta(2\pi i, \log a))^{-1} (\wp''(\log b; 2\pi i, \log a))^3$$

is transcendental. The Fourier expansions for $j$, $\Delta$ and $\wp$ functions (which can be found for instance in Deuring [2] or Hasse [5]) lead to the result stated.

It may be possible to improve the bound for the dimension given by Theorem 2 probably to 1 in all cases. But even a slight improvement in the bound such as $\leqslant \varrho^* + (\varrho_* - \theta)/s$ appears to be very difficult. One may ask another question. Let $a, b, c$ be three non-zero complex numbers such that $\log a$, $\log b$, $\log c$ are linearly independent over the field of rationals. Let $\omega_1, \omega_2, \ldots, \omega_n$ be any complex numbers linearly independent over the field of rationals. Then if $n \geqslant 2$ Theorem 2 shows that the $3n$ numbers

$$a^{\omega_1}, b^{\omega_1}, c^{\omega_1};\ a^{\omega_2}, b^{\omega_2}, c^{\omega_2};\ \ldots;\ a^{\omega_n}, b^{\omega_n}, c^{\omega_n}$$

cannot all be algebraic (in fact $n-1$ of them must be transcendental). It is natural to ask whether among these $3n$ numbers there exist $n-1$