Similarly, by modifying his proof we can deal with the more general proposition

$$\sum_{p\leqslant x}\{f(g(p))\}^k \sim \pi(x)\, S_x^k.$$

In neither of these proofs do we therefore have to appeal to any results from the theory of complex variables.

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Contributions to the theory of transcendental numbers (I)

by

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Dedicated to the memory of

Jacqués Hadamard (1865-1963)

§ 1. Introduction. In this paper we prove the main theorem relating to the set (or a subset) of complex numbers at which a given set of algebraically independent meromorphic functions assume values in a fixed algebraic number field (we actually prove a more general result which may be useful elsewhere). We state a few deductions in § 2 and it is interesting to note that the Main Theorem gives significant results in the case (overlooked by Gelfond) *where the functions concerned do not satisfy algebraic differential equations of the first order with algebraic number coefficients.* Since some of the deductions require lengthy preparations we postpone the proofs of these and other deductions to part II, which is a continuation of this paper. We give a brief history of this theorem in this section.

In the year 1929, A. O. Gelfond made the important discovery that $a^b = e^{b\log a}$ is transcendental for every imaginary quadratic irrationality b and every algebraic a different from zero except for $\log a = 0$ [1]. Assuming the result to be false Gelfond applied the interpolation formula

$$f(z) = a_0F_0(z)+a_1F_1(z)+a_2F_2(z)+\ldots$$

where $F_n(z) = \prod_{l=1}^{n}(z-z_l)$, $a_{n-1} = \sum_{k=1}^{n} f(z_k)\{F_n'(z_k)\}^{-1}$, and $z_1, z_2, \ldots$ is a previously given sequence of complex numbers (for the conditions of validity see Siegel's monograph [6], § 14, Chapter I) to the function $f(z) = e^{z\log a}$ and arrived at the contradiction that the above expansion for $f(z)$ for a suitable sequence $z_1, z_2, \ldots$ must terminate. *It is important to note that the only property of the exponential function required in the proof is the addition theorem* $e^{x+y} = e^x e^y$. Gelfond's proof was carried over to the case of a real quadratic irrationality b by R. O. Kusmin [2] in 1930.

(1) As a consequence we have the remarkable result that the decimal expansion of $e^{\pi} = i^{-2i}$ never terminates. Ref. A. O. Gelfond, *Sur les nombres transcendants*, Comptes Rendus Acad. Sci. Paris, 189(1929), pp. 1224-1226.

The general problem of the transcendency of a^b for arbitrary algebraic irrational b was solved completely in the year 1934, independently of each other by A. O. Gelfond and Th. Schneider. Both the proofs used the arithmetical lemmas in § 2 of Chapter II of [6]. Their proofs had two important differences. First, whereas Gelfond's method *used essentially the differential equation for e^z in addition to the addition theorem, Schneider's method used only the addition theorem.* Secondly Gelfond's method did not lead, at any rate easily, to the determination of a transcendence measure for a^b, i.e. an explicit lower bound for $|P(a^b)|$ where $P(x)$ is a given polynomial with rational integral coefficients not all zero. Schneider's method was readily effective in obtaining a transcendence measure for a^b. However it should be mentioned that Gelfond's proof was shorter and the method was readily applicable to some problems dealing with elliptic functions and related questions. Some applications of this kind were made by Siegel and Schneider and all theorems obtainable by this method are special cases of a general theorem of Schneider (see [4], Satz 12, page 49). Schneider's general theorem gives an upper estimate for the number of points in the complex plane at which the Taylor expansion coefficients, of a given set of algebraically independent (*we shall mean always over the field of complex numbers*) meromorphic functions, lie in a fixed algebraic number field. Under some reasonable conditions (*an essential one is the existence of an algebraic differential equation of the first order with algebraic number coefficients, for each of the functions*) it follows from this theorem that origin is the only possible point at which all the functions take algebraic values. In particular this is true of the following pairs: (z, e^z), $(z, \wp(z))$, (e^z, e^{az}), $(e^z, \wp(z))$, $(\wp_1(z), \wp_2(z))$ where a is an algebraic number, *the invariants g_2, g_3 for the Weierstrass elliptic functions involved are algebraic numbers* (*this convention will always be adopted unless otherwise stated explicitly*) and the functions concerned are algebraically independent. *For the deduction of these results we have also to use the addition theorem.*

It is a pleasure to thank professor C. L. Siegel for going through the manuscript in detail and suggesting this presentation, and to thank professor K. G. Ramanathan for many helpful discussions in connection with the preparation of the manuscript.

§ 2. Statement of some results. However, very little is known about the number $a^b = e^{b\log a}$ (a and b arbitrary complex numbers, $b\log a \neq 0$), for instance π^e. Although there are further developments of Gelfond's method by himself which show, for example, that one at least of the four numbers a^{e^ν}, $a^{e^{2\nu}}$, $a^{e^{3\nu}}$, $a^{e^{4\nu}}$ (a algebraic, ν rational and $\nu\log a \neq 0$) is transcendental (see [1], pp. 131-133) there does not seem to be any result about π^e. By means of a new idea we prove a very general and

important theorem stated in § 3, which we call the Main Theorem. The new idea simplifies considerably the Schneider's method of proof of the transcendency of a^b (referred to in the third paragraph in § 1), at the cost of making the proof, probably, ineffective in questions of transcendence measure. But our method has its advantages and enables us to study the set $S(f, g)$ (Schneider's method as such requires one of the functions f, g to be z) of all complex numbers at which two algebraically independent meromorphic functions $f(z)$ and $g(z)$ take values which are algebraic numbers. For instance if a is any transcendental number it follows from our main theorem that any three numbers of the set $S(e^z, e^{az})$ have a non-trivial linear homogeneous relation with rational numbers as coefficients; in other words we say that the set $S(e^z, e^{az})$ has dimension at most 2. Alternatively if $\alpha, \beta, \gamma, \delta$ are algebraic numbers with the property that $\log\delta/\log\alpha$ is irrational, and $\log\alpha$, $\log\beta$, $\log\gamma$ do not satisfy any non-trivial linear homogeneous relation with rational coefficients, then one at least of the two numbers $\exp\left(\dfrac{\log\delta\log\beta}{\log\alpha}\right)$ and $\exp\left(\dfrac{\log\delta\log\gamma}{\log\alpha}\right)$ is transcendental (²). We can deduce from this a stronger result than Gelfond's, namely that one at least of the four numbers a^b, a^{b^2}, a^{b^3}, a^{b^4} (where a and b are arbitrary complex numbers with $\log a \neq 0$ and b transcendental) is transcendental. In particular, choosing a such that a^b is a given algebraic number A, it follows that one at least one of the three numbers A^b, A^{b^2}, A^{b^3} (b transcendental, A algebraic with $\log A \neq 0$) is transcendental.

There is however, one result of Schneider ([5], II) on elliptic functions which *does not use essentially the differential equation* but uses only the addition theorem; but this result is superseded by the general theorem of Schneider obtained along the lines of Gelfond's method. It seems difficult to extend Schneider's method (which deals with the set $S(z, \wp(az))$, where a is an arbitrary complex number, $a \neq 0$) to study the set $S(e^z, \wp(az))$ or the set $S(\wp_1(z), \wp_2(az))$ since the extension of the method would require good lower bounds for linear forms with arbitrary complex coefficients and there are other difficulties. However, the new idea adopted in this paper is suitable for the discussion of these and other problems since we avoid all such difficulties by using a result of Weyl and others (for references see part II) on the uniform distribution of certain numbers modulo 1. We have also to improve certain estimates of Schneider regarding the size (see § 3 for the definition) of certain algebraic numbers

(²) After writing this manuscript I came to know from professor C. L. Siegel that this is a result first due to Schneider and Siegel. Their result is unpublished. This result is also to be found in a recent paper by S. Lang, *Algebraic values of meromorphic functions*, Topology 5 (4), (1966), pp. 363-370. The results of this paper have something in common with Lang's results.

which appear in connection with the Weierstrass elliptic function. These preparations are rather lengthy and we carry out the details in part II of this paper. It will suffice here to state one or two results. If $\wp_1(z)$ and $\wp_2(az)$ are algebraically independent then the dimension of the set $S(\wp_1(z), \wp_2(az))$ is at most 4. The dimension is at most 3 if $\wp_1(z)$ and $\wp_2(az)$ have a common period and there are other results applicable to two or more functions one of which may be e^z. The analogue of the result for A^b, A^{b^2}, A^{b^3} is as follows: If b is any transcendental number then one at least of the five numbers

$$\wp(\omega_1 b),\ \wp(\omega_1 b^2),\ \wp(\omega_1 b^3),\ \wp(\omega_1 b^4),\ \wp(\omega_1 b^5)$$

is transcendental where ω_1, ω_2 are the periods of $\wp(z) = \wp(z; \omega_1, \omega_2)$. If further ω_2/ω_1 is an imaginary quadratic number then one at least of the three numbers

$$\wp(\omega_1 b),\ \wp(\omega_1 b^2),\ \wp(\omega_1 b^3)$$

is transcendental. An alternative way of stating the same result is that one at least of the three numbers $\{\Delta(1,\tau)\}^{-1}\wp^6(b)$, $\{\Delta(1,\tau)\}^{-1}\wp^6(b^2)$, $\{\Delta(1,\tau)\}^{-1}\wp^6(b^3)$ is transcendental, provided $\wp(z) = \wp(z; 1, \tau)$, τ imaginary quadratic and in this alternative statement it is not necessary to suppose that g_2, g_3 are algebraic. (We have written $\Delta(1,\tau) = g_2^3 - 27g_3^2$ as usual). Combining this result with a result of Mahler [3] and Schneider ([5], II) we can deduce the following result. Let ω_2/ω_1 be an imaginary quadratic number, g_2, g_3 rational and let H be the maximum of the heights of g_2 and g_3 (height of an algebraic number a is as usual the maximum of the absolute values of the coefficients of the irreducible polynomial, with coprime rational integer coefficients, of which a is a zero). Then if b be a complex number for which $\wp(b\omega_1)$ is a rational number with height exceeding $480H^5$, one at least of the two numbers $\wp(\omega_1 b^2)$, $\wp(\omega_1 b^3)$ is transcendental. (It is possible to have both these numbers real and in this case the number $\wp(\omega_1 b^2) + \sqrt{-1}\,\wp(\omega_1 b^3)$ is definitely transcendental). We use the results of Mahler and Schneider to see that b is transcendental.

§ 3. Statement of the Main Theorem. For our later purposes as also for stating our main theorem, it is covenient, from now on, to adopt a fixed notation. It is convenient to begin with

DEFINITION. Let a be any algebraic number and $d(a)$ the least natural number such that $ad(a)$ is an algebraic integer. Then denoting by $\overline{|a|}$ as usual the maximum of the absolute values of the conjugates of a *we define the number* $d(a) + \overline{|a|}$ *as the size of* a and write size a for the size of a. The relation of this notion to the familiar notion of height is not difficult to obtain and details will be given in part II of this paper.

(1) Let $F_1(z) = H_1(z)/G_1(z), \ldots, F_s(z) = H_s(z)/G_s(z)$ be s (>1) algebraically independent (we always mean algebraic independence over the field of complex numbers) meromorphic functions which are quotients of (coprime, i.e. without common zeros) entire functions $H_1(z)$, $G_1(z)$; ...; $H_s(z)$, $G_s(z)$ all of order not exceeding ϱ (finite). Let $M^{(t)}(R)$ ($t = 1, \ldots, s$) denote respectively the quantities

$$M^{(t)}(R) = \Big(1 + \max_{|z|=R} |H_t(z)|\Big)\Big(1 + \max_{|z|=R} |G_t(z)|\Big).$$

(2) Let $\{a_\mu\}$ ($\mu = 1, 2, 3, \ldots$) be a given infinite sequence of distinct complex numbers, arranged in such a way that certain natural numbers n_μ associated with a_μ are non-decreasing. The integers n_μ need not be necessarily distinct and are supposed to have the property that $N(Q)$, the number of numbers a_μ with n_μ not exceeding Q is finite for each $Q = 1, 2, 3, \ldots$ We set $D(Q) = \max_{n_\mu \leqslant Q} |a_\mu|$ and impose the condition

$$\liminf_{Q\to\infty} \frac{\log N(Q)}{\log D(Q)} > \varrho.$$

(3) Let $\{a_{\mu_r}\}$ ($r = 1, 2, 3, \ldots$) be an infinite subsequence of $\{a_\mu\}$ and note that the number $N_1(Q)$ of numbers a_{μ_r} with n_{μ_r} not exceeding Q tends to infinity as Q tends to infinity. This subsequence may be the whole of $\{a_\mu\}$; but we suppose that whenever a polynomial in $F_1(z), \ldots, F_s(z)$ (with complex coefficients) vanishes at all points a_{μ_r} with n_{μ_r} not exceeding Q, it also vanishes at all points a_μ with n_μ not exceeding Q.

(4) Suppose that the numbers $F_t(a_\mu)$ ($t = 1, \ldots, s$; $\mu = 1, 2, 3, \ldots$) are all algebraic numbers (see the remark following the Main Theorem). We denote by $h(Q)$ the degree of the algebraic number field obtained by adjoining the algebraic numbers $F_t(a_\mu)$ ($t = 1, \ldots, s$; $n_\mu \leqslant Q$) to the field of rationals and set

$$M_1^{(t)}(Q) = 1 + \max_{n_\mu \leqslant Q} \{\mathrm{size}(F_t(a_\mu))\} \qquad (t = 1, \ldots, s).$$

Note that $M_1^{(t)}(Q) \geqslant 2$ are non-decreasing functions of Q.

(5) Finally we set

$$M_2^{(t)}(Q) = 1 + \max_{n_\mu \leqslant Q} \frac{1}{|G_t(a_\mu)|} \qquad (t = 1, \ldots, s)$$

and note that $M_2^{(t)}(Q) \geqslant 1$ are non-decreasing functions of Q.

MAIN THEOREM. *Let q be a sufficiently large natural number, and $r_1, \ldots, r_s$ natural numbers related to q asymptotically by*

$$r_1 \ldots r_s \sim h(q)\big(h(q)+1\big)N_1(q).$$

Suppose that the hypotheses (1)-(4) *above are satisfied. Then there exists* *a natural number Q greater than q, such that for every positive quantity R, there holds the inequality*

$$1 \leqslant \Big(\prod_{t=1}^{s} \big(M_1^{(t)}(Q)\big)^{r_t}\Big)^{8h(Q)} \Big(\prod_{t=1}^{s} \big(M_2^{(t)}(Q)\big)^{r_t}\Big) \Big(\prod_{t=1}^{s} \big(M^{(t)}(R)\big)^{r_t}\Big) \Big(\frac{8D(Q)}{R}\Big)^{N(Q-1)}.$$

Remark. The Main Theorem has been proved in a slightly more general form than is required for our purposes since it is possible to do so without complicating the notation or the details. We apply the Main Theorem only in the case where $h(Q)$ is bounded as Q tends to infinity, i.e. when all the numbers $F_t(a_\mu)$ $(t = 1, \ldots, s;\ \mu = 1, 2, 3, \ldots)$ lie in a fixed algebraic number field. It might be mentioned here that if the set $\{a_\mu\}$ has a limit point in a finite part of the plane (for instance when $D(Q)$ is bounded), the restriction that $H_t(z)$, $G_t(z)$ $(t = 1, \ldots, s)$ should be of finite order as also the condition $\liminf\limits_{Q\to\infty} \dfrac{\log N(Q)}{\log D(Q)} > \varrho$ are quite unnecessary; but these conditions are necessary in our applications.

§ 4. Approximation form and the proof of the Main Theorem. We start by stating a most useful arithmetical lemma which is essentially due to Siegel. Our lemma which is a generalization of Siegel's lemma reduces to Siegel's lemma when $h = 1$.

LEMMA. *Suppose that the coefficients of the p linear forms $y_k = a_{k1}x_1 + \ldots + a_{kq_1}x_{q_1}$ $(k = 1, \ldots, p;\ p < q_1)$ are integers in an algebraic number field K of degree h and let $\overline{|a_{kl}|} \leqslant A$. Then there exist rational integers $x_1, \ldots, x_{q_1}$ not all zero satisfying $y_1 = 0, \ldots, y_p = 0$ and such that*

$$|x_k| < 1 + (2q_1A)^{ph(h+1)/[2q_1 - ph(h+1)]} \qquad (h = 1, 2, \ldots, p)$$

provided $2q_1$ exceeds $ph(h+1)$ and $A \geqslant 1$.

Proof. Similar to the proof of Siegel's Lemma 1 (see [6], p. 35). We have to use a rough bound for the number of algebraic integers of degree at most h, not necessarily lying in K but all of whose conjugates do not exceed a given bound. We leave the proof to the reader.

Let q be a sufficiently large natural number and $r_1, \ldots, r_s$ natural numbers whose product is asymptotic to $h(q)\big(h(q)+1\big)N_1(q)$. We set $p = N_1(q)$ and $q_1 = r_1 \ldots r_s$. Our approximation form is

$$R(z) = \sum_{0\leqslant k_1\leqslant r_1-1, \ldots, 0\leqslant k_s\leqslant r_s-1} C_{k_1,\ldots,k_s}\big(F_1(z)\big)^{k_1} \ldots \big(F_s(z)\big)^{k_s}$$

with suitable rational integers $C_{k_1,\ldots,k_s}$ not all zero. We consider the $p = N_1(q)$ linear homogeneous equations

$$R(a_{\mu_r}) = 0 \quad \text{for all } a_{\mu_r} \text{ with } n_{\mu_r} \text{ not exceeding } q,$$

in q_1 (asymptotic to $h(q)\big(h(q)+1\big)p$) unknowns $C_{k_1,\ldots,k_s}$. To make the coefficients algebraic integers in any particular equation we have to multiply the equation by a natural number not exceeding $\big(M_1^{(1)}(q)\big)^{r_1} \ldots \big(M_1^{(s)}(q)\big)^{r_s} = J_1(q)$ say. The absolute values of the conjugates of the algebraic integer coefficients so obtained, do not exceed $\big(J_1(q)\big)^2$. This can be done for each equation. Applying the lemma above we get rational integers $C_{k_1,\ldots,k_s}$ not all zero with

$$|C_{k_1,\ldots,k_s}| < 1 + \big(2r_1 \ldots r_s\big(J_1(q)\big)^2\big)^{ph_1(h_1+1)/[2q_1 - ph_1(h_1+1)]}$$

(where $h_1 = h(q)$ for shortness) such that $R(a_{\mu_r}) = 0$ for all a_{μ_r} with n_{μ_r} not exceeding q. Since q_1 is asymptotic to $h_1(h_1+1)$ the exponent on the right side of the inequality can be replaced by $(1-\varepsilon)^{-1}$ for all $q \geqslant q_0(\varepsilon)$ (we can fix $\varepsilon = \frac{2}{5}$). The sum on the right can be replaced by $2^{1/(1-\varepsilon)}$ times the second term. Using now the trivial inequality $4r_1 \ldots r_s \leqslant 2^{r_1} \ldots 2^{r_s} \leqslant J_1(q)$ valid for all large q since $M_1^{(t)}(q) \geqslant 2$ for $t = 1, \ldots, s$, we have

$$|C_{k_1,\ldots,k_s}| \leqslant \big(J_1(q)\big)^{3/(1-\varepsilon)}, \qquad q \geqslant q_0(\varepsilon).$$

By the hypothesis (3) of the Main Theorem the stronger conclusion

$$R(a_\mu) = 0 \quad \text{for all } a_\mu \text{ with } n_\mu \text{ not exceeding } q,$$

is also valid. Now because of the algebraic independence of the functions $F_1(z), \ldots, F_s(z)$ the approximation form $R(z)$ is not identically zero and moreover if $G(z) = \prod\limits_{t=1}^{s}\big(G_t(z)\big)^{r_t}$, the entire function $R(z)G(z)$ is of order not exceeding ϱ. If $R(z)$ vanishes on the entire sequence $\{a_\mu\}$ so does $R(z)G(z)$, and by the definition of $D(Q)$ it would mean that $\liminf\limits_{Q\to\infty} \dfrac{\log N(Q)}{\log D(Q)}$ does not exceed ϱ. But this contradicts the hypothesis (2) of the Main Theorem. Hence there exist points a_j in $\{a_\mu\}$ for which $R(a_j)$ is different from zero. We choose one such point with least possible n_j, say $n_j = Q$ (*naturally* $Q > q$). Thus $\gamma = R(a_j) \neq 0$ and $R(a_l) = 0$ for all a_l with $n_l < Q$.

By our hypothesis (4) it follows that for some natural number m not exceeding $J_1(Q)$, $m\gamma$ is an algebraic integer of degree at most $h(Q)$ and so

$$N(\gamma) \geqslant \big(J_1(Q)\big)^{-h_2} \quad \text{where} \quad h_2 = h(Q).$$

Also

$$\overline{|\gamma|} \leqslant r_1 \ldots r_s\big(J_1(q)\big)^{3/(1-\varepsilon)} J_1(Q) \leqslant \big(J_1(Q)\big)^{2+3/(1-\varepsilon)}.$$

Finally by integrating on the circle $|z| = R$ with $R \geqslant 2D(Q)$,

$$\gamma = \frac{1}{G(a_j)} \cdot \frac{1}{2\pi i} \int R(z)G(z) \prod_{n_l<Q} \Big(\frac{a_j - a_l}{z - a_l}\Big) \frac{dz}{z - a_j}.$$

Here

$$\frac{1}{|G(a_j)|} \leqslant J_2(Q) = \prod_{t=1}^{s} \left(M_2^{(t)}(Q)\right)^{r_t},$$

$$|R(z)G(z)|_{|z|=R} \leqslant r_1 \ldots r_s \left(J_1(q)\right)^{3/(1-\varepsilon)} J_3(R) \leqslant \left(J_1(Q)\right)^{1+3/(1-\varepsilon)} J_3(R),$$

where

$$J_3(R) = \prod_{t=1}^{s} \left(M^{(t)}(R)\right)^{r_t},$$

$$\left|\prod_{n_l<Q} \left(\frac{a_j-a_l}{z-a_l}\right)\right| \leqslant \left(\frac{2D(Q)}{R-D(Q)}\right)^{N(Q-1)} \leqslant \left(\frac{4D(Q)}{R}\right)^{N(Q-1)},$$

$$\frac{1}{|z-a_j|} \leqslant \frac{1}{R-D(Q)} \leqslant \frac{2}{R}$$

and so

$$|\gamma| \leqslant J_2(Q)\cdot\frac{1}{2\pi}\cdot 2\pi R\left(J_1(Q)\right)^{1+3/(1-\varepsilon)} J_3(R)\cdot\frac{2}{R}\left(\frac{4D(Q)}{R}\right)^{N(Q-1)}$$

$$\leqslant \left(J_1(Q)\right)^{1+3/(1-\varepsilon)} J_2(Q) J_3(R) \left(\frac{8D(Q)}{R}\right)^{N(Q-1)}.$$

Combining all our estimates for γ, we get

$$1 \leqslant \left(J_1(Q)\right)^{h_2+(h_2-1)[2+3/(1-\varepsilon)]+1+3/(1-\varepsilon)} J_2(Q) J_3(R) \left(\frac{8D(Q)}{R}\right)^{N(Q-1)}$$

and since the exponent of $J_1(Q)$ on the right does not exceed $h_2[3+ +3/(1-\varepsilon)] = 8h_2 = 8h(Q)$, for $\varepsilon = \frac{2}{5}$, this proves the validity of the inequality of the Main Theorem for $R \geqslant 2D(Q)$ and the inequality is trivial for $R \leqslant 8D(Q)$. This completes the proof of the Main Theorem.

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Contributions to the theory of transcendental numbers (II)

by

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To the 60-th birthday of A. O. Gelfond

§ 1. A special case of the Main Theorem. In this section we apply a special case of the Main Theorem of the earlier paper I ([9], p. 69) to deduce Theorem 1 below which will be the only theorem to which we shall refer in the later sections of this paper. We begin with some definitions (incidentally we also recall the notation). From now on we deal only with meromorphic functions which are quotients of entire functions of finite order. Given s ($\geqslant 2$) algebraically independent meromorphic functions $F_1(z), \ldots, F_s(z)$ we introduce with respect to these

Definition 1. A *weighted sequence* S (often we write $\{a_\mu\}$ for S) is an infinite sequence $\{a_\mu\}$ ($\mu = 1, 2, \ldots$) of distinct complex numbers together with an infinite subsequence $\{a_{\mu_r}\}$ ($r = 1, 2, 3, \ldots$) (which may be the same as $\{a_\mu\}$) and an infinite sequence $\{n_\mu\}$ ($\mu = 1, 2, 3, \ldots$) of natural numbers not necessarily distinct satisfying the following conditions.

(i) The sequence $\{n_\mu\}$ is non-decreasing.

(ii) For each $Q = 1, 2, 3, \ldots$ there are only finitely many $\{a_\mu\}$ for which n_μ does not exceed Q. We denote this number by $N(Q)$. It follows that there are only finitely many numbers a_{μ_r} for which n_{μ_r} does not exceed Q and this number $N_1(Q)$ does not exceed $N(Q)$.

(iii) The limits

$$\delta = \lim_{Q\to\infty} \frac{\log N(Q)}{\log Q} \quad \text{and} \quad \delta_1 = \lim_{Q\to\infty} \frac{\log N_1(Q)}{\log Q}$$

exist and are finite.

(iv) The upper limit

$$\limsup_{Q\to\infty} \left(\frac{1}{Q} \max_{n_\mu \leqslant Q} |a_\mu|\right)$$

is finite.

(v) Whenever a polynomial in $F_1(z), \ldots, F_s(z)$ with complex coefficients vanishes for all values $z = a_{\mu_r}$ with $n_{\mu_r} \leqslant Q$, it also vanishes for all values $z = a_\mu$ with $n_\mu \leqslant Q$.