Some concluding remarks might not be amiss. It is easily seen that Theorem 1 and as a result its logical corollary Theorem 2, can be relaxed to include forms Q with rational coefficients, the only proviso being that the denominators of the coefficients are relatively prime to k. In particular Theorems 1 and 2 hold for Q and χ replaced by $\bar{Q}$ and $\bar{\chi}'$. In fact since $\bar{\bar{\chi}}' = \chi$ and the discriminant of $\bar{Q}$ is d^{n-1} which is relatively prime to k if and only if $(d, k) = 1$, under the original hypotheses of Theorems 1 and 2 the conclusions are valid not only as stated but for Q, χ and a replaced by $\bar{Q}$, $\bar{\chi}'$ and $a_{\bar{Q}}$ as well ($a_{\bar{Q}}$ is defined from (7) and (8) except that d is replaced by d^{n-1} and χ is replaced by $\bar{\chi}'$). Since $\bar{\bar{Q}} = d^{n-2}Q$ and the discriminant of $\bar{\bar{Q}}$ is $d^{(n-1)^2}$, Theorem 2 for $\bar{Q}$ yields

$$\left(\frac{kd^{(n-1)/n}}{2\pi}\right)^s \Gamma(s) L(s, \bar{\chi}', \bar{Q}) = a_{\bar{Q}} \left(\frac{kd^{(n-1)^2/n}}{2\pi}\right)^{n/2-s} \Gamma\left(\frac{n}{2}-s\right) L\left(\frac{n}{2}-s,\ \chi,\ d^{n-2}Q\right)$$

$$= a_{\bar{Q}}\,\chi(d^{n-2}) \left(\frac{kd^{1/n}}{2\pi}\right)^{n/2-s} \Gamma\left(\frac{n}{2}-s\right) L\left(\frac{n}{2}-s,\ \chi, Q\right).$$

If we compare this with Theorem 2 with s replaced by $n/2-s$ we see that

$$\bar{a} = a_{\bar{Q}}\,\chi(d^{n-2}). \tag{43}$$

There is unfortunately no new information in (43) although it takes considerable algebraic manipulation to prove (43) directly from (7) and (8).

There is always more than one way to derive a functional equation. Theorem 2 can be easily derived from Theorem 1 and the functional equation for the general Epstein zeta function in much the same way that the functional equation of Dirichlet's L-function is derived from Lemma 1 and the functional equation of the Hurwitz zeta function.

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On certain additive functions (II)

by

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A function $f(n)$ which is defined on the set of positive integers is said to be *additive* if for any coprime integers a, b, we have the relation

$$f(ab) = f(a)+f(b).$$

Let $0 < a_1 < a_2 \ldots$ be a set of integers, and let $A(x)$ denote the number of these not exceeding x. It was shown recently [4], that if $A(x)$ is not too small, then in the usual terminology $f(a_i)$ has a normal value. However, in order to prove this result a weak but inconvenient condition was introduced. It is our present purpose to show that this condition can be removed. More especially we prove the following:

THEOREM 1. *For any irreducible polynomial $g(y)$ with integer coefficients, and any integer u, we define $\varrho(u)$ to be the number of residue classes r for which $g(r) = 0 (\operatorname{mod} u)$.*

Let $f(n)$ be an additive function assuming only non-negative values, and for any positive value of x let $\mu_x = \max f(p^a)$ taken over the prime powers not exceeding x, and

$$S_x = \sum_{p^a \leqslant x} \varrho(p^a) f(p^a) p^{-a}.$$

Then if $A(x) > x\exp\left(-\varepsilon(x)\mu_x^{-1}S_x\right)$ for some function $\varepsilon(x)$ which tends to zero as $x\to\infty$, whilst $\mu_x = o(S_x)$, we have the asymptotic relations:

$$\sum_{a_i \leqslant x} f^k\big(g(a_i)\big) \sim A(x) S_x^k, \qquad k = 1, 2, \ldots \tag{1}$$

COROLLARY. *$f\big(g(a_i)\big)$ is normally S_{a_i}.*

We first show that the corollary is satisfied.

It is clear from the theorem that we have

$$\sum_{a_i \leqslant x} \big(f(g(a_i)) - S_x\big)^2 = o\big(A(x) S_x^2\big),$$

and in order to prove the corollary it will be enough to show that

$$\sum_{a_i \leqslant x} (S_x - S_{a_i})^2 = o\big(A(x) S_x^2\big).$$

We do this by dividing the range of summation into two parts.

For the first range we take $0 < a_i \leqslant a = A(x)/\log x$. Over this interval we obtain a contribution which does not exceed

$$S_x^2 \sum_{a_i \leqslant a} 1 \leqslant A(x) S_x^2/\log x.$$

For the remaining values of a_i, it is plain that

$$S_x - S_{a_i} \ll \mu_x \sum_{a < p \leqslant x} \frac{1}{p} \ll \mu_x \left\{\log\left(\frac{\log x}{\log a}\right) + O\left(\frac{1}{\log a}\right)\right\}.$$

In view of the lower bound on $A(x)$, it is easily checked that

$$\log x/\log a = 1 + O\big(\{\log\log x + \mu_x^{-1} S_x\}/\log x\big),$$

so that

$$\sum_{a < a_i \leqslant x} (S_x - S_{a_i})^2 \ll \mu_x^2 A(x) \left\{\left(\frac{\log\log x}{\log x}\right)^2 + \left(\frac{S_x}{\mu_x \sqrt{\log x}}\right)^2\right\}.$$

Putting our results together we obtain the desired inequality. Here we have used the fact [5] that $\varrho(p^a) \ll 1$ holds uniformly for all prime-powers. It is clear that the above result holds even if we weaken our condition on μ_x to $\mu_x \ll S_x$.

Before we begin our proof we show that we may assume that $f(n)$ behaves like a bounded strongly additive function.

Firstly we note, as in [4], that we may assume that $f(n)$ satisfies $f(p^a) < \frac{1}{2}$. This can be easily seen if we use instead of $f(n)$ the additive function $f^*(n)$ defined by

$$f^*(p^a) = \begin{cases} f(p^a)/2\mu_x & \text{if} \quad p^a \leqslant x, \\ 0 & \text{otherwise}. \end{cases}$$

Secondly, we consider the additive function $\bar{f}(n)$ which is defined by

$$\bar{f}(p^a) = f(p) \quad \text{for each value of } a.$$

Clearly:

$$\sum_{p^a \leqslant x} \bar{f}(p^a) \varrho(p^a) p^{-a} = \sum_{p \leqslant x} f(p) \varrho(p) p^{-1} + O(\mu_x)$$
$$= \sum_{p^a \leqslant x} f(p^a) \varrho(p^a) p^{-a} + O(\mu_x) \sim S_x,$$

and thus, if we can show that for any positive integer k the following result is satisfied:

$$(2) \qquad \sum_{a_i \leqslant x} \{f(g(a_i)) - \bar{f}(g(a_i))\}^{2k} = o\big(A(x) S_x^{2k}\big),$$

we can obtain that

$$\sum_{a_i \leqslant x} \{f(g(a_i))\}^k = \sum_{a_i \leqslant x} \{\bar{f}(g(a_i))\}^k + o\big(A(x) S_x^k\big).$$

It is clear from this that it will then suffice to prove Theorem 1 with $\bar{f}(n)$ in place of $f(n)$.

In order to prove (2) we first recall the following

LEMMA 1. *Assuming only that $f(n)$ is non-negative, we have for any positive integer k the inequality*

$$\sum_{n \leqslant x} \Big(\sum_{\substack{p^a \| g(n) \\ p^{a(k+1)} \leqslant x}} f(p^a) \Big)^k \leqslant x (S_x + k\mu_x)^k.$$

Proof. This is an immediate consequence of Lemma 7 of [4]. We apply this result to prove some further lemmas.

LEMMA 2. *Let $\mu_x \ll S_x$ and $A(x) > x\exp(-c_1 S_x/\mu_x)$ be satisfied for some positive constant c_1, and all large x. Then for each $k > 0$, there is a further constant c_2, depending upon c_1 and k, so that*

$$\sum_{a_i \leqslant x} \{f(g(a_i))\}^k \leqslant c_2 A(x) S_x^k.$$

Proof. We apply Lemma 1 with $f(p^a)$ replaced by $\lambda(p^a)$ where

$$\lambda(p^a) = \begin{cases} f(p^a) & \text{if} \quad p^a \leqslant \exp\big(\{\varepsilon\mu_x \log x\}/S_x\big), \\ 0 & \text{otherwise}, \end{cases}$$

and ε is a small positive constant. If now s is any integer satisfying $(s+1)\varepsilon\mu_x S_x^{-1} \leqslant 1$, then

$$\sum_{n \leqslant x} \{\lambda(g(n))\}^s \leqslant x (S_x + s\mu_x)^s.$$

In particular, if $s = k\left[\frac{1}{k}\left(\frac{S_x}{\varepsilon\mu_x} - 1\right)\right]$, which is certainly positive if ε is small enough, we may apply Hölder's inequality to obtain

$$\Big(\sum_{a_i \leqslant x} \{\lambda(g(a_i))\}^k\Big)^{s/k} \leqslant \{A(x)\}^{s/k-1} \sum_{n \leqslant x} \{\lambda(g(n))\}^s$$
$$\leqslant x \{A(x)\}^{s/k-1} (S_x + s\mu_x)^s.$$

Hence

$$\sum_{a_i \leqslant x} \{\lambda(g(a_i))\}^k \leqslant A(x)(S_x + s\mu_x)^k \{x/A(x)\}^{k/s},$$

and

$$\{x/A(x)\}^{k/s} \leqslant \exp\left(\frac{k}{s} c_1 \frac{S_x}{\mu_x}\right) \leqslant \exp\left(c_3 \frac{\varepsilon\mu_x}{S_x} \cdot \frac{S_x}{\mu_x}\right) = c_4;$$

whilst

$$\mu_x s \leqslant \varepsilon^{-1} S_x.$$

By these inequalities

(3) $$\sum_{a_i \leqslant x} \{\lambda(g(a_i))\}^k \leqslant c_5 A(x) S_x^k.$$

Finally

$$|f(g(a_i)) - \lambda(g(a_i))| \leqslant \mu_x S_x / \varepsilon\mu_x,$$

so that

(4) $$\sum_{a_i \leqslant x} |\{f(g(a_i)) - \lambda(g(a_i))\}^k| \leqslant c_6 A(x) S_x^k.$$

The lemma now follows easily from the inequalities (3) and (4).

LEMMA 3. *If the conditions of Theorem 1 are satisfied, and $f'(n)$ is any additive function satisfying $f'(n) \leqslant f(n)$ for all integers n, then in an obvious notation*

$$\sum_{a_i \leqslant x} \{f'(g(a_i))\}^k \leqslant A(x)(S'_x + o(S_x))^k.$$

Proof. The proof follows the lines of Lemma 2 save that we replace ε in the argument by a function $\eta(x)$ which $\to \infty$ with x, but is chosen so that $\eta(x)\varepsilon(x) = o(1)$.

Applying this lemma to (2), with $f'(n) = f(n) - \bar{f}(n)$, we see that under the conditions of Theorem 1,

$$\sum_{a_i \leqslant x} \{f(g(a_i)) - \bar{f}(g(a_i))\}^{2k} \leqslant A(x)\Big\{\sum_{\substack{p^a \leqslant x \\ a \geqslant 2}} \varrho(p^a)|f(p^a) - \bar{f}(p^a)| p^{-a} + o(S_x)\Big\}^{2k}.$$

Since

$$\sum_{\substack{p^a \leqslant x \\ a \geqslant 2}} \varrho(p^a)|f(p^a) - \bar{f}(p^a)| \leqslant c_1 \mu_x \sum_p p^{-2} = o(S_x),$$

we see that the desired result (2) is satisfied. Thus both of our assumptions are justified and we shall make both of them from now on, but we shall preserve our original notation. The condition on S_x then becomes $S_x \to \infty$, and that on $A(x)$ becomes $A(x) > x\exp(-\varepsilon(x) S_x)$.

We need some further lemmas. For convenience we shall denote $\log x/\log y$ when $2 \leqslant y < x$ by D, so that $D > 1$. For any integer m we use $(m)_y$ to denote the product of those prime divisors of m, without multiplicity, which do not exceed y.

LEMMA 4. *Let $\sum'$ denote that the integers n in the sum run over those for which $(g(n))_y > x^{1/2}$. Then we have the estimation*

$$\sum_{n \leqslant x}{}' 1 \leqslant c_1 x \exp(-c_2 D \log D),$$

provided only that

$$(\log x)^{64} \leqslant y^8 < x.$$

Proof. This result can be proved on the lines of Khmirova [6], we do not give the details of the changes which are simple. More exactly one can show that if $u > (\log x)^{32}$ and $u > y^4 \geqslant (\log u)^{32}$, then the number of $n \leqslant x$ for which $(g(n))_y > u$ is at most

$$c_3 x \exp\left(-c_4 \frac{\log u}{\log y} \log\left(\frac{\log u}{\log y}\right)\right).$$

The lemma now follows if we take $u = x^{1/2}$. The inequalities restricting the values of y are needlessly strong, but are sufficient for our purposes. We note that for the present it would be enough to use a weaker form of Lemma 4 which does not have the factor $\log D$, and with merely the condition $D > c_5$ to be satisfied by y. This particular result follows from a more general result of the same type proved as in Barban [3]. The above result will however enable us to sharpen some interesting inequalities.

We now define two new functions. We denote by $f_1(n)$ the additive function defined by

$$f_1(p^a) = \begin{cases} f(p), & \text{if } p \leqslant y, \\ 0 & \text{otherwise.} \end{cases}$$

We then define a multiplicative function $H(\nu)$ by

$$H(p^a) = z^{f_1(p^a)} - z^{f_1(p^{a-1})}, \quad a = 1, 2, \ldots,$$

so that

$$\sum_{\nu | n} H(\nu) = z^{f_1(n)}.$$

Thus $H(\nu)$ is the Möbius inverse of the function $z^{f_1(n)}$.

LEMMA 5. *Let $t(n)$ be a multiplicative function satisfying $t(p^a) \ll a^m$ for some fixed constant $m > 0$, uniformly for all prime powers. Furthermore let $t(p^a) \geqslant t(p^{a-1})$, $a = 1, 2, \ldots$ hold. Then we have the inequality*

$$\sum_{m \leqslant x} t(g(m)) \leqslant c_1 x \exp\left(\sum_{p \leqslant x} \frac{\varrho(p)}{p}\{t(p) - 1\}\right).$$

Proof. This result is a particular case of a result proved by Barban [2]. The conditions $t(p^a) \geqslant t(p^{a-1})$ ensure that the Möbius inverse of $t(n)$ assumes only non-negative values. In particular we need that $t(p) \geqslant 1$.

LEMMA 6. *If c_1, c_2 are positive constants, and $c_1 < z < 1+c_2$, and y satisfies the inequalities of Lemma 4, then*

$$\sum_{\nu > x^{1/2}} |H(\nu)|\, \varrho(\nu)\nu^{-1} \leqslant c_3 \exp\left(c_4|z-1|S_y - \tfrac{1}{3}D\log D\right).$$

Proof. For convenience we denote $\log D/\log y$ by a. Then the sum which we wish to estimate does not exceed

$$x^{-a/2}\sum_{\nu=1}^{\infty} |H(\nu)|\, \varrho(\nu)\nu^{-1+a}.$$

Now

$$\sum_{\nu=1}^{\infty} |H(\nu)|\, \varrho(\nu)\nu^{-1+a} = \prod_{p\leqslant y}\left(1+|H(p)|\, \varrho(p)p^{-1+a}\right) \leqslant \exp\Big(c_5\sum_{p\leqslant y}|H(p)|p^{-1+a}\Big).$$

Let $y' = \exp(\log y/\log D)$. We divide the sum over primes $p \leqslant y$, into two parts. Into the first part we put those primes p which do not exceed y'. These then contribute

$$\sum_{p\leqslant y'} |H(p)|p^{-1}\exp(a\log p) \leqslant e\sum_{p\leqslant y'}|H(p)|p^{-1},$$

since

$$a\log p \leqslant \frac{\log D}{\log y}\log y' = 1.$$

Moreover, in our range for z,

$$|H(p)| = \left|\int_1^z f_1(p)y^{f_1(p)-1}\,dy\right| \leqslant c_6 f_1(p)\,|z-1|.$$

Hence

$$\sum_{p\leqslant y'} |H(p)|p^{-1+a} \leqslant c_7 e\,|z-1|\sum_{p\leqslant y'} f_1(p)p^{-1}.$$

For the second sum we have $y' < p \leqslant y$, so that

$$\sum_{y'<p\leqslant y} |H(p)|p^{-1+a} \leqslant c_7\,|z-1|\sum_{y'<p\leqslant y} p^{-1+a}.$$

Now here

$$p^a = \exp(a\log p) \leqslant \exp\left(\frac{\log D}{\log y}\log y\right) = D,$$

and

$$\sum_{y'<p\leqslant y} p^{-1} \leqslant 1+\log\left(\frac{\log y}{\log y'}\right) \leqslant \log\log D,$$

so that altogether

$$\sum_{p\leqslant y} |H(p)|\, \varrho(p)p^{-1+a} \leqslant c_8\{|z-1|\,S_y + D\log\log D\}.$$

Finally

$$x^{-a/2} = \exp\left(-\tfrac{1}{2}D\log D\right),$$

and thus for large values of D,

$$\sum_{\nu > x^{1/2}} |H(\nu)|\, \varrho(\nu)\nu^{-1} \leqslant c_9\exp\left(c_4|z-1|\,S_y - \tfrac{1}{3}D\log D\right),$$

as required.

LEMMA 7. *If c_1, c_2 are positive constants, and $c_1 < z < 1+c_2$, then for a value of y satisfying the conditions in Lemma 4, we have the inequality*

$$\sum_{n\leqslant x} z^{f_1(g(n))} \leqslant c_3 x\exp\left((z-1)S_y\right) + c_4 x\exp\left(c_5|z-1|\,S_y - c_6 D\log D\right).$$

Proof. We first notice that if $z > 1$ then $z^{f_1(g(n))}$ satisfies the conditions of Lemma 5, since if $k > 1$ or $p > y$,

$$z^{f_1(p^k)} = 1 = z^{f_1(p^{k-1})},$$

whilst if $p \leqslant y$

$$z^{f_1(p)} \geqslant 1.$$

Thus the result stated in the present lemma will follow provided we can show that

$$\sum_{p\leqslant y} (z^{f_1(p)}-1)\, \varrho(p)p^{-1} \leqslant (z-1)\,S_y + O(1).$$

By applying a well-known mean-value theorem we see that the left-hand side here is

$$\sum_{p\leqslant y} (z-1)f_1(p)\, \varrho(p)p^{-1} + \tfrac{1}{2}f_1(p)\{f_1(p)-1\}\, \varrho(p)p^{-1}\eta^2,$$

for some value of η satisfying $0 < \eta < z$. Since $f_1(p) < 1$ we obtain the desired inequality.

We shall therefore assume for the rest of this lemma that $z \leqslant 1$ holds.

Let $N = N(x, y)$ denote the set of integers $n \leqslant x$ for which $(g(n))_y \leqslant x^{1/2}$, and let

$$T = \sum_{\substack{n\leqslant x\\ n\in N}} z^{f_1(g(n))}.$$

Now

$$T = \sum_{\nu \leqslant x^{1/2}} H(\nu) \sum_{\substack{m \leqslant x, g(m) \equiv 0 (\mathrm{mod}\, \nu) \\ m \in N}} 1$$

for $H(\nu) = 0$ unless the primes dividing $g(m)$ do not exceed y, and since m belongs to N we must then have that $\nu \leqslant x^{1/2}$. Thus the inner sum is

$$\frac{x}{\nu} + \theta \varrho(\nu) - \sum_{\substack{m \leqslant x, g(m) \equiv 0 (\mathrm{mod}\, \nu) \\ m \notin N}} 1, \quad |\theta| \leqslant 1.$$

Hence

$$T = x \sum_{\nu \leqslant x^{1/2}} \frac{\varrho(\nu)}{\nu} H(\nu) + O\Big(\sum_{\nu \leqslant x^{1/2}} \varrho(\nu) c^{\omega(\nu)}\Big) - \sum_{\substack{m \leqslant x \\ m \notin N}} \sum_{\substack{\nu | g(m) \\ \nu \leqslant x^{1/2}}} H(\nu),$$

where $\omega(\nu)$ denotes the number of distinct prime divisors of ν. Thus we see that

$$\sum_{n \leqslant x} z^{f_1(g(n))} = x \sum_{\nu \leqslant x^{1/2}} \frac{\varrho(\nu)}{\nu} H(\nu) + O\Big(\sum_{\nu \leqslant x^{1/2}} \varrho(\nu) c^{\omega(\nu)}\Big) - $$
$$- \sum_{\substack{m \leqslant x \\ m \notin N}} \sum_{\substack{\nu | g(m) \\ \nu \leqslant x^{1/2}}} H(\nu) + \sum_{\substack{n \leqslant x \\ n \notin N}} z^{f_1(g(n))},$$
$$= \Sigma_1 + \Sigma_2 - \Sigma_3 + \Sigma_4, \text{ say}.$$

We estimate these sums in turn.

By applying Lemma 6 we can extend the range of summation in Σ_1 so that

$$\Big|\Sigma_1 - x \sum_{\nu=1}^{\infty} \frac{\varrho(\nu)}{\nu} H(\nu)\Big| \leqslant c_7 \exp(c_8 |z-1| S_y - \tfrac{1}{3} D \log D).$$

Moreover

$$\sum_{\nu=1}^{\infty} \frac{\varrho(\nu)}{\nu} H(\nu) = \prod_{p \leqslant y} \Big(1 + \frac{\varrho(p) H(p)}{p}\Big) \leqslant c_9 \exp\Big(\sum_{p \leqslant y} \frac{\varrho(p)}{p} H(p)\Big).$$

We can then estimate this final product as in the above note.

Clearly, for any $\varepsilon > 0$, $\Sigma_2 \ll x^{1/2+\varepsilon}$.

To estimate Σ_3 we first apply the Cauchy-Schwartz inequality,

$$\Sigma_3^2 \leqslant \sum_{\substack{m \leqslant x \\ m \notin N}} 1 \sum_{m \leqslant x} \Big(\sum_{\substack{\nu | g(m) \\ \nu \leqslant x^{1/2}}} |H(\nu)|\Big)^2.$$

Now

$$\sum_{m \leqslant x} \Big(\sum_{\substack{\nu | g(m) \\ \nu \leqslant x^{1/2}}} |H(\nu)|\Big)^2 = \sum_{\nu_1, \nu_2 \leqslant x^{1/2}} \sum |H(\nu_1)| |H(\nu_2)| \sum_{\substack{m \leqslant x \\ g(m) \equiv 0 (\mathrm{mod} [\nu_1, \nu_2])}} 1$$

where $\lambda = [\nu_1, \nu_2]$ denotes the least common multiple of ν_1 and ν_2. Clearly $[\nu_1, \nu_2] \leqslant x$, so that the innermost sum here is at most $x \varrho(\lambda) \lambda^{-1}$. Collecting together terms for which (ν_1, ν_2) the highest common factor of ν_1 and ν_2 has a particular value r, we see that this sum does not exceed

$$x \sum_{t \leqslant x^{1/2}} t \varrho(t) \Big(\frac{H(t)}{t}\Big)^2 \sum_{\substack{s_1, s_2 \leqslant t^{-1} x^{1/2} \\ (s_1, s_2) = 1}} \sum \varrho(s_1 s_2) s_1^{-1} s_2^{-1} |H(s_1)| |H(s_2)|.$$

For, since $H(u) = 0$ unless u is squarefree, the functions considered here are for our purposes completely multiplicative. Clearly the above sum does not exceed

$$x \sum_{t=1}^{\infty} \varrho(t) H^2(t) t^{-1} \Big(\sum_{s=1}^{\infty} \varrho(s) |H(s)| s^{-1}\Big)^2.$$

Before proceeding we note that even if we allow z to assume complex values, then

$$H(p) = f_1(p) \int_1^z \zeta^{f_1(p)-1} d\zeta$$

where the integral is taken over the straight line segment L joining 1 and z. It follows from this representation that

$$|H(p)| \leqslant f_1(p) |z-1| \operatorname*{Max}_L |\zeta|^{f_1(p)-1} \leqslant f_1(p) \mathrm{Max}\Big(\Big|1 - \frac{1}{z}\Big|, |1-z|\Big).$$

Similarly we see that

$$|H(p)|^2 \leqslant f_1(p) \Big(\mathrm{Max}\Big\{\Big|1 - \frac{1}{z}\Big|, |1-z|\Big\}\Big)^2.$$

Putting these estimates into our above considerations we see that

$$\Sigma_3^2 \leqslant \sum_{\substack{m \leqslant x \\ m \notin N}} c_{10} \exp(c_{11} |z-1| S_y).$$

If we write B for the sum in the statement of the lemma which we wish to estimate, then we now have the inequality

$$B \leqslant E + c_{12} \exp(c_{13} |z-1| S_y) \Big(\sum_{\substack{m \leqslant x \\ m \notin N}} 1\Big)^{1/2} + \sum_{\substack{n \leqslant x \\ n \notin N}} z^{f_1(g(n))},$$

with

$$E = c_9 x \exp\big((z-1)S_y\big) + c_{14} x \exp(c_{15}|z-1|S_y - c_{16} D\log D).$$

But once again appealing to the inequality of Cauchy-Schwartz,

$$\Big(\sum_{\substack{n\leqslant x\\ n\nmid N}} z^{f_1(g(n))}\Big)^2 \leqslant \sum_{\substack{m\leqslant x\\ m\nmid N}} 1 \sum_{n\leqslant x} z^{2f_1(g(n))} \leqslant B \sum_{\substack{m\leqslant x\\ m\nmid N}} 1.$$

We can therefore write

$$\Big\{\sqrt{B} - \frac{1}{2}\Big(\sum_{\substack{m\leqslant x\\ m\nmid N}} 1\Big)^{1/2}\Big\}^2 \leqslant E + \frac{1}{4}\sum_{\substack{m\leqslant x\\ m\nmid N}} 1,$$

so that

$$B \leqslant \Big(\sqrt{E} + \Big(\sum_{\substack{m\leqslant x\\ m\nmid N}} 1\Big)^{1/2}\Big)^2 \leqslant 2E + 2\sum_{\substack{m\leqslant x\\ m\nmid N}} 1.$$

Using our above estimate for E, and applying Lemma 4 to give an upper bound for the final sum on the right-hand we obtain the result stated in the lemma.

In view of one of our above remarks we can consider the case when z may assume complex values. If we put $z = e^{\zeta}$ for a complex number ζ, we state the following lemma which is a corollary of the above result.

LEMMA 8. *If $|\zeta| \leqslant c_1$ for a constant $c_1 > 0$, and if y satisfies the conditions of Lemma 4, whilst $S_y < \delta D\log D$ holds for a sufficiently small value of δ depending upon c_1 only, then*

$$\sum_{m\leqslant x} e^{\zeta f_1(g(m))} = x\sum_{\nu=1}^{\infty} \varrho(\nu)H(\nu)\nu^{-1} + O\big(x\exp(-c_2 D\log D)\big).$$

Here we understand that $H(\nu)$ is defined as before but with e^{ζ} in place of z. The constant c_2 may well depend upon the polynomial $g(m)$.

If it is required we can then express the infinite sum in the form

$$F(\zeta)\exp\Big(\sum_{p\leqslant y} p^{-1}\varrho(p)\{e^{\zeta f_1(p)} - 1\}\Big),$$

where, for any fixed value of x, $F(\zeta)$ is an integral function of ζ, and is uniformly bounded for all values of in $|\zeta| \leqslant c_1$.

We can clearly extend this result to cover sequences of the Σ_R-type as considered by Barban [2].

We can now continue with our lemmas for the proof of our theorem.

We set $y = \exp\big(\log x/\{\sqrt{\varepsilon(x)S_x}\}\big)$ from now until the end of our theorems.

This clearly satisfies the conditions of Lemma 4, for if x is large enough $\sqrt{\varepsilon(x)S_x} > 8$ so that $y^8 < x$, whilst $\sqrt{\varepsilon(x)S_x}$ does not exceed $\log\log x + c_1$ so that

$$y > \exp(c_2\log x/\log\log x) > (\log x)^{64}.$$

LEMMA 9. *Let k be a positive integer. Then the following inequality is satisfied:*

$$\sum_{|f_1(g(n)) - S_y| > \{\varepsilon(x)\}^{1/3}S_y} \big(f_1(g(n)) - S_y\big)^{2k} \leqslant c_1 x\exp\big(-\{\varepsilon(x)\}^{2/3}S_y\big).$$

Proof. For any positive integer s and any value of $\varrho > 0$, we have the relation

$$\frac{1}{(2k+2s)!}\sum_{n\leqslant x}\big(f_1(g(n)) - S_y\big)^{2(k+s)} = \frac{1}{2\pi i}\int_{|\zeta|=\varrho}\sum_{m\leqslant x} e^{\zeta(f_1(g(m)) - S_y)}\zeta^{-(2k+2s+1)}\,d\zeta.$$

Now if $|\zeta| \leqslant c_2$, Lemma 7 shows that we can write

$$\sum_{n\leqslant x} e^{\zeta(f_1(g(n)) - S_y)} = h_1(\zeta) + h_2(\zeta),$$

where if $2\varrho < 1$ and $|\zeta| = \varrho$,

$$|h_1(\zeta)| \leqslant c_2 x\exp(\mathrm{Re}\{e^{\zeta} - 1 - \zeta\}S_y) \leqslant c_2 x\exp(c_3\varrho^2 S_y),$$

whilst

$$|h_2(\zeta)| \leqslant c_4 x\exp(c_5\varrho S_y - c_6 D\log D).$$

Corresponding to these expressions we split the integral in the above representation into two parts.

The integral

$$\frac{(2k+2s)!}{\big(\varepsilon(x)^{1/3}S_y\big)^{2s}}\cdot\frac{1}{2\pi i}\int_{|\zeta|=\varrho} h_1(\zeta)\,\zeta^{-(2k+2s+1)}\,d\zeta$$

can be estimated as in Lemma 5 of [4]. Hence we obtain, as there, if $(4\varrho)^4 = \varepsilon(x)$ and $s = -k + [\frac{1}{2}\varrho\varepsilon(x)^{1/3}S_y]$, the upper bound

$$x\exp\big(-c_1\varepsilon(x)^{2/3}S_y\big).$$

During the course of the proof of that lemma certain assumptions were made, namely that $\varepsilon(x)$ did not tend to zero too rapidly. We can clearly make any such assumptions here since this merely weakens our current condition:

$$A(x) > x\exp\big(-\varepsilon(x)S_x\big),$$

to a similar one with a new function $\varepsilon'(x)$. This effects none of our results since we need only that $\varepsilon'(x) \to 0$ as $x \to \infty$.

Secondly we can apply the inequality $a! \leqslant a^a$ to estimate the integral

$$\frac{(2k+2s)!}{\big(\varepsilon(x)^{1/3}S_y\big)^{2s}}\cdot\frac{1}{2\pi i}\int\limits_{|\zeta|=\varrho} h_2(\zeta)\zeta^{-(2k+2s+1)}\,d\zeta.$$

We then obtain the upper bound:

$$c_8 x\exp\Big(\varepsilon(x)^{2/3}S_y+c_8\varepsilon_x^{1/3}S_y-c_6\varepsilon(x)^{1/2}S_x\log\big(\varepsilon(x)^{1/2}S_x\big)\Big)$$

and this expression clearly does not exceed that stated on the right-hand side of the inequality we require.

Proof of Theorem 1. Let F denote the set of integers $n\leqslant x$ for which $|f_1(g(n))-S_y|>\varepsilon(x)^{1/3}S_y$. For any positive integer k Lemma 9 shows that

$$\Big|\sum_{\substack{a_i\leqslant x\\ a_i\in F}}\big(f_1(g(a_i))-S_y\big)^k\Big|^2\leqslant A(x)\sum_{\substack{m\leqslant x\\ m\in F}}\big(f_1(g(m))-S_y\big)^{2k}=o\big(\{A(x)\}^2\big),$$

since $2S_y\geqslant S_x$ (see later), whilst

$$\Big|\sum_{\substack{a_i\leqslant x\\ a_i\notin F}}\big(f_1(g(a_i))-S_y\big)^k\Big|\leqslant(\varepsilon_x^{1/3}S_y)^k\sum_{a_i\leqslant x}1=o\big(A(x)S_x^k\big).$$

Since

$$\sum_{a_i\leqslant x}\{f_1(g(a_i))\}^k=A(x)S_y^k+\sum_{r=1}^{k}\binom{k}{r}\sum_{a_i\leqslant x}S_y^{k-r}\{f_1(g(a_i))-S_y\}^r,$$

we may apply the inequality of Cauchy-Schwartz and that of Lemma 2 to easily obtain the asymptotic equality

$$\sum_{a_i\leqslant x}\{f_1(g(a_i))\}^k\sim A(x)S_y^k.$$

Finally we note that both

$$0\leqslant S_x-S_y\leqslant\sum_{y<p\leqslant x}p^{-1}=\log\Big(\frac{\log x}{\log y}\Big)+o(1)\leqslant\log S_x=o(S_x),$$

and

$$0\leqslant f(g(a_i))-f_1(g(a_i))\leqslant\sum_{\substack{p\mid g(a_i)\\ p>y}}1\leqslant c_8\varepsilon(x)^{1/2}S_x=o(S_x),$$

so that we easily complete the proof of our theorem.

As was pointed out in [4] we cannot obtain an unconditional extension of Theorem 1 by weakening the lower bound on $A(x)$. We can however prove the following result which is perhaps of some interest.

THEOREM 2. *Using the definitions of Theorem* 1 *with the weaker conditions* $\mu_x\leqslant S_x$ *and* $A(x)>x\exp(-cS_x/\mu_x)$ *for some positive constant* c, *the following two statements are equivalent:*

(i) $f(g(a_i))$ *is normally* S_{a_i},

(ii) $\sum_{a_i\leqslant x}\{f(g(a_i))\}^k\sim A(x)S_x^k,\quad k=1,2,\ldots$

Proof. We need only prove that the first of these implies the second, since an inequality of a well known type shows that the converse proposition is true.

For any $\varepsilon>0$ there is a function $\delta(x)$ which $\to 0$ as $x\to\infty$, and which has the property that the number of a_i not exceeding x for which $|f(g(a_i))-S_{a_i}|>\varepsilon S_{a_i}$ is at most $\delta(x)A(x)$. Let us denote the set of these values of a_i by M.

Applying first the inequality of Cauchy-Schwartz and then that of Lemma 2 we obtain the inequality

$$\Big|\sum_{\substack{a_i\leqslant x\\ a_i\in M}}\big(f(g(a_i))-S_{a_i}\big)^k\Big|\leqslant c_1\sqrt{\delta(x)}\,A(x)S_x^k,$$

so that

$$\limsup_{x\to\infty}\Big|\{A(x)S_x^k\}^{-1}\sum_{a_i\leqslant x}\{f(g(a_i))-S_{a_i}\}^k\Big|\leqslant\varepsilon^k.$$

It is now easily seen that for each value of k,

$$\sum_{a_i\leqslant x}\{f(g(a_i))-S_x\}^k=o\big(A(x)S_x^k\big),$$

and from this we readily obtain (ii).

The interest in this result is that for sequences A of the type above, a *necessary and sufficient* condition that $f(g(a_i))$ should normally be S_{a_i} is that

$$\sum_{a_i\leqslant x}\{f(g(a_i))-S_x\}^2=o\big(A(x)S_x^2\big).$$

Thus the method of forming the dispersion of $f(g(a_i))$ is efficient. A very neat example of this is given in Barban [1] where he shows, using sieve arguments alone, that if $\mu_x=o(S_x)$, and f is strongly additive (which it is clear we may assume), and non-negative, then

$$\sum_{p\leqslant x}\{f(p-1)-S_x\}^2=o\Big(\frac{x}{\log x}S_x^2\Big).$$

It is immediately clear that if we denote by $\pi(x)$ the number of primes not exceeding x, and appeal to the Theorem of Tchebycheff which states that $\pi(x)>c_1x/\log x$, then for any integer $k\geqslant 1$,

$$\sum_{p\leqslant x}\{f(p-1)\}^k\sim\pi(x)S_x^k.$$

Similarly, by modifying his proof we can deal with the more general proposition

$$\sum_{p\leqslant x}\{f(g(p))\}^k \sim \pi(x)\, S_x^k.$$

In neither of these proofs do we therefore have to appeal to any results from the theory of complex variables.

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Contributions to the theory of transcendental numbers (I)

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Dedicated to the memory of
Jacqués Hadamard (1865-1963)

§ 1. Introduction. In this paper we prove the main theorem relating to the set (or a subset) of complex numbers at which a given set of algebraically independent meromorphic functions assume values in a fixed algebraic number field (we actually prove a more general result which may be useful elsewhere). We state a few deductions in § 2 and it is interesting to note that the Main Theorem gives significant results in the case (overlooked by Gelfond) *where the functions concerned do not satisfy algebraic differential equations of the first order with algebraic number coefficients.* Since some of the deductions require lengthy preparations we postpone the proofs of these and other deductions to part II, which is a continuation of this paper. We give a brief history of this theorem in this section.

In the year 1929, A. O. Gelfond made the important discovery that $a^b = e^{b\log a}$ is transcendental for every imaginary quadratic irrationality b and every algebraic a different from zero except for $\log a = 0$ (1). Assuming the result to be false Gelfond applied the interpolation formula

$$f(z) = a_0F_0(z)+a_1F_1(z)+a_2F_2(z)+\ldots$$

where $F_n(z) = \prod_{l=1}^{n}(z-z_l)$, $a_{n-1} = \sum_{k=1}^{n} f(z_k)\{F_n'(z_k)\}^{-1}$, and $z_1, z_2, \ldots$ is a previously given sequence of complex numbers (for the conditions of validity see Siegel's monograph [6], § 14, Chapter I) to the function $f(z) = e^{z\log a}$ and arrived at the contradiction that the above expansion for $f(z)$ for a suitable sequence $z_1, z_2, \ldots$ must terminate. *It is important to note that the only property of the exponential function required in the proof is the addition theorem $e^{x+y} = e^x e^y$.* Gelfond's proof was carried over to the case of a real quadratic irrationality b by R. O. Kusmin [2] in 1930.

(1) As a consequence we have the remarkable result that the decimal expansion of $e^\pi = i^{-2i}$ never terminates. Ref. A. O. Gelfond, *Sur les nombres transcendants,* Comptes Rendus Acad. Sci. Paris, 189(1929), pp. 1224-1226.