ACTA ARITHMETICA
XIV (1968)

A new generalization of Schur's second partition theorem*

by

GEORGE E. ANDREWS (University Park, Pa.)

1. Introduction. I. J. Schur was one of the original discoverers of what are known as the Rogers-Ramanujan identities [4]. In 1926, Schur proved the following result which somewhat resembles the Rogers-Ramanujan identities [5].

THEOREM 2. *Let $H_1(n)$ denote the number of partitions of n into parts $\equiv \pm 1 \pmod 6$. Let $F_1(n)$ denote the number of partitions of n into distinct parts $\equiv \pm 1 \pmod 3$. Let $G_1(n)$ denote the number of partitions of n of the form $n = b_1 + \ldots + b_s$, where $b_i - b_{i+1} \geqslant 3$ with strict inequality if $3 \mid b_i$. Then $H_1(n) = F_1(n) = G_1(n)$.*

Recently [2], a general partition theorem of this type has been proved which contains Theorem 2 as a special case. The method of proof is based on the use of q-difference equations and Appell's Comparison Theorem. Since a proof of Theorem 2 has been given utilizing recurrent sequences and Appell's Comparison Theorem [1], it might be expected that if such a technique were suitably generalized, it would yield an alternative proof of the main theorem in [2]. Surprisingly a completely different result is obtained. For example, the main theorem in [2] implies the following result.

THEOREM. *Let $C_2(n)$ denote the number of partitions of n into parts $\equiv 1, 9, 11 \pmod{14}$. Let $D_2(n)$ denote the number of partitions of n into distinct parts $\equiv 1, 2, 4 \pmod 7$. Let $E_2(n)$ denote the number of partitions of n of the form $n = b_1 + \ldots + b_s$, where $b_i - b_{i+1} \geqslant 7$ if $b_{i+1} \equiv 1, 2, 4 \pmod 7$, $b_i - b_{i+1} \geqslant 12$ if $b_{i+1} \equiv 3 \pmod 7$, $b_i - b_{i+1} \geqslant 10$ if $b_{i+1} \equiv 5, 6 \pmod 7$, and $b_i - b_{i+1} \geqslant 15$ if $b_{i+1} \equiv 0 \pmod 7$.*

Theorem 1 of this paper implies the following result.

THEOREM 3. *Let $H_2(n)$ denote the number of partitions of n into parts $\equiv 3, 5, 13 \pmod{14}$. Let $F_2(n)$ denote the number of partitions of n*

* Partially supported by National Science Foundation Grant GP-6663.

into distinct parts $\equiv 3, 5, 6 \pmod 7$. *Let* $G_2(n)$ *denote the number of partitions of* n *of the form* $n = b_1 + \ldots + b_s$, *where* $b_i - b_{i+1} \geqslant 7$ *if* $b_i \equiv 3, 5, 6 \pmod 7$, $b_i - b_{i+1} \geqslant 12$ *if* $b_i \equiv 4 \pmod 7$, $b_i - b_{i+1} \geqslant 10$ *if* $b_i \equiv 1, 2 \pmod 7$, $b_i - b_{i+1} \geqslant 15$, *if* $b_i \equiv 0 \pmod 7$, *and* $b_s \neq 1, 2, 4, 7$. *Then* $H_2(n) = F_2(n) = G_2(n)$.

The striking symmetry between the statements of these two theorems is made even clearer in the relationship between the main theorem of [2] and Theorem 1 of this paper. In Section 2, we shall make certain definitions and state Theorem 1. In Section 3, we shall prove Theorem 1.

2. Preliminaries. The notation needed for our work here is precisely that used in [2]. Throughout this paper we shall write $2(n)$ for 2^n. We consider a set $A = \{a(1), \ldots, a(r)\}$ of r distinct positive integers which will be fixed throughout our discussion and which satisfy $\sum_{i=1}^{k-1} a(i) < a(k)$, $1 \leqslant k \leqslant r$. We note that this last condition implies that the $2(r)-1$ possible sums of distinct elements of A are also distinct; we denote this set of sums by A' and its elements by $a(1) < a(2) < \ldots < a\big(2(r)-1\big)$. From the previously stated inequalities for the a's, it is clear that $a\big(2(i)\big) = a(i+1)$ and that all the a's with $a(k-1) \leqslant a < a(k)$ have $a(k-1)$ as the largest summand in their defining sum. We let N be a positive integer with $N \geqslant a\big(2(r)-1\big) = a(1) + a(2) + \ldots + a(r)$. We further define $a\big(2(r)\big) = a(r+1) = N + a(1)$. Let $-A_N$ be the set of all positive integers which are congruent to some $-a(i) \pmod N$. Let $-A'_N$ be the set of all positive integers which are congruent to some $-a(i) \pmod N$. Let $\beta_N(m)$ denote the least positive residue of $m \pmod N$. If $m \in A'$, let $w(m)$ be the number of terms appearing in the defining sum of m, and let $v(m)$ denote the smallest $a(i)$ appearing in this sum. With these definitions, we are now prepared to state Theorem 1.

THEOREM 1. *Let* $F(-A_N; n)$ *denote the number of partitions of* n *into distinct parts taken from* $-A_N$. *Let* $G(-A'_N; n)$ *denote the number of partitions of* n *into parts taken from* $-A'_N$ *of the form* $n = b_1 + \ldots + b_s$, $b_i \geqslant b_{i+1}$, $b_i - b_{i+1} \geqslant N \cdot w\big(\beta_N(-b_i)\big) + v\big(\beta_N(-b_i)\big) - \beta_N(-b_i)$, $b_s \geqslant N \cdot \big(w\big(\beta_N(-b_s)\big) - 1\big)$. *Then* $G(-A'_N; n) = F(-A_N; n)$.

Let us now note how Theorems 2 and 3 are derived from Theorem 1.

To prove Theorem 2, take $N = 3$, $a(1) = 1$, $a(2) = 2$. Then immediately $F(-A_N; n)$ becomes $F_1(n)$. We also note that $-A'_N$ is the set of all positive integers. Finally if $b_i \equiv 1 \pmod 3$, then $b_i - b_{i+1} \geqslant 3 \cdot 1 + 2 - 2 = 3$; if $b_i \equiv 2 \pmod 3$, then $b_i - b_{i+1} \geqslant 3 \cdot 1 + 1 - 1 = 3$; if $b_i \equiv 0 \pmod 3$, then $b_i - b_{i+1} \geqslant 3 \cdot 2 + 1 - 3 = 4$. Thus $G(-A'_N; n) = G_1(n)$.

The fact that $H_1(n) = F_1(n)$ follows directly from

$$\prod_{j=0}^{\infty} (1+q^{3j+1})(1+q^{3j+2}) = \prod_{j=0}^{\infty} (1-q^{6j+1})^{-1}(1-q^{6j+5})^{-1}.$$

The condition $b_s \geqslant N \cdot \big(w\big(\beta_N(-b_s)\big) - 1\big)$ is superfluous in this case.

To prove Theorem 3, take $N = 7$, $a(1) = 1$, $a(2) = 2$, $a(3) = 4$. Then immediately $F(-A_N; n) = F_2(n)$. Also we note again that $-A'_N$ is the set of all positive integers. Finally

if $b_i \equiv 1 \pmod 7$, then $b_i - b_{i+1} \geqslant 7 \cdot 2 + 2 - 6 = 10$ and $b_i \geqslant 7 \cdot 1 = 7$;
if $b_i \equiv 2 \pmod 7$, then $b_i - b_{i+1} \geqslant 7 \cdot 2 + 1 - 5 = 10$ and $b_i \geqslant 7 \cdot 1 = 7$;
if $b_i \equiv 3 \pmod 7$, then $b_i - b_{i+1} \geqslant 7 \cdot 1 + 4 - 4 = 7$ and $b_i \geqslant 7 \cdot 0 = 0$;
if $b_i \equiv 4 \pmod 7$, then $b_i - b_{i+1} \geqslant 7 \cdot 2 + 1 - 3 = 12$ and $b_i \geqslant 7 \cdot 1 = 7$;
if $b_i \equiv 5 \pmod 7$, then $b_i - b_{i+1} \geqslant 7 \cdot 1 + 2 - 2 = 7$ and $b_i \geqslant 7 \cdot 0 = 0$;
if $b_i \equiv 6 \pmod 7$, then $b_i - b_{i+1} \geqslant 7 \cdot 1 + 1 - 1 = 7$ and $b_i \geqslant 7 \cdot 0 = 0$;
if $b_i \equiv 7 \pmod 7$, then $b_i - b_{i+1} \geqslant 7 \cdot 3 + 1 - 7 = 15$ and $b_i \geqslant 7 \cdot 2 = 14$.

Thus

$$G(-A'_N; n) = G_2(n).$$

The fact that $H_2(n) = F_2(n)$ follows directly from

$$\prod_{j=0}^{\infty} (1+q^{7j+3})(1+q^{7j+5})(1+q^{7j+6}) = \prod_{j=0}^{\infty} (1-q^{14j+3})^{-1}(1-q^{14j+5})^{-1}(1-q^{14j+13})^{-1}.$$

3. Proof of Theorem 1. Let $\pi(m, n)$ denote the number of partitions of n of the type enumerated by $G(-A'_N; n)$ with the added restriction that no part exceeds m.

LEMMA 1. *If* $j \geqslant 1$,

$$\pi\big(jN - a(m), n\big) = \pi\big(jN - a(m+1), n\big) + \pi\Big(jN - w\big(a(m)\big) \cdot N - v\big(a(m)\big), n - jN + a(m)\Big). \tag{3.1}$$

Proof. We break the set of partitions enumerated by $\pi\big(jN - a(m), n\big)$ into two sets: 1) those with largest part $< jN - a(m)$ and 2) those with largest part $= jN - a(m)$. The partitions in the first of these sets are enumerated by $\pi\big(jN - a(m+1), n\big)$. If we remove the summand $jN - a(m)$ from the partitions in the second set, then we see that we are now partitioning $n - jN + a(m)$, and by the conditions defining these partitions the largest part is now $\leqslant Nj - a(m) - N \cdot w\big(a(m)\big) - v\big(a(m)\big) + a(m) = jN - w\big(a(m)\big) \cdot N - v\big(a(m)\big)$. Thus the partitions in the second set are in one-to-one correspondence with those partitions enumerated by $\pi\big(jN-w(a(m))\cdot N-v(a(m)),\, n-jN+a(m)\big)$. Thus (3.1) is established.

Define

$$d(m)\equiv d(m;q)=1+\sum_{n=1}^{\infty}\pi(m,n)q^n,\qquad |q|<1,\ m\geqslant 0,$$

We now wish to derive a functional equation for the $d(m)$ utilizing Lemma 1. In order to have this functional equation valid when one or more of the arguments is negative, we further define

$$d(m)=\begin{cases}1 & \text{for}\quad -N\leqslant m<0,\\ 0 & \text{for}\quad m<-N.\end{cases}$$

This definition makes the following equation consistent with (3.1) and the condition on the partitions that $b_s\geqslant N\cdot\big(w(\beta_N(-b_s))-1\big)$

$$\begin{aligned}(3.2)\qquad & d\big(jN-a(m)\big)\\ &=d\big(jN-a(m+1)\big)+q^{jN-a(m)}d\Big(jN-w\big(a(m)\big)\cdot N-v\big(a(m)\big)\Big),\end{aligned}$$

provided the argument on the left hand side is $>-N$.

Since $a\big(2(i)\big)=a(i-1)$, we may add the equations (3.2) together for $1\leqslant m\leqslant 2(k+1)-1$, and we obtain

$$(3.3)\quad d\big(jN-a(1)\big)=d\big(jN-a(k)\big)+\sum_{a<a(k)}q^{jN-a}d\big(jN-w(a)\cdot N-v(a)\big).$$

If now we add the equations (3.2) together for $2(k)\leqslant m\leqslant 2(k+1)-1$, we obtain

$$\begin{aligned}(3.4)\qquad & d\big(jN-a(k-1)\big)\\ &=d\big(jN-a(k)\big)+\sum_{a(k-1)\leqslant a<a(k)}q^{jN-a}d\big(jN-w(a)\cdot N-v(a)\big).\end{aligned}$$

Now every a in the interval $\big(a(k-1),\,a(k)\big)$ is of the form $a(k-1)+a'$ where $a'<a(k-1)$. Hence

$$\begin{aligned}(3.5)\qquad d\big(jN-a(k-1)\big)&=d\big(jN-a(k)\big)+q^{jN-a(k-1)}d\big(N\cdot(j-1)-a(k-1)\big)+\\ &\quad+\sum_{a'<a(k-1)}q^{jN-a(k-1)-a'}d\Big(N\cdot\big(j-1-w(a')\big)-v(a')\Big)\\ &=d\big(jN-a(k)\big)+q^{jN-a(k-1)}d\big(N\cdot(j-1)-a(k-1)\big)+\\ &\quad+q^{N-a(k-1)}\Big(d\big((j-1)\cdot N-a(1)\big)-d\big((j-1)\cdot N-a(k-1)\big)\Big)\\ &=d\big(jN-a(k)\big)+q^{N-a(k-1)}d\big((j-1)\cdot N-a(1)\big)-\\ &\quad-q^{N-a(k-1)}\big(1-q^{(j-1)N}\big)d\big(N\cdot(j-1)-a(k-1)\big).\end{aligned}$$

LEMMA 2. *If* $1\leqslant k\leqslant r+1$,

$$\begin{aligned}(3.6)\qquad & d\big(jN-a(1)\big)\\ &=d\big(jN-a(k)\big)+\sum_{i=1}^{k-1}\Big(\sum_{\substack{a<a(k)\\ w(a)=i}}q^{jN-a}\Big)\prod_{h=1}^{i-1}\big(1-q^{N(j-h)}\big)d\big((j-i)\cdot N-a(1)\big).\end{aligned}$$

Proof. For $k=1$, (3.6) reduces to $d\big(jN-a(1)\big)=d\big(jN-a(1)\big)$. Assume (3.6) true for a particular $k<r+1$. Then

$$\begin{aligned}& d\big(jN-a(1)\big)-d\big(jN-a(k+1)\big)\\ &=d\big(jN-a(1)\big)-d\big(jN-a(k)\big)+d\big(jN-a(k)\big)-d\big(jN-a(k+1)\big)\\ &=d\big(jN-a(1)\big)-d\big(jN-a(k)\big)+q^{N-a(k)}d\big((j-1)\cdot N-a(1)\big)-\\ &\quad-q^{N-a(k)}\big(1-q^{(j-1)N}\big)d\big(N\cdot(j-1)-a(k)\big)\\ &=d\big(jN-a(1)\big)-d\big(jN-a(k)\big)+q^{N-a(k)}d\big((j-1)\cdot N-a(1)\big)\\ &\quad-q^{N-a(k)}\big(1-q^{(j-1)N}\big)\Big(d\big((j-1)\cdot N-a(1)\big)-\\ &\quad-\sum_{i=1}^{k-1}\Big(\sum_{\substack{a<a(k)\\ w(a)=i}}q^{(j-1)\cdot N-a}\Big)\prod_{h=1}^{i-1}\big(1-q^{N\cdot(j-h-1)}\big)d\big((j-1-i)\cdot N-a(1)\big)\Big)\\ &=d\big(jN-a(1)\big)-d\big(jN-a(k)\big)+q^{jN-a(k)}d\big((j-1)\cdot N-a(1)\big)+\\ &\quad+\sum_{i=1}^{k}\Big(\sum_{\substack{a<a(k)\\ w(a)=i}}q^{jN-a-a(k)}\Big)\prod_{h=1}^{i}\big(1-q^{N(j-h)}\big)d\big((j-i-1)\cdot N-a(1)\big).\end{aligned}$$

Now if a runs over all elements of A' less than $a(k)$, then $a'=a+a(k)$ runs over all elements of A' in the interval $\big(a(k),\,a(k+1)\big)$. Hence

$$\begin{aligned}& d\big(jN-a(1)\big)-d\big(jN-a(k+1)\big)\\ &=d\big(jN-a(1)\big)-d\big(jN-a(k)\big)+\\ &\quad+\sum_{i=0}^{k-1}\Big(\sum_{\substack{a(k)\leqslant a'<a(k+1)\\ w(a')=i+1}}q^{jN-a'}\Big)\prod_{h=1}^{i}\big(1-q^{N\cdot(j-h)}\big)d\big((j-i-1)\cdot N-a(1)\big)\\ &=\sum_{i=1}^{k-1}\Big(\sum_{\substack{a<a(k)\\ w(a)=i}}q^{jN-a}\Big)\prod_{h=1}^{i-1}\big(1-q^{N\cdot(j-h)}\big)d\big((j-i)\cdot N-a(1)\big)+\\ &\quad+\sum_{i=1}^{k}\Big(\sum_{\substack{a(k)\leqslant a'<a(k+1)\\ w(a')=i}}q^{jN-a'}\Big)\prod_{h=1}^{i-1}\big(1-q^{N\cdot(j-h)}\big)d\big((j-i)\cdot N-a(1)\big)\\ &=\sum_{i=1}^{k}\Big(\sum_{\substack{a<a(k+1)\\ w(a)=i}}q^{jN-a}\Big)\prod_{h=1}^{i-1}\big(1-q^{N\cdot(j-h)}\big)d\big((j-i)\cdot N-a(1)\big).\end{aligned}$$

Thus we obtain (3.6) for $k+1$, and the lemma is proved.

Proof of Theorem 1. If we define

$$t_j = d\big(jN - a(1)\big),$$

then, by Lemma 2 with $k = r+1$,

$$t_j = t_{j-1} + \sum_{i=1}^{r}\Big(\sum_{\substack{a\in A' \\ w(a)=i}} q^{jN-a}\Big)\prod_{h=1}^{i-1}\big(1-q^{N\cdot(j-h)}\big)\,t_{j-i}.$$

By definition of $d(m)$ for negative m, $t_0 = 1$ and $t_{-n} = 0$ for $n > 0$. Hence by Theorem 2 of [1], p. 129,

$$1+\sum_{n=1}^{\infty} G(-A'_N; n)q^n = \lim_{j\to\infty} t_j = \prod_{m=1}^{\infty}\Big(1+\sum_{i=1}^{r}\Big(\sum_{\substack{a\in A' \\ w(a)=i}} q^{-a}\Big)q^{imN}\Big)$$

$$= \prod_{m=1}^{\infty}\big(1+q^{mN-a(1)}\big)\big(1+q^{mN-a(2)}\big)\ldots\big(1+q^{mN-a(r)}\big)$$

$$= 1+\sum_{n=1}^{\infty} F(-A_N; n)q^n.$$

Thus $G(-A'_N; n) = F(-A_N; n)$.

References

[1] G. E. Andrews, *On Schur's second partition theorem*, Glasgow Mathematical Journal 8 (1967), pp. 127-132.

[2] — *A general theorem on partitions with difference conditions*, Amer. J. Math. (to appear).

[3] P. Dienes, *The Taylor Series*, New York 1957.

[4] I. J. Schur, *Ein Beitrag zur additiven Zahlentheorie*, S.-B. Akad. Wiss. Berlin (1917), pp. 301-321.

[5] — *Zur additiven Zahlentheorie*, S.-B. Akad. Wiss. Berlin (1926), pp. 488-495.

THE PENNSYLVANIA STATE UNIVERSITY

Reçu par la Rédaction le 3. 11. 1967

ACTA ARITHMETICA
XIV (1968)

Corrigendum to the paper "On the zeros of L-functions"

(Acta Arithmetica 11 (1965), pp. 67-96)

by

E. Fogels (Riga)

The formula (34) of the paper in question should be replaced by the following:

$$(34^*)\qquad V < \frac{e^{c_7\lambda}}{\log^2 T}\sum_{T^B<n<T^{3B}}\frac{\Lambda(n)}{n}\sum_{\substack{T^B<m<T^{3B}\\ m\equiv n \,(\mathrm{mod}\, D)}} h\,\frac{\Lambda(m)}{m}\Big|\sum_{1\leqslant j\leqslant V}\Big(\frac{m}{n}\Big)^{iw_j}\Big|.$$

Proof. By the arguments of § 9 and § 5 we have

$$V < e^{c_6\lambda}\sum_{1\leqslant j\leqslant V}\Big|\sum_{T^B<n<T^{3B}}\frac{\chi_j(n)\Lambda(n)R(n)}{n^{1+i(T_0+w_j)}}\Big|^2$$

$$\leqslant e^{c_6\lambda}\sum_{1\leqslant j\leqslant V}\sum_{\chi}\Big|\sum_{T^B<n<T^{3B}}\frac{\chi(n)\Lambda(n)R(n)}{n^{1+i(T_0+w_j)}}\Big|^2$$

$$= e^{c_6\lambda}\sum_{1\leqslant j\leqslant V}\sum_{\chi}\sum_{T^B<n<T^{3B}}\frac{\chi(n)\Lambda(n)R(n)}{n^{1+i(T_0+w_j)}}\sum_{T^B<m<T^{3B}}\frac{\overline{\chi(m)}\,\Lambda(m)\overline{R(m)}}{m^{1-i(T_0+w_j)}}$$

$$= e^{c_6\lambda}\sum_{1\leqslant j\leqslant V}\sum_{T^B<n<T^{3B}}\frac{\Lambda(n)R(n)}{n}\sum_{\substack{T^B<m<T^{3B}\\ m\equiv n \,(\mathrm{mod}\, D)}} h\,\frac{\Lambda(m)}{m}\overline{R(m)}\Big(\frac{m}{n}\Big)^{i(T_0+w_j)}$$

$$= e^{c_6\lambda}\sum_{T^B<n<T^{3B}}\frac{\Lambda(n)R(n)}{n}\sum_{\substack{T^B<m<T^{3B}\\ m\equiv n \,(\mathrm{mod}\, D)}} h\,\frac{\Lambda(m)}{m}\overline{R(m)}\sum_{1\leqslant j\leqslant V}\Big(\frac{m}{n}\Big)^{i(T_0+w_j)}$$

$$\leqslant e^{c_6\lambda}\sum_{T^B<n<T^{3B}}\frac{\Lambda(n)\,|R(n)|}{n}\sum_{\substack{T^B<m<T^{3B}\\ m\equiv n \,(\mathrm{mod}\, D)}} h\,\frac{\Lambda(m)}{m}|R(m)|\Big|\sum_{1\leqslant j\leqslant V}\Big(\frac{m}{n}\Big)^{i(T_0+w_j)}\Big|,$$

whence (34^*) follows. Using (34^*) instead of (34) we can proceed as in § 9. The formula at the end of § 10 undergoes a similar exchange.

Reçu par la Rédaction le 15. 12. 1967