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On Siegel's Theorem

by

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1. Given a character $\chi \bmod k$, we denote by $\delta(\chi)$ the distance between 1 and the nearest real zero of $\mathscr{L}(s, \chi)$.

Dirichlet showed that $\delta(\chi) \neq 0$, Page [7] improved on it by giving the lower bound $\delta(\chi) > (c_1 k^{1/2} \log k)^{-1}$ (1), and finally Siegel proved [9] that for every $\varepsilon > 0$ we have

$$\delta(\chi) > \frac{c(\varepsilon)}{k^{\varepsilon}}; \tag{1.1}$$

the former result is deduced in a very simple way from the following

Theorem A (2). *If χ_1 is an arbitrary, real character* $\bmod k_1$, *and $\mathscr{L}(s, \chi_1)$ has a real zero at β_1 with $1-\varepsilon < \beta_1 < 1$, then*

$$\delta(\chi) > k^{-\varepsilon c_2}, \tag{1.2}$$

if

$$k > c(\varepsilon, k_1). \tag{1.3}$$

The aim of this paper is to give a still different proof of Theorem A; we even assert the following, stronger

Theorem B. *If χ_1 is an arbitrary real or complex character* $\bmod k_1$, *and $\mathscr{L}(s, \chi_1)$ has a zero at $\beta_1 + i\gamma_1$ with $1-\varepsilon < \beta_1 < 1$, then* (1.2) *follows if*

$$k > c(\varepsilon, k_1, \gamma_1). \tag{1.4}$$

Although the bound (1.4) can be given explicitly, as was also the case with (1.3) in [9], [3] and [1], it is fairly obvious that Theorems A and B alone cannot produce (1.1) with an explicit $c(\varepsilon)$. We add that Linnik proved (1.1) in an elementary way (see [6], also [4]); however, his $c(\varepsilon)$ is as ineffective as it used to be in the previous proofs.

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(1) c_1, and further $c_2, c_3, \ldots$, denote positive numerical constants.

(2) Essentially due to Siegel [9]; for alternative proofs see Estermann [3] (also [8]) and Chowla [1] (also [2]).

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Our proof of Theorem B will be based on the following theorem of Turán (see [10]; the original form was Theorem X in [11]):

If $z_1, z_2, \ldots, z_N$ *with*

$$|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_N|$$

are arbitrary complex numbers, $g > 0$ *is arbitrary, and* $N < M$, *then there exists an integer* ω *with*

$$g \leqslant \omega \leqslant g+M \tag{1.5}$$

such that

$$|z_1^\omega + z_2^\omega + \ldots + z_N^\omega| \geqslant \left(\frac{M}{23(g+M)}\right)^M |z_1|^\omega. \tag{1.6}$$

2. Preliminaries and notation. Without any loss of generality we can suppose that χ is a real character, $\mathscr{L}(s, \chi)$ has a real zero at β, $\delta = \delta(\chi) = 1-\beta$, and

$$\delta < \frac{1}{c_3 \log k}, \tag{2.1}$$

with c_3 sufficiently large; further, it can be supposed that $\varepsilon > 0$ is sufficiently small, characters χ_1 and χ are not equivalent, and

$$k > k_1, \qquad k > |\gamma_1|. \tag{2.2}$$

We consider

$$f(s) \stackrel{\text{def}}{=} \mathscr{L}(s, \chi_1)\mathscr{L}(s, \chi_1\chi)$$

and assume that $f(s)$ has at most

$$N(\varepsilon, \gamma_1) \leqslant \varepsilon c_4 \log k \tag{2.3}$$

zeros in

$$\begin{cases} 1-3\varepsilon \leqslant \sigma \leqslant 1, \\ |t-\gamma_1| \leqslant 30\varepsilon \end{cases} \tag{2.4}$$

(the inequality (2.3) follows from Lemma 2; we need the constant c_4 in our further notation).

We put

$$A = 1/\varepsilon^2, \qquad B = 30/\varepsilon, \tag{2.5}$$

and consider an integer ω satisfying

$$c_5 \varepsilon \log k \leqslant \omega \leqslant (c_5 + c_4)\varepsilon \log k, \tag{2.6}$$

where $c_5 \geqslant c_4$ is supposed to be big enough. Finally, we consider the integral

$$I_\omega = \frac{1}{2\pi i} \int_{(1)} e^{A\omega s^2 + B\omega s} \frac{f'}{f}(s+1+i\gamma_1 - 3\varepsilon)\,ds. \tag{2.7}$$

We will also use the classical inequality of Dirichlet

$$\mathscr{L}(1, \chi) > \frac{1}{c_6\sqrt{k}}. \tag{2.8}$$

3. LEMMA 1 [3]. *Write*

$$\zeta(s)\mathscr{L}(s, \chi) = \sum_n \frac{a_n}{n^s};$$

we have $a_n \geqslant 0$, *and, with* c_3 *in* (2.1) *sufficiently large,*

$$\sum_{n \leqslant k^2} \frac{a_n}{n} > \frac{\mathscr{L}(1, \chi)}{c_7\delta}. \tag{3.1}$$

Proof. Note that

$$a_n = \sum_{d|n} \chi(d) = \prod_{p^m | n,\, p^{m+1} \nmid n} p\{1+\chi(p)+\ldots+\chi(p^m)\}, \tag{3.2}$$

from which

$$0 \leqslant a_n \leqslant \sum_{d|n} 1 \leqslant n. \tag{3.3}$$

Consider the formula

$$\sum_n a_n n^{-\beta} e^{-xn} = \frac{1}{2\pi i} \int_{(2)} x^{\beta - s} \Gamma(s-\beta)\zeta(s)\mathscr{L}(s, \chi)\,ds,$$

where $x = k^{-3/2}$. Moving the line of integration from $\sigma = 2$ to $\sigma = -1/2$, and using the theorem of residues, we obtain

$$\sum_n a_n n^{-\beta} e^{-xn} = k^{3\delta/2}\Gamma(\delta)\mathscr{L}(1, \chi) + O(k^{-1/2}). \tag{3.4}$$

Also, by (3.3),

$$\sum_{n > k^2} a_n n^{-\beta} e^{-xn} \leqslant \sum_{n > k^2} n e^{-xn} < \frac{c_8}{k},$$

and so by (3.4)

$$\sum_{n \leqslant k^2} \frac{a_n}{n} = \sum_{n \leqslant k^2} \frac{a_n}{n^\beta} n^{-\delta} \geqslant k^{-2\delta} \sum_{n \leqslant k^2} \frac{a_n}{n^\beta} e^{-xn} > e^{-2/c_3}\left(\Gamma(\delta)\mathscr{L}(1, \chi) - \frac{c_9}{\sqrt{k}}\right).$$

Hence, using (2.8), we come to (3.1).

LEMMA 2. *Under* (2.2), *the number of zeros of* $f(s)$ *in the rectangle* (2.4) *is* $O(\varepsilon \log k)$.

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[3] This lemma is implicitly contained in [8], pp. 362-363, and seems to go back to Linnik [5]. We reproduce it here for the sake of completeness.

Proof. Using the approximate formula ([8], p. 225, Satz 4.1)

$$\frac{f'}{f}(1+\varepsilon+i\gamma_1) = \sum_{|\gamma-\gamma_1|\leqslant 1} \frac{1}{1+\varepsilon+i\gamma_1-\varrho} + O(\log k),$$

where $\varrho = \beta+i\gamma$ runs through the zeros of $f(s)$, further using the (simple) inequalities

$$\operatorname{Re}\frac{f'}{f}(1+\varepsilon+i\gamma_1) \leqslant \frac{c_{10}}{\varepsilon}$$

and

$$\operatorname{Re}\sum_{|\gamma-\gamma_1|\leqslant 1} \frac{1}{1+\varepsilon+i\gamma_1-\varrho} \geqslant \operatorname{Re}\sum_{\varrho \text{ in } (2.4)} \frac{1}{1+\varepsilon+i\gamma_1-\varrho} \geqslant \frac{N(\varepsilon,\gamma_1)}{1000\varepsilon},$$

we obtain the result.

LEMMA 3. *There exists a* $\vartheta = \vartheta(\varepsilon)$ *with*

$$(3.5) \qquad 1-3\varepsilon \leqslant \vartheta \leqslant 1-2\varepsilon$$

such that

$$(3.6) \qquad \left|\frac{f'}{f}(\vartheta+it)\right| \leqslant c_{11}\log^2 k, \qquad |t-\gamma_1| \leqslant 21\varepsilon;$$

similarly, there exists a $\vartheta' = \vartheta'(\varepsilon)$ *with*

$$(3.7) \qquad 20\varepsilon \leqslant \vartheta' \leqslant 21\varepsilon$$

such that

$$(3.8) \qquad \left|\frac{f'}{f}(\sigma+i\gamma_1\pm i\vartheta')\right| \leqslant c_{12}\log^2 k, \qquad 1-3\varepsilon \leqslant \sigma \leqslant 1.$$

Proof. Standard and simple (using [8], p. 225, Satz 4.1).

4. We return to the integral (2.7); pushing the line $\sigma = 1$ to

(i) $\quad s = \vartheta-1+3\varepsilon+it, \quad |t| \leqslant \vartheta',$

(ii) $\quad s = \sigma\pm i\vartheta', \quad \vartheta-1+3\varepsilon \leqslant \sigma \leqslant 3\varepsilon,$

(iii) $\quad s = 3\varepsilon+it, \quad \vartheta' \leqslant |t| < \infty,$

we estimate the corresponding integrals

$$\int_{(i)}, \quad \int_{(ii)}, \quad \int_{(iii)}.$$

By (3.5), (3.6),

$$\left|\int_{(i)}\right| \leqslant c_{13}\log^2 k\cdot e^{A\omega\varepsilon^2+B\omega\varepsilon},$$

further by (3.7), (3.8)

$$\left|\int_{(ii)}\right| \leqslant c_{14}\log^2 k\cdot e^{A\omega(9\varepsilon^2-\vartheta'^2)+3B\varepsilon\omega} < e^{-100\omega}.$$

Finally using the estimate ([8], p. 132)

$$\left|\frac{f'}{f}(1+it)\right| \leqslant c_{15}\log\big(k(|t|+2)\big),$$

we find

$$\left|\int_{(iii)}\right| < c_{16}\int_{20\varepsilon}^{\infty} e^{100\omega-\omega t^2/\varepsilon^2}\log k(|t|+2)\,dt < c_{17}\int_{20}^{\infty} e^{100\omega-\omega u^2}\log(ku)\,du < e^{-200\omega}.$$

Hence

$$(4.1) \qquad I_\omega = \sum_{\varrho}{}' e^{A\omega(\varrho-1+3\varepsilon-i\gamma_1)^2+B\omega(\varrho-1+3\varepsilon-i\gamma_1)} + O(\log^2 k\cdot e^{A\omega\varepsilon^2+B\omega\varepsilon}),$$

where $\sum_{\varrho}'$ denotes summation over some ϱ's in the region (2.4). Now using Turán's theorem (1.5), (1.6) with

$$z_j = e^{A(\varrho-1+3\varepsilon-1\gamma_1)^2+B(\varrho-1+3\varepsilon-i\gamma_1)}$$

(and remembering $1-\varepsilon < \beta_1 < 1$, which makes

$$|z_1| \geqslant e^{A(\beta_1-1+3\varepsilon)^2+B(\beta_1-1+3\varepsilon)} > e^{4A\varepsilon^2+2B\varepsilon},$$

and with $g = c_5\varepsilon\log k$, $M = c_4\varepsilon\log k$, we get from (4.1)

$$|I_\omega| > e^{4A\omega\varepsilon^2+2B\omega\varepsilon}\left(e^{-\varepsilon c_4\log\left(\frac{23(c_4+c_5)}{c_4}\right)^{\log k}} - c_{18}\log^2 k\cdot e^{-3A\omega\varepsilon^2-B\omega\varepsilon}\right),$$

on choosing a suitable ω in (2.6). Hence, making c_5 big enough,

$$(4.2) \qquad |I_\omega| > k^{\varepsilon/c_{19}}.$$

5. Computing the integral (2.7) directly, we obtain

$$I_\omega = -\sum_n \frac{\Lambda(n)\chi_1(n)\big(1+\chi(n)\big)}{n^{1+i\gamma_1-3\varepsilon}}\cdot\frac{1}{2\sqrt{\pi A\omega}}\, e^{-\frac{1}{4A\omega}\log^2(ne^{-B\omega})},$$

whence

$$(5.1) \qquad |I_\omega| \leqslant \frac{1}{2\sqrt{\pi A\omega}}\sum_n \frac{\Lambda(n)\big(1+\chi(n)\big)}{n^{1-3\varepsilon}}\, e^{-\frac{1}{4A\omega}\log^2(ne^{-B\omega})}.$$

We estimate the contribution of n's with

$$n < e^{B\omega/2}.$$

Here the exponential factor is

$$< e^{-\frac{B^2\omega}{16A}} = e^{\frac{900}{16}\omega} < e^{-50\omega},$$

whence the whole contribution does not exceed

$$c_{20}\frac{B\omega}{\sqrt{A\omega}}e^{-50\omega}\sum_{n<e^{B\omega/2}} n^{-1-3\varepsilon} \leqslant c_{21}\frac{B\sqrt{\omega}}{\sqrt{A}}e^{-50\omega}\cdot\frac{1}{\varepsilon}e^{B\omega\varepsilon/2} \leqslant e^{-30\omega}.$$

As to the contribution of n's with

$$n > e^{3B\omega},$$

we estimate it by

$$c_{22}\frac{1}{\sqrt{A\omega}}\int_{e^{2B\omega}}^{\infty}\frac{\log x}{x^{1-3\varepsilon}}e^{-\frac{1}{4A\omega}\log^2(xe^{-B\omega})}dx$$

$$\leqslant \frac{c_{23}}{\sqrt{A\omega}}\int_{e^{B\omega}}^{\infty}\frac{e^{3\varepsilon B}\log u}{u^{1-3\varepsilon}}e^{-\frac{1}{4A\omega}\log^2 u}du = \frac{c_{23}}{\sqrt{A\omega}}\int_{B\omega}^{\infty} te^{3\varepsilon B\omega+3\varepsilon t-t^2/(4A\omega)}dt$$

$$< \frac{c_{23}}{\sqrt{A\omega}}e^{3\varepsilon B\omega}\int_{B\omega}^{\infty}e^{7\varepsilon t/2-t^2/(4A\omega)}dt < \frac{c_{23}}{\sqrt{A\omega}}e^{3\varepsilon B\omega}\int_{B\omega}^{\infty}e^{-t^2/(8A\omega)}dt$$

$$= c_{24}e^{3\varepsilon B\omega}\int_{B\omega(8A\omega)^{-1/2}}^{\infty}e^{-y^2}dy < c_{24}e^{3\varepsilon B\omega-B^2\omega/(8A)} = c_{24}e^{90\omega-900\omega/8} < e^{-20\omega}.$$

Hence (4.2) and (5.1) give

$$\frac{1}{2}k^{\varepsilon/c_{19}} < \frac{1}{2\sqrt{\pi A\omega}}\sum_{e^{B\omega/2}\leqslant n\leqslant e^{3B\omega}}\frac{\Lambda(n)\big(1+\chi(n)\big)}{n^{1-3\varepsilon}},$$

and further

(5.2) $$\sum_{k^{c_{25}}\leqslant p^m\leqslant k^{c_{26}}}\frac{1+\chi(p^m)}{p^m} > k^{-\varepsilon c_{27}} \quad (p \text{ primes}),$$

where $c_{25} > 15$ (if we only make $c_5 > 1$).

The contribution of p's with $p|k$ to the sum in (5.2) is

$$\leqslant \sum_{p|k} 2k^{-c_{25}} < c_{28}(\log k)k^{-15},$$

so that

(5.3) $$\sum_{\substack{k^{c_{25}}\leqslant p^m\leqslant k^{c_{26}}\\ \chi(p^m)=1}}\frac{1}{p_m} > k^{-\varepsilon c_{29}}.$$

6. We obtain from (5.3)

$$k^{-\varepsilon c_{27}}\sum_{n_1\leqslant k^2}\frac{a_{n_1}}{n_1} < \sum_{\substack{k^{c_{25}}\leqslant p^m\leqslant k^{c_{26}}\\ \chi(p^m)=1}}\frac{1}{p^m}\cdot\sum_{n_1\leqslant k^2}\frac{a_{n_1}}{n_1} \leqslant \sum\frac{2}{p^{m(p)}}\cdot\sum_{n_1\leqslant k^2}\frac{a_{n_1}}{n_1},$$

where $m(p)$ is the minimal exponent such that $k^{c_{25}} \leqslant p^{m(p)} \leqslant k^{c_{26}}$ and $\chi(p^{m(p)}) = 1$.

Observing that numbers $n \leqslant k^{c_{26}+2}$ can be represented in $(c_{26}+2)/c_{25}$ ways at most as $n_1 p^{m(p)}$, and using (3.2), we get

$$k^{-\varepsilon c_{27}}\sum_{n_1\leqslant k^2}\frac{a_{n_1}}{n_1} < 2\frac{c_{26}+2}{c_{25}}\sum_{k^{c_{25}}\leqslant n\leqslant k^{c_{26}+2}}\frac{a_n}{n},$$

whence

(6.1) $$\sum_{k^{c_{25}}\leqslant n\leqslant k^{c_{26}+2}}\frac{a_n}{n} > k^{-\varepsilon c_{28}}\sum_{n\leqslant k^2}\frac{a_n}{n},$$

where $c_{25} > 15$. Now we use the formula (see [8], p. 362)

$$\sum_{n\leqslant x}\frac{a_n}{n} = \mathscr{L}(1,\chi)(\log x + c_{29}) + \mathscr{L}'(1,\chi) + O\left(k\frac{\log x}{x^{1/2}}\right),$$

which gives

$$\sum_{k^{c_{25}}\leqslant n\leqslant k^{c_{26}+2}}\frac{a_n}{n} = \mathscr{L}(1,\chi)\log k(c_{26}+2-c_{25}) + O(k^{-6}),$$

thus

$$\sum_{k^{c_{25}}\leqslant n\leqslant k^{c_{26}+2}}\frac{a_n}{n} < c_{30}\mathscr{L}(1,\chi)\log k.$$

This and (6.1) give

(6.2) $$\sum_{n\leqslant k^2}\frac{a_n}{n} < k^{\varepsilon c_{31}}\mathscr{L}(1,\chi),$$

which together with (3.1) implies (1.2).

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Demonstration d'une conjecture de P. Erdös

par

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§ 1. Introduction. Pour une suite de nombres réels $(u_n)_{n\in N}$ (1) et une partie quelconque A de R, $\pi(A, n)$ $(n \in N)$ désigne le nombre de points appartenant à A parmi les n premiers points de la suite.

Supposons que pour tout $n \in N: 0 \leqslant u_n < 1$; soit β un nombre réel $0 < \beta < 1$; le *n-ième reste de la suite pour l'intervalle* $[0, \beta[$ est:

$$E(\beta, n) = \pi([0, \beta[, n) - n\beta.$$

Paul Erdös, dans [2], se demande s'il existe des suites (u_n) telles que la suite des restes $n \to E(\beta, n)$ est bornée pour tout β. Il pense qu'il n'en est rien; nous démontrerons, en effet:

Théorème. *Soient $(u_n)_{n\in N}$ une suite de nombres de l'intervalle réel $[0, 1[$ et θ un intervalle quelconque de $[0, 1[$. Alors il existe un ensemble continu de points $\beta \in \theta$ tels que la suite $n \to E(\beta, n)$ soit non bornée.*

La démonstration utilise le résultat suivant qui fut conjecturé par van der Corput:

(a) La fonction:

$$(\beta, n) \to E(\beta, n)$$

qui applique $[0, 1[\times N$ dans R est *non bornée.*

(a) est conséquence immédiate d'un résultat plus précis dû à Mme. Aardene-Ehrenfest [1], qui a été amélioré par K. F. Roth [3].

§ 2. Transformation de la propriété (a). Il convient tout d'abord de généraliser la notation E. Soient $\gamma, \delta \in R$, $\gamma < \delta$, deux nombres réels et $(v_n)_{n\in N}$ une suite de nombres réels tels que pour tout $n \in N$, $\gamma \leqslant v_n < \delta$. Pour β réel, $\gamma < \beta < \delta$, posons:

$$E_{\gamma,\delta}(\beta, n) = \pi([\gamma, \beta[, n) - n(\beta - \gamma)/(\delta - \gamma).$$

(1) $N = 1, 2, 3, \ldots$