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On the twin-prime problem III

by

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1. In their fundamental paper [6] Hardy and Littlewood gave in 1922 the first quantitative form of the Goldbach- and twin-prime conjectures (among others). Denoting by A_0 the constant [1]

$$(1.1)\qquad 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$$

they announced the

CONJECTURE A. *If $R_2(n)$ stands for the number of Goldbach decompo -itions of n then for even n's for $n\to\infty$ the asymptotic representation*

$$(1.2)\qquad R_2(n)\sim A_0\frac{n}{\log^2 n}\prod_{\substack{p>2\\ p|n}}\frac{p-1}{p-2}$$

holds.

Equivalent forms of (1.2) are

$$(1.3)\qquad \sum_{p_1+p_2=n}\log p_1\log p_2\sim A_0 n\prod_{\substack{p>2\\ p|n}}\frac{p-1}{p-2}$$

or

$$(1.4)\qquad \sum_{n_1+n_2=n}\Lambda(n_1)\Lambda(n_2)\sim A_0 n\prod_{\substack{p>2\\ p|n}}\frac{p-1}{p-2}.$$

Further

CONJECTURE B. *If $P_d(n)$ stands for the number of such primes $p\leqslant n$ for which $p+d$ is also a prime then for fixed even d and $n\to\infty$ the asym-*

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(1) The letter p will be reserved for rational primes, $A_0, A_1, \ldots$ specified, c unspecified positive numerical constants. Empty sum means 0, empty product 1. The complex variable is $s=\sigma+it$, $\sum^*_{\chi^* \bmod k}$ will stand for a summation with respect to *primitive* characters only, k^* for the conductor of k.

ptotic representation

$$P_d(n) \sim A_0 \frac{n}{\log^2 n} \prod_{\substack{p>2\\ p|d}} \frac{p-1}{p-2} \tag{1.5}$$

holds.

Their paper became fundamental though it contained either conditional or heuristical results only; the conditional results used unproved assumptions on the nontrivial zeros of the Dirichlet $L(s, k, \chi)$-functions (2) and when even the strongest assumption, the assumption of the truth of Riemann-Piltz conjecture did not help, they worked boldly with main terms only. In the frame of a different method (a sketch of which I gave in [12]) I worked out in [13] the theorems corresponding to these conjectures. If

$$\varrho = \varrho(k, \chi) = \beta + i\gamma \tag{1.6}$$

stand for the nontrivial zeros of $L(s, k, \chi)$, then they run as follows (in a slightly specialized form).

THEOREM A. *For $M/2 \leqslant n \leqslant M$ and even n the representation*

$$R_2(n) = \{1+O(\log\log M)^{-1}\} A_0 \frac{n}{\log^2 n} \prod_{\substack{p>2\\ p|n}} \frac{p-1}{p-2} - \tag{1.7}$$

$$- \frac{1+O(\log^{-1/2} M)}{\log^2 M} \sum_{\substack{k\leqslant M\\ (k,n)=1}}^{(k)} \frac{\mu(k)\log(M/k)}{\varphi(k)} \sum_{\chi \bmod k}^{(\chi)} \bar{\chi}(n, k) \times$$

$$\times \sum_{|\gamma|\leqslant \log^2 M}^{\varrho(\chi)} \frac{n^{\varrho} - n^{\varrho/100}}{\varrho(1+\varrho/\log M)^{1+\left[\frac{2\log M}{\log\log M}\right]}}$$

holds unconditionally. The O-sign refers to $M \to \infty$ uniformly in n.

THEOREM B. *For even $d \leqslant n/\log^{10} n$ the representation*

(1.8)

$$P_d(n) = \{1+O(\log\log M)^{-1}\} A_0 \frac{n}{\log^2 n} \prod_{\substack{p>2\\ p|d}} \frac{p-1}{p-2} - \frac{1+O(\log^{-1/2} n)}{\log^2 n} \times$$

$$\times \sum_{\substack{k\leqslant n+d\\ (k,d)=1}}^{(k)} \frac{\mu(k)\log(n/k)}{\varphi(k)} \sum_{\chi \bmod k}^{(\chi)} \bar{\chi}(-d, k) \sum_{|\gamma|\leqslant \log^2 n}^{\varrho(\chi)} \frac{n^{\varrho} - n^{\varrho/100}}{\varrho(1+\varrho/\log n)^{1+\left[\frac{2\log n}{\log\log n}\right]}}$$

holds unconditionally. The O-sign refers to $n \to \infty$ uniformly in d.

(2) The usual notation $L(s, \chi)$ of these functions was good for fixed k. Here k will be variable.

These formulae seem to play the same role in the additive prime-number theory what Riemann's "exact" prime-number formula played in the theory of distribution of primes and show at the same time they depend only on "small" zeros of $L(s, k, \chi)$ functions. The aim of this note is to show that using the "large-sieve" method of Linnik [7], the range of nontrivial zeros can considerably be narrowed as to "width"; it is enough to retain zeros "near" to the line $\sigma = \frac{1}{2}$. More exactly we assert the

THEOREM I. *The formulae* (1.7) *and* (1.8) *remain unconditionally true if for an arbitrarily small $\varepsilon > 0$ the range of summation is replaced by*

$$\tfrac{1}{3} \leqslant \beta \leqslant \tfrac{5}{6}, \quad |\gamma| \leqslant \log^2 M, \quad M^{1/2-\varepsilon} \leqslant k \leqslant M, \tag{1.9}$$

respectively

$$\tfrac{1}{3} \leqslant \beta \leqslant \tfrac{5}{6}, \quad |\gamma| \leqslant \log^2 n, \quad n^{1/2-\varepsilon} \leqslant k \leqslant n. \tag{1.10}$$

The O-sign depends also upon ε.

With little extra trouble, using properly the inequality (b) of Lemma I, we could replace $\frac{5}{6}$ by $\frac{4}{5}$ (even a bit less). Using an inequality of Bombieri (l.c. [1], p. 225, without detailed proof) the range in (1.9) e.g. could be replaced, as indicated in [12], by

$$\tfrac{1}{3} \leqslant \beta \leqslant \tfrac{3}{4}+\varepsilon, \quad |\gamma| \leqslant \log^2 M, \quad M^{1/2-\varepsilon} \leqslant k \leqslant M. \tag{1.11}$$

The proof of Bombieri's density hypothesis (l.c. [1], p. 205) would even lead to the range

$$\tfrac{2}{5} \leqslant \beta \leqslant \tfrac{2}{3}, \quad |\gamma| \leqslant \log^2 M. \tag{1.12}$$

I did not work these further reductions out from two reasons. Firstly it is not desirable to obscure the simplicity of the basic ideas by more technical improvements. Secondedly I do not think quite impossible to avoid the necessity to enter into the half-plane $\sigma < \frac{5}{6}$ at all. To this or other possibilities I shall return in the subsequent papers of this series with my usual low speed however.

Further I mention without proof a further reduction of the domain of summation as to its "height". This runs for the simpler case of the twin-primes as follows.

THEOREM II. *For even d the formula*

$$n^{-3/4}\log^{-1/2} n \sum_{\substack{p_1-p_2=d\\ p_1\leqslant n}} \log p_1 \log p_2 \exp\left\{-\frac{\log^2(\sqrt{n}/p_2)}{\log n}\right\}$$

$$= \pi\sqrt{2} A_0(1+o(1)) \prod_{\substack{p>2\\ p|d}} \frac{p-1}{p-2} + \sum_{\substack{k\leqslant n+d\\ (k,d)=1}} \frac{\mu(k)\log k}{\varphi(k)} \sum_{\chi \bmod k} \bar{\chi}(-d, k) \sum_{\varrho\in D_1} n^{\frac{\varrho}{2}+\frac{\varrho^2}{4}-\frac{3}{4}}$$

holds unconditionally; here D_1 means the rectangle: $\frac{1}{3} \leqslant \beta \leqslant \frac{3}{4}$, $|\gamma| \leqslant \frac{7}{4}$.

In Theorem I (and could have been done in Theorem II too) only zeros of L-functions belonging to "large" k moduli occur; the fact that for the binary Goldbach problem only the zeros of L-functions with large moduli are relevant (in contrary to the ternary Goldbach-problem) was previously remarked by Linnik on a completely different way (see [8]).

2. It will be enough to prove Theorem I for the Goldbach case. Important role is played by the following theorem of Bombieri-Davenport-Halberstam-Gallagher (³) (see Davenport [3], p. 160) proved by the large sieve method of Linnik.

Denoting by $N(\alpha, T, k, \chi)$ the number of zeros (according to multiplicity) of $L(s, k, \chi)$ in the parallelogram

$$\sigma \geqslant \alpha, \quad |t| \leqslant T \tag{2.1}$$

the inequality

$$G(\alpha, T, X) \overset{\text{def}}{=} \sum_{k \leqslant X} \sum_{\chi^* \bmod k}^{*} N(\alpha, T, k, \chi) < cT(X^2 + XT)^{\frac{4(1-\alpha)}{3-2\alpha}} \log^{10}(X+T) \tag{2.2}$$

holds for $\frac{1}{2} \leqslant \alpha \leqslant 1$, $X > 1$, $T > 1$.

Putting

$$S(\alpha, T, X) \overset{\text{def}}{=} \sum_{k \leqslant X} \frac{|\mu(k)|}{\varphi(k)} \sum_{\chi \bmod k} N(\alpha, T, k, \chi) \tag{2.3}$$

we assert the

LEMMA I. *The inequalities*

(a) $$S(\alpha, T, X) < c\log^{23} X \quad \textit{for} \quad \tfrac{5}{6} \leqslant \alpha \leqslant 1,$$

(b) $$S(\alpha, T, X) < cX^{\frac{5-6\alpha}{3-2\alpha}} \log^{23} X \quad \textit{for} \quad \tfrac{1}{2} \leqslant \alpha \leqslant \tfrac{5}{6}$$

hold for $T \leqslant \log^{10} X$.

(³) That sort of theorems occured at first in Rényi's paper [9]. Essentially the inequality (2.2) occured in Bombieri's paper [1], however with a factor very inconvenient near the line $\sigma = 1$. The elimination of this factor was made possibl[e] by the work of Davenport-Halberstam and Gallagher on the large sieve; the actua[l] inequality (2.2) appeared in Davenport's booklet [3].

For the proof we remark that from (2.3)

$$\begin{aligned} S(\alpha, T, X) &= \sum_{k \leqslant X} \frac{|\mu(k)|}{\varphi(k)} \sum_{k^* | k} \sum_{\chi^* \bmod k^*}^{*} N(\alpha, T, k^*, \chi^*) \\ &= \sum_{k^* \leqslant X} \sum_{\chi^* \bmod k^*}^{*} N(\alpha, T, k^*, \chi^*) \sum_{\substack{k \leqslant X \\ k \equiv 0 \bmod k^*}} \frac{|\mu(k)|}{\varphi(k)} \\ &= \sum_{k^* \leqslant X} \frac{|\mu(k^*)|}{\varphi(k^*)} \sum_{\chi^* \bmod k^*}^{*} N(\alpha, T, k^*, \chi^*) \sum_{\substack{k_1 \leqslant X/k^* \\ (k_1, k^*) = 1}} \frac{|\mu(k_1)|}{\varphi(k_1)} \\ &< c\log X \sum_{k^* \leqslant X} \frac{|\mu(k^*)|}{\varphi(k^*)} \sum_{\chi^* \bmod k^*}^{*} N(\alpha, T, k^*, \chi^*) \\ &< c\log X \log\log X \sum_{k^* \leqslant X} \frac{1}{k^*} \sum_{\chi^* \bmod k^*}^{*} N(\alpha, T, k^*, \chi^*) \\ &= c\log X \log\log X \int_{1/2}^{X} \frac{1}{y} dG(\alpha, T, y) \end{aligned}$$

and hence

$$S(\alpha, T, X) < c\log^2 X \left\{ \frac{G(\alpha, T, X)}{X} + \int_{1/2}^{X} \frac{G(\alpha, T, y)}{y^2} dy \right\}. \tag{2.4}$$

Applying (2.2) we get

$$S(\alpha, T, X) < c\log^{12} X \left\{ X^{\frac{5-6\alpha}{3-2\alpha}} + \int_{1/2}^{X} (y^2 + yT)^{\frac{4(1-\alpha)}{3-2\alpha}} \frac{dy}{y^2} \right\}.$$

The last integral is for $\alpha \geqslant \frac{5}{6}$

$$\int_{1/2}^{T} + \int_{T}^{X} < c\left(T^{\frac{4(1-\alpha)}{3-2\alpha}} \int_{1/2}^{T} \frac{dy}{y} + \int_{1/2}^{X} y^{\frac{2-4\alpha}{3-2\alpha}} dy \right) < c\log^{11} X$$

which proves (a). Similarly with (b).

3. We shall need the following special case of more general results of Gronwall [5] and Titchmarsh [11].

For real z the functions $L(s, k, \chi)$ with $k \leqslant z$ which can vanish in the domain

$$\sigma \geqslant 1 - \frac{A_1}{\log z}, \quad |t| \leqslant \log^3 z \tag{3.1}$$

can only have real zeros.

We shall also use Siegel's theorem [10] in the weaker form that $L(s, k, \chi) \neq 0$ on the segment

$$1-B_1 k^{-1/1000} \leqslant s \leqslant 1 \tag{3.2}$$

with a suitable B_1 (ineffective) constant (as later $B_2, \ldots$).

Hence if

$$1-\frac{A_1}{\log z} \geqslant 1-B_1 k^{-1/1000},$$

i.e.

$$k \leqslant B_2 \log^{1000} z$$

then the rectangle (3.1) contains no zeros of $L(s, k, \chi)$. Choosing

$$z = \exp\left\{\frac{A_1}{100}\cdot\frac{\log M}{\log\log M}\right\}, \qquad M > c$$

we get

Lemma II. *No $L(s, k, \chi)$ functions with*

$$k \leqslant B_2\left(\frac{A_1}{100}\right)^{1000}\left(\frac{\log M}{\log\log M}\right)^{1000} \tag{3.3}$$

can vanish in the domain

$$\sigma \geqslant 1-100\frac{\log\log M}{\log M}, \qquad |t| \leqslant \log^2 M. \tag{3.4}$$

4. Next we assert the

Lemma III. *For $S(a, T, X)$ in (2.3) we assert the inequality*

$$S\left(1-100\frac{\log\log M}{\log M}, \log^2 M, M\right) < B_3 \log^{-500} M.$$

For the proof we use the inequality (2.4), using also the fact that owing to Lemma II

$$G\left(1-100\frac{\log\log M}{\log M}, \log^2 M, X\right) = 0$$

if

$$X \leqslant B_2\left(\frac{A_1}{100}\right)^{1000}\left(\frac{\log M}{\log\log M}\right)^{1000},$$

and also (2.2). These give

$$S\left(1-100\frac{\log\log M}{\log M}, \log^2 M, M\right)$$

$$< c\log^{14} M\left\{\frac{1}{\sqrt{M}} + \int\limits_{B_2\left(\frac{A_1}{100}\right)^{1000}\left(\frac{\log M}{\log\log M}\right)^{1000}}^{M} y^{-2+800\frac{\log\log M}{\log M}}\,dy\right\} < B_3 \log^{-500} M$$

indeed.

5. Now we turn to the proof of (1.9). First we consider the contribution of the L-zeros in

$$\sigma > 1-100\frac{\log\log M}{\log M}, \qquad |t| \leqslant \log^2 M \tag{5.1}$$

to the critical sum in (1.7). As one can easily see its absolute value cannot exceed (with the notation (2.3))

$$c\frac{M}{\log M} S\left(1-100\frac{\log\log M}{\log M}, \log^2 M, M\right)$$

which is

$$o\left(\frac{M}{\log^2 M}\right) \tag{5.2}$$

owing to Lemma III.

Next we consider the contribution of the L-zeros with

$$\frac{5}{6} \leqslant \sigma \leqslant 1-100\frac{\log\log M}{\log M}, \qquad |t| \leqslant \log^2 M$$

to the critical sum. This cannot exceed absolutely

$$c\frac{M^{1-100\frac{\log\log M}{\log M}}}{\log M} S\left(\frac{5}{6}, \log^2 M, M\right)$$

which is

$$o\left(\frac{M}{\log^2 M}\right) \tag{5.3}$$

owing to the inequality (a) of Lemma I.

Next we consider the contribution of the L-zeros with

$$0 < \beta \leqslant \tfrac{1}{3}, \qquad |\gamma| \leqslant \log^2 M$$

to the critical sum. Since here the inequality

$$\left|\frac{n^{\varrho} - n^{\varrho/100}}{\varrho}\right| < cM^{1/3}\log M$$

holds, we get, using the functional equation, for the absolute value of this contribution the upper bound

$$cM^{1/3} S(\tfrac{2}{3}, \log^2 M, M)$$

which is, owing to the inequality (b) of Lemma I,

$$o\left(\frac{M}{\log^2 M}\right). \tag{5.4}$$

Finally we want to estimate the contribution of the k's with $k \leqslant M^{1/2-\varepsilon}$. We shall split first the range

$$\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{5}{6}, \quad |t| \leqslant \log^2 M \tag{5.5}$$

into parallelograms

$$K_\nu: \quad \frac{1}{2}+\frac{\nu-1}{\log M} \leqslant \sigma < \frac{1}{2}+\frac{\nu}{\log M} \overset{\text{def}}{=} \frac{1}{2}+\xi, \quad |t| \leqslant \log^2 M, \qquad 0 \leqslant \frac{\nu-1}{\log M} < \xi \leqslant \frac{1}{3}.$$

The contribution of the zeros in K_ν absolutely cannot exceed

$$\frac{c}{\log M} M^{1/2+\xi} S\left(\frac{1}{2}+\xi-\frac{1}{\log M}, \log^2 M, M^{1/2-\varepsilon}\right)$$

which is, owing to inequality (b) of Lemma I,

$$< \frac{c}{\log M} M^{\frac{1}{2}+\xi} \left(M^{\frac{1}{2}-\varepsilon}\right)^{\frac{1-3\xi}{1-\xi}} \log^{23} M.$$

Since the exponent of M is

$$1-\frac{\xi^2}{1-\xi}-\frac{1-3\xi}{1-\xi}\varepsilon < 1-\frac{\varepsilon}{4}$$

the total contribution of the zeros in (5.5) belonging to K_ν cannot exceed

$$c\log^{22} M \cdot M^{1-\varepsilon/4} c\log M = o\left(\frac{M}{\log^2 M}\right) \tag{5.6}$$

and analogously for the range

$$\tfrac{1}{3} \leqslant \sigma \leqslant \tfrac{1}{2}, \quad |t| \leqslant \log^2 M.$$

This together with (5.2), (5.3), (5.4) and (5.6) proves Theorem I (for (1.9)).

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