On the twin-prime problem III

by

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1. In their fundamental paper [6] Hardy and Littlewood gave in 1922 the first quantitative form of the Goldbach- and twin-prime conjectures (among others). Denoting by \( A_k \) the constant (\(^{(*)}\))

\[
2 \prod_{p \leq 2} \left(1 - \frac{1}{(p-1)^{k+1}}\right)
\]

they announced the

CONJECTURE A. If \( R_k(n) \) stands for the number of Goldbach decompositions of \( n \) then for even \( n \)'s for \( n \rightarrow \infty \) the asymptotic representation

\[
R_k(n) \sim A_k \frac{n}{\log^2 n} \prod_{p \leq n} \frac{p-1}{p-2}
\]

holds.

Equivalent forms of (1.2) are

\[
\sum_{n \leq x} \log p_1 \log p_2 \sim A_k x \prod_{p \leq x} \frac{p-1}{p-2}
\]

or

\[
\sum_{n \leq x} A(n_1) A(n_2) \sim A_k x \prod_{p \leq x} \frac{p-1}{p-2}.
\]

Further

CONJECTURE B. If \( P_k(n) \) stands for the number of such primes \( p \leq n \) for which \( p + d \) is also a prime then for fixed even \( d \) and \( n \rightarrow \infty \) the asym-

\(^{(\ast)}\) The letter \( p \) will be reserved for rational primes, \( A_2, A_4, \ldots \) specified, \( c \) unspecified positive numerical constants, empty sum means 0, empty product 1. The complex variable is \( s = \sigma + it \), \( \zeta(s) \) will stand for a summation with respect to primes characters only, \( k \ast \) for the conductor of \( k \).
holds.

Their paper became fundamental though it contained either conditional or heuristical results only; the conditional results used unproved assumptions on the nontrivial zeros of the Dirichlet \( L(s, k, \chi) \) functions \(^{(2)}\) and when even the strongest assumption, the assumption of the truth of Riemann's "exact" prime-number formula in the theory of distribution of primes and show at the same time they depend only on "small" zeros of \( L(s, k, \chi) \) functions. The aim of this note is to show that using the "large-sieve" method of Linnik \([1]\), the range of nontrivial zeros can considerably be narrowed as to "width"; it is enough to retain zeros "near" to the line \( \sigma = \frac{1}{2} \). More exactly we assert the

**Theorem I.** The formulae (1.7) and (1.8) remain unconditionally true if for an arbitrarily small \( \epsilon > 0 \) the range of summation is replaced by

\[
\frac{1}{2} \leq \beta \leq \frac{1}{2} + \epsilon, \quad |\gamma| \leq \log^2 a, \quad M^{\alpha/\epsilon} \leq k \leq M.
\]

The \( \smallO \)-sign depends also upon \( \epsilon \).

With little extra trouble, using properly the inequality (b) of Lemma I, we could replace \( \frac{1}{2} \) by \( \frac{1}{2} + \epsilon \) (even a bit less). Using an inequality of Bombieri (l.c. [1], p. 225, without detailed proof) the range in (1.9) e.g. could be replaced, as indicated in [12], by

\[
\frac{1}{2} \leq \beta \leq \frac{1}{2} + \epsilon, \quad |\gamma| \leq \log^2 a, \quad M^{\alpha/\epsilon} \leq k \leq M.
\]

The proof of Bombieri's density hypothesis (l.c. [1], p. 205) would even lead to the range

\[
\frac{1}{2} \leq \beta \leq \frac{1}{2}, \quad |\gamma| \leq \log^2 M.
\]

I did not work these further reductions out from two reasons. Firstly it is not desirable to obscure the simplicity of the basic ideas by more technical improvements. Secondly I do not think quite impossible to avoid the necessity to enter into the half-plane \( \sigma < \frac{1}{2} \) at all. To this or other possibilities I shall return in the subsequent papers of this series with my usual low speed however.

Further I mention without proof a further reduction of the domain of summation as to its "height". This runs for simpler case of the twin-primes as follows.

**Theorem II.** For even \( d \) the formula

\[
n^{-N/2} \log^{-1} n \sum_{\text{prime } p_1, p_2 \leq n} \log p_1 \log p_2 \exp \left( - \frac{\log^2 (\sqrt{n}/p_1 p_2)}{\log n} \right) = \sqrt{2} \sum_{\text{prime } p_1 \leq n} \frac{p_1}{p_2} - \sum_{\text{prime } p_2 \leq n} \frac{p_2}{p_1} \sum_{d \leq n} \mu(d) \log k \sum_{m | d} \chi(-d, k) \sum_{n \leq a} \frac{n^2 - n^{2+\epsilon}}{n^{2+\epsilon}}
\]

holds unconditionally; here \( D_1 \) means the rectangle: \( \frac{1}{2} \leq \beta \leq \frac{1}{4}, \quad |\gamma| \leq \frac{1}{4} \).

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In Theorem I (and could have been done in Theorem II too) only zeros of $L$-functions belonging to “large” $a$ moduli occur; the fact that for the binary Goldbach problem only the zeros of $L$-functions with large moduli are relevant (in contrary to the ternary Goldbach-problem) was previously remarked by Linnik on a completely different way (see [5]).

2. It will be enough to prove Theorem I for the Goldbach case. Important role is played by the following theorem of Bombieri-Davenport-Halberstam-Gallagher (3) (see Davenport [3], p. 160) proved by the large sieve method of Linnik.

Denoting by $N(a, T, k, \chi)$ the number of zeros (according to multiplicity) of $L(s, k, \chi)$ in the parallelogram

$$\sigma \geq a, \quad |t| \leq T$$

the inequality

$$G(a, T, X) \text{ def } \sum_{k \leq X} \sum_{a \mod \psi(k)} |\mu(k)| N(a, T, k, \chi)$$

$$< eT(X^a + XT) \frac{11/2}{11/2 - 12/7} \log^b(X + T)$$

holds for $\frac{1}{3} \leq a \leq 1$, $X > 1$, $T > 1$.

Putting

$$S(a, T, X) \text{ def } \sum_{k \leq X} \sum_{a \mod \psi(k)} |\mu(k)| N(a, T, k, \chi)$$

we assert the

**Lemma I. The inequalities**

(a) $S(a, T, X) < \log^{23} X$ for $\frac{1}{3} \leq a \leq 1$,

(b) $S(a, T, X) < eX^{\frac{5-3a}{2}} \log^{33} X$ for $\frac{1}{3} \leq a \leq \frac{1}{2}$

hold for $T \leq \log^{10} X$.

(3) That sort of theorems occurred at first in Rényi’s paper [9]. Essentially the inequality (2.3) occurred in Bombieri’s paper [1], however with a factor very inconvenient near the line $a = 1$. The elimination of this factor was made possible by the work of Davenport-Halberstam and Gallagher on the large sieve; the last inequality (2.3) appeared in Davenport’s booklet [3].

For the proof we remark that from (2.3)

$$S(a, T, X) = \sum_{k \leq X} \sum_{a \mod \psi(k)} |\mu(k)| N(a, T, k, \chi)$$

$$= \sum_{k \leq X} \sum_{a \mod \psi(k)} N(a, T, k, \chi)$$

$$< eT \sum_{k \leq X} \sum_{a \mod \psi(k)} |\mu(k)| N(a, T, k, \chi)$$

$$< e \log X \sum_{k \leq X} \sum_{a \mod \psi(k)} N(a, T, k, \chi)$$

$$< e \log X \log X \sum_{k \leq X} \sum_{a \mod \psi(k)} N(a, T, k, \chi)$$

and hence

$$S(a, T, X) < \log^{23} X \left( \frac{G(a, T, X)}{X} + \int_{i\gamma}^{X} G(a, T, y) \frac{dy}{y^2} \right).$$

Applying (2.2) we get

$$S(a, T, X) < e \log^{43} X \left( X^{\frac{5-3a}{2}} + \int_{i\gamma}^{X} (y^{\frac{5-3a}{2}} + \int_{i\gamma}^{X} \frac{dy}{y^{\frac{5-3a}{2}}} \right).$$

The last integral is for $\alpha > 1$

$$\int_{i\gamma}^{X} \frac{dy}{y^{\frac{5-3a}{2}}} < e \left( T^{\frac{5-3a}{2}} + \int_{i\gamma}^{X} \frac{dy}{y^{\frac{5-3a}{2}}} \right) < \log^{21} X$$

which proves (a). Similarly with (b).

3. We shall need the following special case of more general results of Grenwald [5] and Titchmarsh [11].

For real $\varepsilon$ the functions $L(s, k, \chi)$ with $k \leq \varepsilon$ which can vanish in the domain

$$\sigma \geq 1 - \frac{A}{\log \varepsilon}, \quad |t| \leq \log^2 \varepsilon$$

can only have real zeros.
We shall also use Siegel's theorem \([10]\) in the weaker form that \(L(s, k, \chi) \neq 0\) on the segment
\[
1 - B_1 k^{-1/1000} \leq s \leq 1
\]
with a suitable \(B_1\) (ineffective) constant (as later \(B_1, \ldots\)).

Hence if
\[
1 - \frac{A_1}{\log z} \geq 1 - B_1 k^{-1/1000},
\]
i.e.
\[
k \leq B_1 \log^{1000} z
\]
then the rectangle (3.1) contains no zeros of \(L(s, k, \chi)\). Choosing
\[
z = \exp\left\{ A_1 \frac{\log M}{100 \log \log M} \right\}, \quad M > e
\]
we get

**Lemma II.** No \(L(s, k, \chi)\) functions with
\[
k \leq B_1 \left( \frac{A_1}{100} \right)^{100} \left( \frac{\log M}{\log \log M} \right)^{1000}
\]
can vanish in the domain
\[
\sigma \geq 1 - 100 \frac{\log \log M}{\log M}, \quad |t| \leq \log^2 M.
\]

4. Next we assert the

**Lemma III.** For \(S(a, Z, Y)\) in (2.4) we assert the inequality
\[
S\left( 1 - 100 \frac{\log \log M}{\log M}, \log^2 M, M \right) < B_1 \log^{1000} M.
\]
For the proof we use the inequality (2.4), using also the fact that owing to Lemma II
\[
\theta \left( 1 - 100 \frac{\log \log M}{\log M}, \log^2 M, X \right) = 0
\]
if
\[
X \leq B_1 \left( \frac{A_1}{100} \right)^{1000} \left( \frac{\log M}{\log \log M} \right)^{1000},
\]
and also (2.2). These give
\[
S\left( 1 - 100 \frac{\log \log M}{\log M}, \log^2 M, M \right)
\leq c \log^{14} M \left\{ \frac{1}{YM} + \int \frac{M}{\log (M_{(t, \chi)} \log \log M) \log \log M} dy \right\} < B_1 \log^{1000} M
\]
indeed.

5. Now we turn to the proof of (1.9). First we consider the contribution of the \(L\)-zeros in

\[
|t| \leq \log^{2} M
\]
to the critical sum in (1.7). As one can easily see its absolute value cannot exceed (with the notation (2.3))
\[
\epsilon \frac{M}{\log M} S\left( 1 - 100 \frac{\log \log M}{\log \log M}, \log^{2} M, M \right)
\]
which is
\[
\frac{M}{\log^{2} M}
\]
owing to Lemma III.

Next we consider the contribution of the \(L\)-zeros with
\[
\frac{5}{6} \leq |t| \leq 1 - 100 \frac{\log \log M}{\log \log M}, \quad |t| \leq \log^{2} M
\]
to the critical sum. This cannot exceed absolutely
\[
\epsilon \frac{M}{\log M} S\left( \frac{5}{6}, \log^{2} M, M \right)
\]
which is
\[
\frac{M}{\log^{2} M}
\]
owing to the inequality (a) of Lemma I.

Next we consider the contribution of the \(L\)-zeros with
\[
0 \leq |t| \leq \frac{5}{6}, \quad |t| \leq \log^{2} M
\]
to the critical sum. Since here the inequality
\[
\left| \frac{M - \log \log M}{\log M} \right| < c M^{1/2} \log M
\]
holds, we get, using the functional equation, for the absolute value of this contribution the upper bound
\[
\epsilon M^{1/2} S\left( \frac{5}{6}, \log^{2} M, M \right)
\]
which is, owing to the inequality (b) of Lemma I,

\[
\frac{M}{\log^{2} M}
\]
Finally we want to estimate the contribution of the \( k \)'s with \( k \leq M^{|t|} \). We shall split first the range

\[
\frac{1}{2} \leq \sigma \leq \frac{1}{4}, \quad |t| \leq \log^2 M
\]

into parallelograms

\[
\frac{1}{2} + \frac{r-1}{\log M} \leq \sigma \leq \frac{1}{2} + \frac{r}{\log M}, \quad \frac{1}{2} + \frac{\sigma |t|}{2} \leq \xi, \quad |t| \leq \log^2 M,
\]

\( K_r \),

\[
0 \leq \frac{r-1}{\log M} < \frac{1}{3}.
\]

The contribution of the zeros in \( K_r \) absolutely cannot exceed

\[
\frac{c}{\log M} M^{1+\varepsilon} \left( \frac{1}{2} + \frac{\xi}{\log M} \right)^{\frac{1}{2}} \left( \log^2 M, M^{1-\varepsilon} \right),
\]

which is, owing to inequality (b) of Lemma I,

\[
< \frac{c}{\log M} M^{1+\varepsilon} \left( \log^2 M, M^{1-\varepsilon} \right).
\]

Since the exponent of \( M \) is

\[
1 - \frac{\xi}{1 - \xi} - \frac{1 - S_1}{1 - \xi} < 1 - \frac{\varepsilon}{4}
\]

the total contribution of the zeros in \( K_r \) belonging to \( K \), cannot exceed

\[
\log^{16} M \cdot M^{1-\varepsilon} \log M = \sigma \left( \frac{M}{\log^2 M} \right)
\]

and analogously for the range

\[
\frac{1}{2} \leq \sigma \leq \frac{1}{4}, \quad |t| \leq \log^2 M.
\]

This together with (5.2), (5.3), (5.4) and (5.6) proves Theorem I (for (1.9)).

References