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Diophantine inequalities with a non-integral exponent*

by

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1. Introduction. Employing estimates due to Vinogradov, Davenport and Roth [1] proved the following:

Let $\lambda_1, \ldots, \lambda_s$ be non-zero real numbers not all of the same sign and not all in rational ratios, and let β be any real number. Let m be an integer such that $m \geqslant 12$. Then there is an absolute constant $c > 0$ such that if $s > cm\log m$, the inequality

$$(1)\qquad |\lambda_1 k_1^m + \ldots + \lambda_s k_s^m + \beta| < \varepsilon$$

has infinitely many solutions in positive integers k_i, where $\varepsilon > 0$ is an arbitrary real number.

In this paper we shall show that their methods can be adapted to prove a similar result when the exponent m is not an integer. In this case it no longer needs to be assumed that some ratio λ_i/λ_j is irrational. When m is an integer this assumption is necessary, since then the values of $\lambda_1 k_1^m + \ldots + \lambda_s k_s^m$ are discrete, and so (1) could not be solvable for all β and all $\varepsilon > 0$. It is clearly necessary to assume that not all of the λ_i have the same sign in order to solve (1) for all real β. This condition is also necessary when m is not an integer. Finally, it will be assumed that $m \geqslant 12$ so that Vinogradov's strong estimates for certain exponential sums can be used.

We prove the following

THEOREM. *Let $\lambda_1, \ldots, \lambda_s$ be non-zero real numbers not all of the same sign, and let β be any real number. Let $\tau > 12$ and τ not an integer. Then there is an absolute constant $c > 0$ such that if $s > c\tau\log\tau$, the inequality*

$$(2)\qquad |\lambda_1 k_1^\tau + \ldots + \lambda_s k_s^\tau + \beta| < \varepsilon$$

has infinitely many solutions in positive integers k_i for all $\varepsilon > 0$.

The proof of the above theorem follows that of Davenport and Roth, and the main divergence from their proof is in Lemma 11. The proof

* This paper is part of a thesis submitted as partial fulfilment of the requirements for the degree of Doctor of Philosophy at the University of Michigan.

of this lemma requires a more complicated analysis than the proof of the corresponding lemma in the case where τ is an integer.

Note that it suffices to prove (2) in the special case when $\varepsilon = 1$, as the proof of the general case follows by replacing $\lambda_1, \ldots, \lambda_s, \beta$ with $\lambda_1/\varepsilon, \ldots, \lambda_s/\varepsilon, \beta/\varepsilon$. Without loss of generality, it may be assumed that $\lambda_1, \lambda_2 > 0$ and $\lambda_3 < 0$, since we have assumed that not all of the λ_i are of the same sign. Finally, (2) will be proved for the case when $\beta = 0$, since the proof of the general case is, with minor changes, identical to the proof of the case $\beta = 0$ (see [1]).

For convenience it will be assumed that s is an odd integer, say $s = 2r+1$.

2. Notation and necessary lemmas. We use the following notation:

(3) $$a = |\lambda_3|/4(\lambda_1+\lambda_2),$$

(4) $$\theta = 1/\tau,$$

(5) $$\theta(l) = (1-\theta)^l, \quad 1 \leqslant l \leqslant r-1.$$

We divide the $s = 2r+1$ numbers $\lambda_1, \ldots, \lambda_s$ into three sets:

(6) $$\{\lambda_1, \lambda_2, \lambda_3\},$$

(7) $$\{\mu_1, \ldots, \mu_{r-1}\} = \{\lambda_4, \ldots, \lambda_{r+2}\},$$

(8) $$\{\nu_1, \ldots, \nu_{r-1}\} = \{\lambda_{r+3}, \ldots, \lambda_{2r+1}\}.$$

Let α be a real number which will later be treated as a variable of integration, and let x be a large positive number. Then we define

(9) $$S(\lambda_j \alpha) = \sum_{ax < k^\tau \leqslant 8ax} e(\lambda_j \alpha k^\tau), \quad j = 1, 2,$$

(10) $$T(\lambda_3 \alpha) = \sum_{x/2 < k^\tau \leqslant 2x} e(\lambda_3 \alpha k^\tau),$$

(11) [1] $$U(\mu_l \alpha) = \sum_{x^{\theta(l)} < k^\tau \leqslant 2x^{\theta(l)}} e(\mu_l \alpha k^\tau), \quad 1 \leqslant l \leqslant r-1,$$

(12) $$U(\nu_l \alpha) = \sum_{x^{\theta(l)} < k^\tau \leqslant 2x^{\theta(l)}} e(\nu_l \alpha k^\tau), \quad 1 \leqslant l \leqslant r-1,$$

(13) $$P(\alpha) = S(\lambda_1 \alpha) S(\lambda_2 \alpha) T(\lambda_3 \alpha),$$

(14) $$Q(\alpha) = \prod_{1 \leqslant l \leqslant r-1} U(\mu_l \alpha),$$

(15) $$R(\alpha) = \prod_{1 \leqslant l \leqslant r-1} U(\nu_l \alpha),$$

(16) $$F(x, \alpha) = F(\alpha) = P(\alpha) Q(\alpha) R(\alpha).$$

---

[1] Strictly we should write $U_l(\mu_l \alpha)$, since the range of summation depends on l, but no confusion will arise.

Lemma 1. *For any real number t, we have*

$$\int_{-\infty}^{\infty} e(t\alpha) \left(\frac{\sin \pi\alpha}{\pi \alpha}\right)^2 d\alpha = \begin{cases} 1-|t|, & |t| \leqslant 1, \\ 0, & |t| > 1. \end{cases}$$

Proof. See [2], page 158.

Henceforth we shall denote $\left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2$ by $D(\alpha)$. Then, by Lemma 1 and (16), we have

(17) $$\Delta(x) = \int_{-\infty}^{\infty} F(\alpha) D(\alpha)\, d\alpha = \sum_{|K| < 1} (1-|K|),$$

where

(18) $$K = \lambda_1 h_1^\tau + \ldots + \lambda_3 h_3^\tau + \mu_1 g_1^\tau + \ldots + \mu_{r-1} g_{r-1}^\tau + \nu_1 k_1^\tau + \ldots + \nu_{r-1} k_{r-1}^\tau$$

and where the positive integers h_1, h_2, h_3, g_l, k_l $(1 \leqslant l \leqslant r-1)$ are summed over

(19) $$ax < h_1^\tau, h_2^\tau \leqslant 8ax,$$

(20) $$x/2 < h_3^\tau \leqslant 2ax,$$

(21) $$x^{\theta(l)} < g_l^\tau, k_l^\tau \leqslant 2x^{\theta(l)}.$$

The proof of the theorem depends on showing that $\Delta(x)$ tends to infinity as x tends to infinity; for if the sum in (17) tends to infinity, then there must be infinitely many values of K with $|K| < 1$.

In order to show this, the interval $(-\infty, \infty)$ is divided into three disjoint sets J_1, J_2, J_3. It will be shown that the contribution to (17) from those α in J_1 is $\gg$ [2] $x\big(1+\theta-2(1-\theta)^r\big)$, where we have put

(22) $$x\big(1+\theta-2(1-\theta)^r\big) \stackrel{\mathrm{df}}{=} x^{1+\theta-2(1-\theta)^r}.$$

(For the remainder of the paper the notation $x^y = x(y)$ will be used exclusively for complicated powers of the variable x.) Also, it will be shown that the contribution to (17) from all α in $J_2 \cup J_3$ is $\ll x\big(1+\theta-2(1-\theta)^r-\delta\big)$ for some small, but fixed, $\delta > 0$. This will show that (17) tends to infinity.

---

[2] If $g(x) \geqslant 0$ for all $x > 0$, the notations $f(x) = O(g(x))$ and $f(x) \ll g(x)$ both mean that there is a positive constant c such that $|f(x)| < cg(x)$ for all sufficiently large x.

We define

(23) $$J_1 = \{a\colon |a| \leqslant x(-1+\theta-\theta^2)\},$$

(24) $$J_2 = \{a\colon x(-1+\theta-\theta^2) < |a| \leqslant x\},$$

(25) $$J_3 = \{a\colon |a| > x\},$$

(26) $$L(\lambda_j a) = \theta \int_{ax}^{8ax} e(a\lambda_j \xi)\xi^{\theta-1}d\xi, \quad j = 1, 2,$$

(27) $$M(\lambda_3 a) = \theta \int_{x/2}^{2x} e(a\lambda_3 \xi)\xi^{\theta-1}d\xi.$$

LEMMA 2. *Let $f(t)$ be a real-valued, differentiable function on the interval (t_1, t_2) such that $f'(t)$ is monotonic and $|f'(t)| \leqslant d < 1$. Then*

$$\sum_{t_1<n\leqslant t_2} e(f(n)) = \int_{t_1}^{t_2} e(f(t)) + O(1).$$

Proof. See [3], Section 8, Hilfssatz 2.

LEMMA 3. *For a in J_1, we have*

$$S(\lambda_j a) = L(\lambda_j a) + O(1), \quad j = 1, 2; \quad T(\lambda_3 a) = M(\lambda_3 a) + O(1),$$

for all sufficiently large x.

Proof. The lemma will be proved for $S(\lambda_1 a)$. The proofs of the other two cases are similar. From (9), we have

(28) $$S(\lambda_1 a) = \sum_{(ax)^\theta < k \leqslant (8ax)^\theta} e(\lambda_1 a k^\tau).$$

We now apply Lemma 2 to the right hand side of (28) with $f(t) = \lambda_1 a t^\tau$. For $a \in J_1$, $a \geqslant 0$, we have

(29) $$f''(t) = \lambda_1 a\tau(\tau-1)t^{\tau-2} \geqslant 0,$$

(30) $$\left|\frac{d}{dt}(\lambda_1 a t^\tau)\right| \leqslant |\lambda_1 \tau x(-1+\theta-\theta^2)(8ax)^{1-\theta}| \ll x(-\theta^2),$$

since $(ax)^\theta < t \leqslant (8ax)^\theta$. So, for all sufficiently large x, $\left|\frac{d}{dt}(\lambda_1 a t^\tau)\right| \leqslant \frac{1}{2}$. The conditions of Lemma 2 are satisfied by (29) and (30), so we have

$$S(\lambda_1 a) = \int_{(ax)^\theta}^{(8ax)^\theta} e(\lambda_1 a t^\tau)dt + O(1) = \theta \int_{ax}^{8ax} e(\lambda_1 a\xi)\xi^{\theta-1}d\xi + O(1),$$

where we have made the change of variable $\xi = t^\tau$. This proves the result for $a \in J_1$, $a \geqslant 0$. The proof for $a \in J_1$, $a < 0$ is similar.

LEMMA 4. *For all a*

$$L(\lambda_j a) \ll \min(x^\theta, x^{\theta-1}/|a|), \quad j = 1, 2,$$
$$M(\lambda_3 a) \ll \min(x^\theta, x^{\theta-1}/|a|).$$

Proof. Clearly,

$$|L(\lambda_1 a)| \ll \int_{ax}^{8ax} \xi^{\theta-1}d\xi \ll x^\theta.$$

By the Second Mean Value Theorem, there is a ξ_1 such that $ax < \xi_1 < 8ax$ and

$$L(\lambda_1 a) = (ax)^{\theta-1}\int_{ax}^{\xi_1} e(a\lambda_1\xi)d\xi + (8ax)^{\theta-1}\int_{\xi_1}^{8ax} e(a\lambda_1\xi)d\xi \ll x^{\theta-1}/|a|.$$

This proves the lemma for $L(\lambda_1 a)$. The proofs for $L(\lambda_2 a)$ and $M(\lambda_3 a)$ are similar.

LEMMA 5.

$$\int_{a\in J_1} F(a)D(a)da \gg x(1+\theta-2(1-\theta)^\tau).$$

Proof. The proof of Lemma 5 will proceed in steps. We will prove the following three inequalities:

(31) $$H_1 = \int_{-\infty}^{\infty} Q(a)R(a)L(\lambda_1 a)L(\lambda_2 a)M(\lambda_3 a)D(a)da \gg x(1+\theta-2(1-\theta)^\tau),$$

(32) $$H_2 = \int_{a\notin J_1} Q(a)R(a)L(\lambda_1 a)L(\lambda_2 a)M(\lambda_3 a)D(a)da \ll x(1-\theta-2(1-\theta)^\tau+2\theta^2),$$

(33) $$H_3 = \int_{a\in J_1} Q(a)R(a)[P(a)-L(\lambda_1 a)L(\lambda_2 a)M(\lambda_3 a)]D(a)da \ll x(1-2(1-\theta)^\tau).$$

This suffices for the proof of the lemma, since

$$\int_{a\in J_1} F(a)D(a)da = H_3 + H_1 - H_2.$$

In (31) interchange the order of integration and summation and integrate with respect to a. By Lemma 1, we obtain

(34) $$H_1 = \sum\int A\,d\xi_1 d\xi_2 d\xi_3,$$

where

$$A = A(g_1, \ldots, g_{r-1};\ k_1, \ldots, k_{r-1};\ \xi_1, \xi_2, \xi_3) = \begin{cases} (1-|B|)(\xi_1\xi_2\xi_3)^{\theta-1} & \text{if } |B| \leqslant 1, \\ 0 & \text{if } |B| > 1, \end{cases}$$

and

$$B = \mu_1 g_1^\tau + \ldots + \mu_{r-1} g_{r-1}^\tau + \nu_1 k_1^\tau + \ldots + \nu_{r-1} k_{r-1}^\tau + \lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_3 \xi_3,$$

and where the summation in (34) is over positive integers g_l, k_l satisfying (21) and (22), and the integration in (34) is over the ranges

$$ax < \xi_1,\ \xi_2 \leqslant 8ax, \tag{35}$$

$$x/2 < \xi_3 \leqslant 2x. \tag{36}$$

For $3ax \leqslant \xi_1, \xi_2 \leqslant 6ax$, we have

$$3|\lambda_3|x/4 = 3ax(\lambda_1+\lambda_2) \leqslant \lambda_1\xi_1 + \lambda_2\xi_2 \leqslant 6ax(\lambda_1+\lambda_2) = 3|\lambda_3|x/2. \tag{37}$$

We also have

$$|B - (\lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3)| \ll x^{1-\theta}, \tag{38}$$

since $g_l^\tau, k_l^\tau \ll x^{\theta(l)}$ for $1 \leqslant l \leqslant r-1$. Hence, for $3ax \leqslant \xi_1, \xi_2 \leqslant 6ax$ and for all sufficiently large x, we have

$$(3/4-\eta)|\lambda_3|x \leqslant M - \lambda_3\xi_3 \leqslant (3/2+\eta)|\lambda_3|x$$

by (37) and (38), where η is a very small positive number. Since the range of integration for ξ_3 is $(x/2, 2x]$, there is an interval of length $1/2$, say $\mathfrak{Q} = \mathfrak{Q}(g_1, \ldots, g_{r-1}; k_1, \ldots, k_{r-1}; \xi_1, \xi_2)$ for ξ_3, lying completely within $(x/2, 2x]$, such that $|M| \leqslant 1/2$ on this interval; and this holds for all ξ_1, ξ_2 such that $3ax \leqslant \xi_1, \xi_2 \leqslant 6ax$ and all g_l, k_l satisfying (21) and (22). Hence, for ξ_1, ξ_2, ξ_3 in the region

$$\mathfrak{R} = [3ax, 6ax] \times [3ax, 6ax] \times \mathfrak{Q},$$

we have $A \geqslant (\xi_1\xi_2\xi_3)^{\theta-1}/2$, uniformly for all g_l, k_l satisfying (21) and (22). So for any choice of g_l, k_l, $1 \leqslant l \leqslant r-1$, we have

$$\int_{\mathfrak{R}} A\, d\xi_1 d\xi_2 d\xi_3 \gg x^2 x^{-3(1-\theta)} = x^{-1+3\theta}. \tag{39}$$

The number of terms in the sum in (34) is

$$\gg x\Big(2\theta \sum_{1 \leqslant l \leqslant r-1} (1-\theta)^r\Big) = x\big(2(1-\theta) - 2(1-\theta)^r\big),$$

by (21) and (22). Therefore

$$H_1 \gg x\big(-1+3\theta+2(1-\theta)-2(1-\theta)^r\big) = x\big(1+\theta-2(1-\theta)^r\big),$$

which proves (31).

Turning to the proof of (32), we have the estimates

$$L(\lambda_j\alpha) \ll x^{\theta-1}/|\alpha| \quad \text{for} \quad j = 1, 2,$$

and

$$M(\lambda_3\alpha) \ll x^{\theta-1}/|\alpha|$$

by Lemma 4. Therefore

$$L(\lambda_1\alpha)L(\lambda_2\alpha)M(\lambda_3\alpha) \ll x^{3(\theta-1)}/|\alpha|^3. \tag{40}$$

Now if $Q(\alpha)R(\alpha)$ is multiplied out, the number of terms obtained is

$$\ll x\Big[2\theta \sum_{1 \leqslant l \leqslant r-1} (1-\theta)^l\Big] = x\big(2(1-\theta) - 2(1-\theta)^r\big),$$

and each term is $\leqslant 1$ in absolute value. Therefore

$$|Q(\alpha)R(\alpha)| \ll x\big(2(1-\theta)-2(1-\theta)^r\big).$$

Since $|D(\alpha)| \ll 1$, we have by (40)

$$|H_2| \ll x\big(2(1-\theta)-2(1-\theta)^r\big) \int x^{3(\theta-1)}/|\alpha|^3 d\alpha,$$

where the integration is over those α not in J_1. Therefore

$$|H_2| \ll x\big(2(1-\theta)-2(1-\theta)^r + 3(\theta-1) + 2(1+\theta^2-\theta)\big)$$
$$= x\big(1-\theta-2(1-\theta)^r+2\theta^2\big),$$

which proves (32).

To prove (33), we use Lemma 3 for α in J_1 to obtain

$$S(\lambda_j\alpha) = L(\lambda_j\alpha) + O(1), \quad j = 1, 2,$$

and

$$T(\lambda_3\alpha) = M(\lambda_3\alpha) + O(1).$$

So, by Lemma 4,

$$|P(\alpha) - L(\lambda_1\alpha)L(\lambda_2\alpha)M(\lambda_3\alpha)| \ll \min(x^{2\theta}, x^{2(\theta-1)}/|\alpha|^2).$$

Estimating H_3 by the methods used in estimating H_2, we have

$$|H_3| \ll x\big(2(1-\theta)-2(1-\theta)^r\big)\Big(\int x^{2\theta} d\alpha + \int x^{2(\theta-1)}/|\alpha|^2 d\alpha\Big),$$

where the first integral is over $|\alpha| \leqslant 1/x$, and the second is over $1/x < |\alpha| \leqslant x(-1+\theta-\theta^2)$. Thus

$$|H_3| \ll x\big(2(1-\theta)-2(1-\theta)^r+2\theta-1\big) = x\big(1-2(1-\theta)^r\big),$$

which proves (33) and hence the lemma.

Lemma 6.

$$\int_{-\infty}^{\infty} |S(\lambda_1\alpha)Q(\alpha)|^2 D(\alpha)\, d\alpha \ll x\big(1-(1-\theta)^r\big),$$

$$\int_{-\infty}^{\infty} |T(\lambda_3\alpha)R(\alpha)|^2 D(\alpha)\, d\alpha \ll x\big(1-(1-\theta)^r\big).$$

Proof. The integral appearing in the first line of the lemma is the number of solutions of the inequality

$$(41)\qquad |\lambda_1(h^\tau-m^\tau)+\mu_1(g_1^\tau-m_1^\tau)+\ldots+\mu_{r-1}(g_{r-1}^\tau-m_{r-1}^\tau)|<1$$

in integers h, m, g_l, m_l satisfying

$$(42)\qquad ax<h^\tau,\ m^\tau\leqslant 8ax,$$

$$(43)\qquad x^{\theta(l)}<g_l^\tau,\ m_l^\tau\leqslant 2x^{\theta(l)},\quad 1\leqslant l\leqslant r-1.$$

Now for a particular value of h satisfying (42), the inequality (41) determines m^τ with an error which is $\ll x(\tau-1)$; and hence determines m^τ to within a bounded number of possibilities, i.e., to within $O(1)$. Similarly, given particular values of h^τ, m^τ, g_1^τ satisfying (42) and (43), m_1^τ is determined within $O(1)$ possibilities, etc. Therefore, the number of solutions of (41) in integers h, m, g_l, m_l satisfying (42) and (43) is

$$\ll x\Big(\theta+\theta\sum_{1\leqslant l\leqslant r-1}(1-\theta)^l\Big)=x\big(1-(1-\theta)^r\big),$$

which proves the lemma for the first integral. The second inequality is proved similarly.

LEMMA 7. *Let*

$$G(a)=\sum e\big(af(x_1,\ldots,x_m)\big),$$

where the sum is over a finite set of integer values of $x_1,\ldots,x_m$ and f is real. Then for $W>4$, we have

$$\int_{|a|>W}|G(a)|^2D(a)\,da\leqslant\frac{16}{W}\int_{-\infty}^{\infty}|G(a)|^2D(a)\,da.$$

Proof. See [2], page 82.

LEMMA 8.

$$\int_{a\in J_3}F(a)D(a)\,da\ll x\big(\theta-(1-\theta)^r\big).$$

Proof. Since the number of terms in the sum defining $T(\lambda_3 a)$ is $\ll x^\theta$ and since each such term is $\leqslant 1$ in absolute value, we have $|T(\lambda_3 a)|\ll x^\theta$. Applying this estimate and the Cauchy-Schwarz inequality to the integral

$$\int_{a\in J_3}|Q(a)R(a)S(\lambda_1 a)T(\lambda_3 a)|D(a)\,da,$$

we obtain

$$\int_{a\in J_3}F(a)D(a)\,da\ll x^\theta F_1F_2,$$

where

$$F_1=\Big(\int_{|a|>x}|S(\lambda_1 a)Q(a)|^2D(a)\,da\Big)^{1/2},$$

$$F_2=\Big(\int_{|a|>x}|T(\lambda_3 a)R(a)|^2D(a)\,da\Big)^{1/2}.$$

The result follows by Lemmas 6 and 7.

LEMMA 9 (Vinogradov). *Let N and P be integers, P being large and positive. Let $f(u)$ be a real-valued function on $N\leqslant u\leqslant N+P$, and having in this interval a continuous $(n+1)$-st derivative satisfying*

$$\frac{1}{z}\leqslant\left|\frac{f^{(n+1)}(u)}{(n+1)!}\right|\leqslant\frac{c'}{z},\quad\textit{where}\quad n\geqslant 11\ \textit{and}\ P\ll z\ll P^{2+2/n}.$$

Let

$$S=\sum_{N\leqslant k\leqslant N+P}e\big(f(k)\big).$$

Then we have

$$|S|\ll P^{1-\Phi'},\quad\textit{where}\quad\Phi'=\frac{1}{3n^2\log(125n)}.$$

Proof. See [4], Theorem 2a, page 109.

LEMMA 10 (van der Corput). *Let $f(u)$ be a real-valued function defined on $b+1\leqslant u\leqslant b+p$. For $n\geqslant 2$, suppose $f^{(n)}(u)\geqslant\gamma>0$ (or $f^{(n)}(u)\leqslant\gamma<0$) for all u in $[b+1, b+p]$. Then*

$$\Big|\sum_{b+1\leqslant k\leqslant b+p}e\big(f(k)\big)\Big|\ll p\big[(\gamma/\Gamma^2)^{-1/(2^n-2)}+(\gamma p^n)^{-1/2^{n-1}}+(\gamma p/\Gamma)^{-1/2^{n-1}}\big],$$

where

$$\Gamma=(1/p)\big[f^{(n-1)}(b+p)-f^{(n-1)}(b+1)\big].$$

Proof. See [3], Section 8, Satz 3.

LEMMA 11. *For a in J_2, we have*

$$S(\lambda_1 a)\ll x\big(\theta(1-\Phi)\big),$$

where

$$\Phi=\min\big((1-\theta)/(2^{11}-2),\ 1/3\varphi^2\log(125\varphi)\big),$$

and

$$\varphi=1+[2\tau].$$

Proof. It suffices to prove the lemma for $a\in J_2$, $a>0$. So, for $a>0$, a in J_2, put $a=x(\theta y)$. Hence

$$y=\tau\log a/\log x;$$

and since
$$x(-1+\theta-\theta^2) < a \leqslant x,$$
we have

(44) $$\tau(-1+\theta-\theta^2) < y \leqslant \tau.$$

Now divide the positive values of a in J_2 into two disjoint sets J_{21} and J_{22}, where
$$J_{21} = \{a \in J_2\colon\ a > 0;\ [\tau+y] \geqslant 10\},$$
$$J_{22} = \{a \in J_2\colon\ a > 0\} - J_{21}.$$

For a specific value of a in J_{21} apply Lemma 9 to $S(\lambda_1 a)$ with $f(u) = \lambda_1 a u^{\tau}$ and $n = 1+[\tau+y]$. For this value of n, letting $E = |\lambda_1\tau(\tau-1)\ldots(\tau-n)/(n+1)!|$, we have for all u in $(ax)^{\theta} < u \leqslant (8ax)^{\theta}$, the inequalities
$$Ea^{\theta(\tau-n+1)}x^{\theta(y+\tau-n-1)} \leqslant |f^{(n+1)}(u)/(n+1)!| \leqslant E(8a)^{\theta(\tau-n+1)}x^{\theta(y+\tau-n-1)}.$$
If we put

(45) $$1/z = Ea^{\theta(\tau-n-1)}x^{\theta(y+\tau-n-1)}, \qquad c' = 8^{\theta(\tau-n-1)},$$

then we have

(46) $$1/z \leqslant |f^{(n+1)}(u)/(n+1)!| \leqslant c'/z$$

for all u in $ax < u \leqslant 8ax$. From (45), we have
$$z = x\big(\theta(n+1-y-\tau)\big)/Ea^{\theta(\tau-n-1)} \gg x\big(\theta(n+1-y-\tau)\big) \gg x^{\theta},$$
since $n+1-y-\tau = [\tau+y]+2-(\tau+y) \geqslant 1$. Also, $2+2/n > 2 \geqslant [\tau+y]+ +2-(\tau+y)$. Since P is proportional to x^{θ}, we obtain

(47) $$P^{2+2/n} \gg z \gg P.$$

So by (46) and (47), and since $n \geqslant 11$, we see that the hypotheses of Lemma 9 are satisfied. Therefore,
$$|S(\lambda_1 a)| \ll P^{1-\Phi'} = x\big(\theta(1-\Phi')\big)$$
holds for $a \in J_{21}$ with $n = [\tau+y]+1$, where
$$\Phi' = \frac{1}{3n^2\log(125n)}.$$

Note that as a varies over J_{21}, $y = \tau\log a/\log x$ also changes. Consequently, $n = 1+[\tau+y]$ will also change. So in order to obtain a uniform estimate for $S(\lambda_1 a)$ for all a in J_{21}, we must choose the value of n which is greatest. From (44), this is seen to be $1+[2\tau]$. Therefore, for all a in J_{21},

(48) $$|S(\lambda_1 a)| \ll x\big(\theta(1-\Phi'')\big)$$

where
$$\Phi'' = \frac{1}{3[2\tau+1]^2\log(125[2\tau+1])}.$$

For a in J_{22} we cannot apply the estimate of Lemma 9 to $S(\lambda_1 a)$, since in this case $n \leqslant 10$. We can apply Lemma 10, however. To do this, put $a = x(\theta y)$ as before. Since a is in J_{22}, we have $[\tau+y] \leqslant 9$; i.e., $y < 10-\tau$. From (44) we also have $\tau(\theta-1-\theta^2) < y$. So

(49) $$1-\tau-\theta < y < 10-\tau,$$

since $\theta\tau = 1$.

We apply Lemma 10 to $S(\lambda_1 a)$ with $n = [\tau+y]+2$ and $f(u) = \lambda_1 a u^{\tau}$, and $b = [(ax)^{\theta}]$, $p = [(8ax)^{\theta}]-[(ax)^{\theta}]$.

Note that $x^{\theta}/p \asymp 1$ [3] for $x \geqslant 1$. Also, for all u in $(ax)^{\theta} < u \leqslant (8ax)^{\theta}$, we have
$$|f^{(n)}(u)| = |\lambda_1 a\tau(\tau-1)\ldots(\tau-n+1)u^{\tau-n}|$$
$$\geqslant |\lambda_1 a\tau\ldots(\tau-n+1)a^{1-\theta n}x^{1+\theta y-\theta n}| = \gamma.$$
If we let $\Gamma = (1/p)[f^{(n-1)}(b+p)-f^{(n-1)}(b+1)]$, then we have

(51) $$\Gamma/\gamma \asymp 1,$$

by an elementary calculation. Therefore
$$|S(\lambda_1 a)| \ll p[(\gamma/\Gamma^2)^{-1/(2^n-2)}+(\gamma p^n)^{-1/2^{n-1}}+(\gamma p/\Gamma)^{-1/2^{n-1}}],$$
by Lemma 10 and (50) and (51). Consequently
$$|S(\lambda_1 a)| \ll x^{\theta}\big[x\big((1+\theta y-\theta n)/(2^n-2)\big)+x\big(-(1+\theta y)/2^{n-1}\big)+x(-\theta/2^{n-1})\big].$$
We have by (49)

(52) $$1+\theta y-\theta n = 1-\theta([\tau+y]+2-y) < 1-\theta(\tau+y-1+2-y) = -\theta,$$

(53) $$1+\theta y > \theta-\theta^2.$$

So, by (52) and (53), we have
$$|S(\lambda_1 a)| \ll x^{\theta}x\big(-(\theta-\theta^2)/(2^n-2)\big).$$
As a varies over J_{22}, $[\tau+y]+2 = n$ also varies; and in order to obtain a uniform estimate for $S(\lambda_1 a)$ for all a in J_{22}, we have to find the largest value of n used to estimate $S(\lambda_1 a)$ by Lemma 10. This value is easily seen to be $n = 11$ by the definition of J_{22}. Therefore, for a in J_{22}, we have

(54) $$|S(\lambda_1 a)| \ll x\Big(\theta\big(1-(1-\theta)/(2^{11}-2)\big)\Big).$$

Combining (48) and (54), we have Lemma 11.

[3] The notation $f(t) \asymp 1$ means that there exist constants $0 < c' < c''$ such that $c' \leqslant |f(t)| \leqslant c''$.

LEMMA 12.

$$\int_{a\in J_2} F(a)D(a)\,da \ll x\big(1+\theta-(1-\theta)^r-\theta\Phi\big),$$

where Φ is the value given in the statement of Lemma 11.

Proof. By (16),

$$\int_{a\in J_2} F(a)D(a)\,da = \int_{a\in J_2} P(a)Q(a)R(a)D(a)\,da$$
$$\leqslant \max_{a\in J_2}|S(\lambda_1 a)| \int_{a\in J_2} |T(\lambda_3 a)S(\lambda_2 a)Q(a)R(a)|\,D(a)\,da$$
$$\ll x\big(\theta(1-\Phi)\big)I_1 I_2,$$

where

$$I_1 = \Big\{\int_{a\in J_2} |T(\lambda_3 a)Q(a)|^2 D(a)\,da\Big\}^{1/2}, \qquad I_2 = \Big\{\int_{a\in J_2} |S(\lambda_2 a)R(a)|^2 D(a)\,da\Big\}^{1/2},$$

by Lemma 11 and the Cauchy-Schwarz inequality. But by Lemma 6 we have

$$I_1, I_2 \ll \big(x\big(1-(1-\theta)^r\big)\big)^{1/2}.$$

This proves the lemma.

3. Proof of the theorem. By comparing the results of Lemmas 5, 8, 12, we see that the inequality

$$|\lambda_1 k_1^\tau+\ldots+\lambda_s k_s^\tau| < 1$$

will have infinitely many solutions provided

$$1+\theta-2(1-\theta)^r > \theta-(1-\theta)^r,$$

and

$$1+\theta-2(1-\theta)^r > 1+\theta-(1-\theta)^r-\theta\Phi,$$

or, equivalently, provided

(55) $$\theta\Phi > (1-\theta)^r,$$

since $1 > (1-\theta)^r$.

The problem now is to choose r as a positive integer so that r is as small as possible, yet (55) holds. It is clear from the definition of Φ that

$$\Phi > 1/3(2\tau+1)^2\log\big(125(2\tau+1)\big) = 1/12\tau^2\log(250\tau)V,$$
$$V = (1+1/2\tau)^2\log\big(125(2\tau+1)\big)/\log(250\tau) > 1/108\tau^2\log(250\tau),$$

since $V < 9$ for $\tau \geqslant 12$. So if we choose r so that

(56) $$\theta\Phi > \theta^3/108\log(250\tau) > (1-\theta)^r,$$

then the theorem will hold for $s = 2r+1$. Taking the logarithm of both sides of (56), we obtain

$$3\log\theta-\log(108)-\log\log(250\tau) > r\log(1-\theta),$$

and since $\log(1-\theta) < 0$, we have

$$r > 3\,[\log\theta/\log(1-\theta)]\big(1+o(1)\big) \quad \text{as} \quad \tau\to\infty,$$

since

$$\big(\log(108)+\log\log(250\tau)\big)/3\log\theta = o(1) \quad \text{as} \quad \tau\to\infty.$$

Now

$$3\log\theta/\log(1-\theta) = 3\tau\log\tau\big(1+O(1/\tau)\big) \quad \text{as} \quad \tau\to\infty.$$

So there is an absolute constant $c > 0$ such that for all non-integral $\tau > 12$, and for $s = 2r+1 > c\tau\log\tau$, the inequality

$$|\lambda_1 k_1^\tau+\ldots+\lambda_s k_s^\tau| < 1$$

has infinitely many solutions in positive integers k_i. This proves the theorem.

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