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A theorem on linear forms

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1. Introduction. In a recent paper [2] we proved a theorem on approximation to real numbers by quadratic irrationals. This theorem was an easy consequence of the following result on linear forms which, however, was not stated explicitly in our paper.

Write $\boldsymbol{x} = (x, y, z)$ *and* $|\boldsymbol{x}| = \max(|x|, |y|, |z|)$. *Suppose* ξ *is real and* $L(\boldsymbol{x})$ *is the linear form*

$$L(\boldsymbol{x}) = \xi^2 x + \xi y + z$$

and $P(\boldsymbol{x})$ *the linear form*

$$P(\boldsymbol{x}) = 2\xi x + y.$$

Then there are infinitely many integer points $\boldsymbol{x}$ *which satisfy*

$$|L(\boldsymbol{x})| \leqslant c_1(\xi)|P(\boldsymbol{x})|\, |\boldsymbol{x}|^{-3},$$

where $c_1(\xi)$ *is a certain positive constant depending only on* ξ.

Our proof of this result did not depend on the actual form of $L(\boldsymbol{x})$ and $P(\boldsymbol{x})$. In fact the same conclusion holds if L, P are any two independent linear forms in x, y, z and if $c_1(\xi)$ is replaced by a constant $c_2(L, P)$.

In the present paper we shall prove the following more general theorem. We shall be concerned with integer points

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$$

and we write

$$|\boldsymbol{x}| = \max(|x_1|, \ldots, |x_n|).$$

THEOREM 1. *Suppose* $m \geqslant 1$ *and*

$$n \geqslant m+2. \tag{1}$$

Let $L, P_1, P_2, \ldots, P_m$ *be independent linear forms in* $\boldsymbol{x}$. *Then there are infinitely many integer points* $\boldsymbol{x}$ *with*

$$|L(\boldsymbol{x})| \leqslant c_3(L, P_1, \ldots, P_m)\max\big(|P_1(\boldsymbol{x})|, \ldots, |P_m(\boldsymbol{x})|\big)|\boldsymbol{x}|^{-m-2}. \tag{2}$$

On the other hand, there exist independent linear forms $L, P_1, \ldots, P_m$ such that for every $\varepsilon > 0$ and every integer point $\boldsymbol{x}$ one has

$$|L(\boldsymbol{x})| \geqslant c_4(\varepsilon)\max\big(|P_1(\boldsymbol{x})|, \ldots, |P_m(\boldsymbol{x})|\big)|\boldsymbol{x}|^{-m-2-\varepsilon}. \tag{3}$$

The case $m = 1, n = 3$ is that of our previous paper, and as remarked above, it sufficed to settle the question of approximation to a real number by algebraic numbers of degree at most 2. For the corresponding question with algebraic numbers of degree at most 3, it is natural to define the linear forms

$$L(\boldsymbol{x}) = \xi^3x_1+\xi^2x_2+\xi x_3+x_4 \tag{4}$$

and

$$P(\boldsymbol{x}) = 3\xi^2x_1+2\xi x_2+x_3. \tag{5}$$

Suppose that one knows that the inequality

$$|L(\boldsymbol{x})| \leqslant c_5(\xi)|P(\boldsymbol{x})|\,|\boldsymbol{x}|^{-\gamma} \tag{6}$$

has infinitely many integer solutions $\boldsymbol{x}$. Then one can easily show that there are infinitely many algebraic numbers α of degree at most 3 which satisfy

$$|\xi-\alpha| \leqslant c_6(\xi)H(\alpha)^{-\gamma},$$

where $H(\alpha)$ denotes the height of α. The case $m = 1$, $n = 4$ of Theorem 1 allows us to take $\gamma = 3$. This result is rather weak, and we conjecture that in the special case when L, P are given by (4), (5), one may take γ equal to 4 or at least arbitrarily close to 4. But it is plain from the second assertion of Theorem 1 that any proof of this conjecture must be based on the special shape of the linear forms (4) and (5).

As we shall prove in § 6, our theorem does, however, have the following application to approximation by algebraic numbers.

COROLLARY. *Suppose $h > 1$ and suppose ξ is real and not algebraic of degree at most h. Then there is a number k in the range*

$$1 \leqslant k \leqslant h-1 \tag{7}$$

and there are infinitely many polynomials $f(x)$ of degree h with integer coefficients and with roots $a_1, \ldots, a_h$ which can be ordered so that

$$|(\xi-\alpha_1)(\xi-\alpha_2)\ldots(\xi-\alpha_k)| \leqslant c_8(\xi)H(f)^{-h-1}. \tag{8}$$

Here $H(f)$ denotes the height of f, i.e. the maximum of the absolute values of the coefficients of f.

Probably this result is true for every ξ and for every k between 1 and $h-1$. When $h = 2$, (7) gives $k = 1$, and we obtain essentially the theorem proved in our previous paper [2].

2. An auxiliary theorem. The following theorem will be needed in the proof of the second part of Theorem 1, but is perhaps also of independent interest.

THEOREM 2. *Let $\psi(t)$ be a positive function defined for $t = 1, 2, \ldots$ Suppose that $k > 1$. Then there exist k real numbers $a_1, \ldots, a_k$ such that*

(1) $1, a_1, \ldots, a_k$ are linearly independent over the rationals;

(2) for every sufficiently large t there is an integer q with

$$1 \leqslant q \leqslant t \tag{9}$$

satisfying

$$\|a_iq\| \leqslant \psi(t) \tag{10}$$

for each i in $1 \leqslant i \leqslant k$ with one possible exception $i_0 = i_0(t)$.

Further, the sets $(a_1, \ldots, a_k)$ with these properties are everywhere dense in k dimensional space.

One has to allow an exceptional i_0 if $\psi(t) = o(1/t)$. For suppose (10) were true for each i in $1 \leqslant i \leqslant k$ for every large t. Then for each fixed i, the inequalities

$$1 \leqslant q \leqslant t \quad \text{and} \quad \|a_iq\| \leqslant \varepsilon/t$$

would be soluble for every $\varepsilon > 0$ and every $t > t_0(\varepsilon)$. But by well known results in the theory of continued fractions (see e.g. Koksma [4], pp. 36-37), this is possible only if a_i is rational.

The proof of Theorem 2 is given in § 8.

COROLLARY. *Let n be an integer greater than 2. There exist linear forms*

$$L(\boldsymbol{x}) = a_1x_1+\ldots+a_nx_n,$$

having $L(\boldsymbol{x}) \neq 0$ for every integer point $\boldsymbol{x} \neq 0$, such that for every sufficiently large t there are $n-2$ linearly independent integer points $\boldsymbol{x}$ with

$$|\boldsymbol{x}| \leqslant t \tag{11}$$

and

$$|L(\boldsymbol{x})| \leqslant \psi(t). \tag{12}$$

This extends a theorem of Khintchine [3], who proved the existence of linear forms L such that for every large t the inequalities (11), (12) have at least one integer solution $\boldsymbol{x} \neq 0$.

Proof of the Corollary. We may assume that $\psi(t) \leqslant \frac{1}{2}$. Put $k = n-1$ and let $a_1, \ldots, a_k$ be a k-tuple with the properties described in Theorem 2 and satisfying $|a_i| \leqslant \frac{1}{2}$ $(i = 1, \ldots, k)$. Set $a_n = -1$ and $L(\boldsymbol{x}) = a_1x_1+\ldots+a_kx_k-x_n$. Suppose now that (9) holds and that (10) holds for $i = 1, 2, \ldots, k-1$, say. There are integers $p_1, \ldots, p_{k-1}$ with $|a_iq-p_i| \leqslant \psi(t)$ for $i = 1, \ldots, k-1$. Then

$$|p_i| \leqslant |a_iq|+\psi(t) \leqslant \tfrac{1}{2}t+\tfrac{1}{2} < t \quad (i = 1, \ldots, k-1).$$

The points

$$x_1 = (q, 0, \ldots, 0, 0, p_1),$$
$$x_2 = (0, q, \ldots, 0, 0, p_2),$$
$$\ldots\ldots\ldots\ldots$$
$$x_{k-1} = (0, 0, \ldots, q, 0, p_{k-1})$$

are $k-1 = n-2$ linearly independent solutions of (11) and (12).

3. The sequence of minimal points. It will suffice to prove the first part of Theorem 1 when

(13) $$n = m+2.$$

For in general, when $n \geqslant m+2$, we may assume without loss of generality that the restrictions of the linear forms $L, P_1, \ldots, P_m$ to the subspace of points

$$x = (x_1, \ldots, x_{m+2}, 0, \ldots, 0)$$

are independent. We may then apply the case $n = m+2$ of the theorem to these restrictions.

To prove the first assertion of Theorem 1 we may assume that $L(x) \neq 0$ if $x \neq 0$ is an integer point. We may further assume that the linear forms

$$x_1,\ L(x),\ P_1(x),\ \ldots,\ P_m(x)$$

in $x_1, \ldots, x_n$ are independent. Then if we put

(14) $$\langle x \rangle = \max(|x_1|, |L(x)|, |P_1(x)|, \ldots, |P_m(x)|),$$

we have

(15) $$|x| \ll \langle x \rangle \ll |x|.$$

Here and later, $A \ll B$ will mean that $A \leqslant cB$, where c is a constant depending only on $L, P_1, \ldots, P_m$. In view of (15) it will suffice to prove the existence of infinitely many integer points x having

(16) $$|L(x)| \ll \max(|P_1(x)|, \ldots, |P_m(x)|)\langle x \rangle^{-m-2}.$$

For each real $X > 0$ we consider the finite set of integer points $x \neq 0$ satisfying

$$\langle x \rangle \leqslant X.$$

Assume X to be so large that this set is non-empty. The values of $L(x)$ at the points of this set are distinct, since $L(x)$ does not vanish at any integer point other than the origin. We choose the unique point x for which $|L(x)|$ has its least value and the first non-vanishing coordinate x is positive, and we call this the *minimal point* corresponding to X.

It is obvious that if x is the minimal point corresponding to both X' and X'' it is also the minimal point corresponding to any X between X' and X''. Hence there is a sequence of numbers

(17) $$X_1 < X_2 < \ldots$$

which tend to infinity, and a sequence of points

(18) $$x_1,\ x_2,\ \ldots,$$

such that x_i is the minimal point corresponding to all X in the range $X_i \leqslant X < X_{i+1}$ but to no X outside this range. We obviously have

(19) $$\langle x_i \rangle = X_i.$$

We write for brevity

(20) $$L_i = L(x_i)$$

and

(21) $$Q_i = \max(|P_1(x_i)|, \ldots, |P_m(x_i)|).$$

Plainly

(22) $$|L_1| > |L_2| > \ldots$$

By our construction of the sequences (17) and (18), there is no integer point $x \neq 0$ satisfying

(23) $$\langle x \rangle < X_{i+1} \quad \text{and} \quad |L(x)| < |L_i|.$$

The inequalities (23) define a symmetrical convex set of volume $\gg X_{i+1}^{n-1}|L_i|$, and hence by Minkowski's Theorem we have $X_{i+1}^{n-1}|L_i| \ll 1$, whence

(24) $$|L_i| \ll X_{i+1}^{1-n} = X_{i+1}^{-m-1}.$$

The first assertion of Theorem 1 is proved if we can show that

$$|L_i| \ll Q_i X_i^{-m-2}$$

for infinitely many i. The proof will be indirect; we shall assume that (1)

(25) $$Q_i = o(|L_i| X_i^{m+2}),$$

and we shall eventually reach a contradiction.

By (24) and (25) we have

(26) $$Q_i = o(X_i^{m+2} X_{i+1}^{-m-1}) = o(X_i).$$

(1) In reality we assume an inequality with a constant factor and reach a contradiction if the constant is sufficiently small. But the o notation is a convenience.

Therefore by the definition of the notation $\langle\ \rangle$, and since $|L_i| = o(X_i)$, we have $|x_{i1}| = X_i$ if i is large. Since x_{i1} is non-negative, we have in fact

(27) $$x_{i1} = X_i.$$

4. Two lemmas. The supposition (25) implies, in view of (24), that the $m = n-2$ 'components' $P_1(\boldsymbol{x}_i), \ldots, P_m(\boldsymbol{x}_i)$ of $\boldsymbol{x}_i$ are small compared with X_i. Thus in a sense the two most important components are $x_{i1} = X_i$, which is large, and $L(\boldsymbol{x}_i) = L_i$, which is small. There is a limited degree of analogy between the sequence of points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ and a sequence of convergents in the theory of continued fractions. It is this general idea which underlies the two following lemmas.

LEMMA 1. *The signs of the L_i alternate when i is large.*

Proof. Consider the point

$$\boldsymbol{y} = \boldsymbol{x}_{i+1} - \boldsymbol{x}_i.$$

It follows from (27) that

$$0 < y_1 = x_{i+1,1} - x_{i1} = X_{i+1} - X_i < X_{i+1}.$$

Since $|L(\boldsymbol{y})| \ll 1$ and since

$$\max\big(|P_1(\boldsymbol{y})|, \ldots, |P_m(\boldsymbol{y})|\big) \ll Q_i + Q_{i+1} = o(X_{i+1}),$$

we have $\langle \boldsymbol{y} \rangle < X_{i+1}$ provided i is large. Since (23) has no solution, we must have

$$|L_{i+1} - L_i| = |L(\boldsymbol{y})| \geqslant |L_i|.$$

Since $0 < |L_{i+1}| < |L_i|$, the numbers L_i, L_{i+1} have opposite signs.

LEMMA 2. *Suppose i is large and*

(28) $$Q_{i+1} \leqslant \tfrac{1}{2} X_i.$$

Then

(29) $$\boldsymbol{x}_{i+1} = t\boldsymbol{x}_i + \boldsymbol{x}_{i-1},$$

where t is a positive integer.

Proof. Define positive integers t, u by

(30) $$t = [X_{i+1}/X_i], \qquad u = [|L_{i-1}|/|L_i|],$$

and integer points $\boldsymbol{y}, \boldsymbol{z}$ by

(31) $$\boldsymbol{y} = \boldsymbol{x}_{i+1} - t\boldsymbol{x}_i, \qquad \boldsymbol{z} = \boldsymbol{x}_{i-1} + u\boldsymbol{x}_i.$$

Clearly $\boldsymbol{x}_{i+1}$ and $\boldsymbol{x}_i$ are independent, and therefore $\boldsymbol{y} \neq 0$. Similarly $\boldsymbol{z} \neq 0$. We have

$$\max\big(|L(\boldsymbol{y})|, |P_1(\boldsymbol{y})|, \ldots, |P_m(\boldsymbol{y})|\big) \leqslant \max\big(|L_{i+1}| + (X_{i+1}/X_i)|L_i|, Q_{i+1} + (X_{i+1}/X_i)Q_i\big).$$

By (24), (26) and (28) this maximum is

$$\leqslant \tfrac{1}{2}X_i + o(X_{i+1}X_i^{-1}X_i^{m+2}X_{i+1}^{-m-1}) < \tfrac{3}{4}X_i,$$

if i is large. By (27) we have

(32) $$0 \leqslant y_1 = X_{i+1} - tX_i < X_i,$$

and therefore

$$\langle \boldsymbol{y} \rangle < X_i.$$

Since (23) has no non-trivial solution for any i, we have

(33) $$|L(\boldsymbol{y})| \geqslant |L_{i-1}|.$$

Next, observe that

$$\max\big(|L(\boldsymbol{z})|, |P_1(\boldsymbol{z})|, \ldots, |P_m(\boldsymbol{z})|\big) \leqslant \max\big(|L_{i-1}| + u|L_i|, Q_{i-1} + uQ_i\big).$$

By (24), (25) and (26) this maximum is

$$o(X_{i-1} + u|L_i|X_i^{m+2}) = o(X_{i+1} + |L_{i-1}|X_i^{m+2}) = o(X_{i+1}),$$

and is therefore less than X_{i+1} if i is large.

Since L_i and L_{i-1} have opposite signs, we have

(34) $$|L(\boldsymbol{z})| = |L_{i-1} + uL_i| < |L_i|.$$

Using again the fact that (23) has no non-trivial solution, we obtain

$$\langle \boldsymbol{z} \rangle \geqslant X_{i+1}.$$

Since $|L(\boldsymbol{z})|, |P_1(\boldsymbol{z})|, \ldots, |P_m(\boldsymbol{z})|$ are smaller than X_{i+1}, we must have

(35) $$|z_1| \geqslant X_{i+1}.$$

The inequality (33) yields

$$|L_{i-1}| \leqslant |L_{i+1}| + t|L_i|,$$

and therefore

$$u \leqslant |L_{i-1}|/|L_i| \leqslant t + |L_{i+1}|/|L_i| < t+1.$$

On the other hand, (35) yields

$$X_{i+1} \leqslant X_{i-1} + uX_i,$$

and therefore

$$t \leqslant X_{i+1}/X_i \leqslant u + X_{i-1}/X_i < u+1.$$

Since t and u are integers, we obtain

(36) $$t = u.$$

Consider the point

(37) $$\boldsymbol{w} = \boldsymbol{x}_{i+1} - t\boldsymbol{x}_i - \boldsymbol{x}_{i-1} = \boldsymbol{y} - \boldsymbol{x}_{i-1} = \boldsymbol{x}_{i+1} - \boldsymbol{z}.$$

By what we have said about $\boldsymbol{y}$ and by (26),

$$\max\big(|L(\boldsymbol{w})|, |P_1(\boldsymbol{w})|, \ldots, |P_m(\boldsymbol{w})|\big) < \tfrac{3}{4}X_i + o(X_{i-1}) < X_i,$$

if i is large. By (32) we have

$$|w_1| = |y_1 - X_{i-1}| < X_i,$$

and therefore

$$\langle \boldsymbol{w} \rangle < X_i < X_{i+1}.$$

By virtue of (34) and since $L(\boldsymbol{z})$ has the same sign as L_{i-1} and hence the same sign as L_{i+1}, we obtain

$$|L(\boldsymbol{w})| = |L_{i+1} - L(\boldsymbol{z})| < |L_i|.$$

The point $\boldsymbol{w}$ satisfies the inequalities (23), hence it must be the origin. This proves the lemma.

5. The first assertion of Theorem 1. Suppose (28) is satisfied for some large i. Then by (27) and by Lemma 2,

$$|X_i L_{i+1} - X_{i+1} L_i| = |X_{i-1} L_i - X_i L_{i-1}|$$

and

$$|P_j(\boldsymbol{x}_i) L_{i+1} - P_j(\boldsymbol{x}_{i+1}) L_i| = |P_j(\boldsymbol{x}_{i-1}) L_i - P_j(\boldsymbol{x}_i) L_{i-1}| \quad (j = 1, \ldots, m).$$

Now suppose (28) is satisfied for every i in the interval $h < i < k$. Then

(38) $$|X_h L_{h+1} - X_{h+1} L_h| = |X_{k-1} L_k - X_k L_{k-1}|$$

and

$$|P_j(\boldsymbol{x}_h) L_{h+1} - P_j(\boldsymbol{x}_{h+1}) L_h| = |P_j(\boldsymbol{x}_{k-1}) L_k - P_j(\boldsymbol{x}_k) L_{k-1}| \quad (j = 1, \ldots, m),$$

whence, on choosing j so that $|P_j(\boldsymbol{x}_{h+1})| = Q_{h+1}$, we obtain

(39) $$Q_{h+1}|L_h| \leqslant Q_h |L_{h+1}| + Q_{k-1}|L_k| + Q_k |L_{k-1}|.$$

These relations and inequalities remain trivially true if $k = h+1$.

The left hand side of (38) equals $|L(X_h \boldsymbol{x}_{h+1} - X_{h+1}\boldsymbol{x}_h)|$, and therefore is not zero. On the other hand, by (24), the right hand side of (38) tends to zero as k tends to infinity. Hence (28) cannot be satisfied for every i greater than some i^*, and so there are infinitely many integers i for which (28) does not hold.

Now suppose (28) is satisfied for every i in the interval $h < i < k$ but not for $i = h$ or $i = k$. Then $Q_{h+1} > \frac{1}{2}X_h$, and by (39) and by (24), (25) we get

$$\begin{aligned}\tfrac{1}{2}X_h|L_h| &\leqslant Q_h|L_{h+1}| + Q_{k-1}|L_k| + Q_k|L_{k-1}| \\ &= o(X_h^{m+2}|L_h L_{h+1}| + X_k^{m+2}|L_{k-1}L_k|) = o(X_h|L_h| + X_k|L_k|).\end{aligned}$$

Therefore

$$X_h|L_h| = o(X_k|L_k|),$$

and if h and hence k is large, this implies

$$X_h|L_h| < X_k|L_k|.$$

But this is impossible, since it leads to an infinite sequence of values of ν for which $X_\nu|L_\nu|$ increases, whereas we know that $X_\nu|L_\nu| \to 0$ as $\nu \to \infty$ by (24). By what was said above in § 3, this contradiction proves the first assertion of Theorem 1.

6. The corollary to Theorem 1. Let ξ be a real number which is not algebraic of degree at most h, and put $n = h+1$. Write

$$\begin{aligned}L(\boldsymbol{x}) &= \xi^h x_1 + \xi^{h-1}x_2 + \ldots + \xi x_h + x_n, \\ P_1(\boldsymbol{x}) &= h\xi^{h-1}x_1 + (h-1)\xi^{h-2}x_2 + \ldots + x_h, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ P_{h-1}(\boldsymbol{x}) &= h!\,\xi x_1 + (h-1)!\,x_2.\end{aligned}$$

We may apply the first part of Theorem 1 with $n = h+1$ and $m = h-1 = n-2$. There are infinitely many integer points $\boldsymbol{x}$ which satisfy (2). There is an integer k in the interval (7) such that

(40) $$|L(\boldsymbol{x})| \leqslant c_9(\xi)|P_k(\boldsymbol{x})|\,|\boldsymbol{x}|^{-h-1}$$

is satisfied by infinitely many integer points $\boldsymbol{x}$.

For every such $\boldsymbol{x}$, let $g_{\boldsymbol{x}}$ be the polynomial

$$g_{\boldsymbol{x}}(t) = t^h x_1 + t^{h-1}x_2 + \ldots + t x_h + x_n,$$

and let $d = d(\boldsymbol{x})$ be the degree of $g_{\boldsymbol{x}}$. By (40) we have

(41) $$0 < |g_{\boldsymbol{x}}(\xi)| \leqslant c_9(\xi)|g_{\boldsymbol{x}}^{(k)}(\xi)|\,|\boldsymbol{x}|^{-h-1}.$$

Therefore $g_{\boldsymbol{x}}^{(k)}$ is not identically zero and $d = d(\boldsymbol{x}) \geqslant k$. Let $g_{\boldsymbol{x}}(t)$ have the roots $t = a_1, \ldots, a_d$. Then the identity

$$\frac{g_{\boldsymbol{x}}^{(k)}(t)}{g_{\boldsymbol{x}}(t)} = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant d} \frac{k!}{(t-a_{i_1})\ldots(t-a_{i_k})}$$

holds. We can order the roots $a_1, \ldots, a_d$ of $g_{\boldsymbol{x}}$ in such a way that, when $t = \xi$, the term

$$\frac{k!}{(\xi - a_1)\ldots(\xi - a_k)}$$

is the greatest in absolute value. Then, by (41),

$$|(\xi - a_1)\ldots(\xi - a_k)| \leqslant c_8(\xi)|\boldsymbol{x}|^{-h-1} = c_8(\xi)H(g_{\boldsymbol{x}})^{-h-1}.$$

The polynomials f_x defined by

$$f_x(t) = t^{h-d(x)} g_x(t)$$

have all the properties described in the corollary.

7. The second assertion of Theorem 1. We are now going to prove the existence of forms $L, P_1, \ldots, P_m$ such that (3) holds for every integer point x. This is very simple if $n = m+2$. For there certainly exist linear forms $L(x) = a_1x_1 + \ldots + a_nx_n$ with the property that

$$|L(x)| \geqslant c_{10}|x|^{-n+1} = c_{10}|x|^{-m-1}$$

for every integer point x (see, e.g. [1], ch. I, Theorem 8). For such forms L and for any given forms $P_1, \ldots, P_m$ one has

$$|L(x)| \geqslant c_{11}\max\big(|P_1(x)|, \ldots, |P_m(x)|\big)|x|^{-m-2}.$$

We may therefore assume that

(42) $$n > m+2.$$

LEMMA 3. *Let ϱ be real, $\varepsilon > 0$ and $m \geqslant 1$. Almost every* (2) *m-tuple $\beta_1, \ldots, \beta_m$ of real numbers has only finitely many integer points $y = (y_0, y_1, \ldots, y_m)$ with*

$$(y_1, \ldots, y_m) \neq (0, \ldots, 0)$$

and

(43) $$\|\varrho y_0 + \beta_1 y_1 + \ldots + \beta_m y_m\| \leqslant |y|^{-m-1-\varepsilon}.$$

Proof. We may restrict $\beta_1, \ldots, \beta_m$ to the unit cube $0 \leqslant \beta_i \leqslant 1$ $(i = 1, \ldots, m)$. For any particular y as above, the m-tuples $\beta_1, \ldots, \beta_m$ satisfying (43) form a set of measure $\ll |y|^{-m-1-\varepsilon}$. The number of points y having $|y| = r$, where r is any positive integer, is $\ll r^m$. Hence for given r the set of m-tuples $\beta_1, \ldots, \beta_m$ satisfying (43) for some y with $|y| = r$ is of measure $\ll r^{-1-\varepsilon}$. Since

$$\sum_{r=1}^{\infty} r^{-1-\varepsilon}$$

is convergent, the m-tuples $\beta_1, \ldots, \beta_m$ which satisfy (43) for infinitely many points y as above form a set of measure zero. This proves the lemma.

Now we apply Theorem 2 with

(44) $$k = n-m-1$$

and with

$$\psi(t) = t^{-\log t}.$$

Note that $k > 1$ by (42).

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(2) In the sense of measure theory.

Let $a_1, \ldots, a_k$ be a k-tuple with the properties stated in Theorem 2. By Lemma 3 there are numbers $\beta_1, \ldots, \beta_m$ such that (43) has only finitely many solutions

$$y = (y_0, y_1, \ldots, y_m) \quad \text{with} \quad (y_1, \ldots, y_m) \neq (0, \ldots, 0),$$

for each of the numbers $\varrho = a_1, \varrho = a_2, \ldots, \varrho = a_k$, and for every $\varepsilon > 0$. Put

$$L(x) = a_1x_1 + \ldots + a_kx_k + x_{k+1} + \beta_1x_{k+2} + \ldots + \beta_mx_n,$$
$$P_1(x) = \mu_1(a_1x_1 + \ldots + a_kx_k + x_{k+1}) + \mu_{11}x_{k+2} + \ldots + \mu_{1m}x_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P_m(x) = \mu_m(a_1x_1 + \ldots + a_kx_k + x_{k+1}) + \mu_{m1}x_{k+2} + \ldots + \mu_{mm}x_n,$$

where $\mu_1, \ldots, \mu_m, \mu_{11}, \ldots, \mu_{mm}$ are any numbers with $\mu_1, \ldots, \mu_m \neq 0, \ldots, 0$ and such that the forms $L, P_1, \ldots, P_m$ are independent. In particular one can choose the numbers β_j, μ_j, μ_{ij} so that the coefficients of each of the forms $L, P_1, \ldots, P_m$ are linearly independent over the rationals.

Obviously, an inequality of the type (3) holds when $x_{k+2} = \ldots \ldots = x_n = 0$, since then $L, P_1, \ldots, P_m$ are proportional. We are now going to show that

(45) $$|L(x)| \geqslant |x|^{-m-1-\varepsilon}$$

if

(46) $$(x_{k+2}, \ldots, x_n) \neq (0, \ldots, 0)$$

and if $|x| > c_{12}(\varepsilon)$. This will complete the proof of Theorem 1, since

$$\max\big(|P_1(x)|, \ldots, |P_m(x)|\big) \ll |x|.$$

Suppose x is an integer point satisfying (46), and set $t = [|x|^{\varepsilon/8m}]$. If $|x|$ and hence t is large, there will be an integer q with

(47) $$1 \leqslant q \leqslant t \leqslant |x|^{\varepsilon/8m}$$

and

(48) $$\|a_i q\| \leqslant \psi(t) = t^{-\log t} \leqslant |x|^{-m-3}$$

for at least $k-1$ of the k numbers $i = 1, 2, \ldots, k$. This follows from our choice of $a_1, \ldots, a_k$. Let us assume that (48) is true for $i = 1, 2, \ldots, k-1$. Then

$$\begin{aligned}|qL(x)| &\geqslant \|qL(x)\| \\ &= \|a_1qx_1 + \ldots + a_{k-1}qx_{k-1} + a_kqx_k + qx_{k+1} + \beta_1qx_{k+2} + \ldots + \beta_mqx_n\| \\ &\geqslant \|a_kqx_k + \beta_1qx_{k+2} + \ldots + \beta_mqx_n\| - \|a_1qx_1\| - \ldots - \|a_{k-1}qx_{k-1}\| \\ &\geqslant \|a_kqx_k + \beta_1qx_{k+2} + \ldots + \beta_mqx_n\| - k|x|^{1-m-3}.\end{aligned}$$

By our choice of $\beta_1, \ldots, \beta_m$ and by (46), we have further

$$|L(\boldsymbol{x})| \geqslant q^{-1}\{(q|\boldsymbol{x}|)^{-m-1-\varepsilon/4} - k|\boldsymbol{x}|^{-m-2}\}$$
$$\geqslant |\boldsymbol{x}|^{-m-1-3\varepsilon/4} - k|\boldsymbol{x}|^{-m-2} \geqslant |\boldsymbol{x}|^{-m-1-\varepsilon}$$

if $|\boldsymbol{x}|$ is large, say if $|\boldsymbol{x}| > c_{12}(\varepsilon)$. This proves the result stated earlier.

8. Proof of Theorem 2. To avoid a cumbersome notation we shall prove the theorem only in the case $k = 3$, which is quite typical. We shall write α, β, γ instead of $\alpha_1, \alpha_2, \alpha_3$, in the enunciation of the theorem. The numbers $1, \alpha, \beta, \gamma$ are linearly independent over the rationals if and only if the point (α, β, γ) of three dimensional space lies on no rational plane Π. Arrange the rational planes into a sequence $\Pi_1, \Pi_2, \ldots$

Given an interval $I: a \leqslant x \leqslant b$, let I^- be the interval $a \leqslant x \leqslant a + \frac{1}{3}(b-a)$ and I^+ the interval $b - \frac{1}{3}(b-a) \leqslant x \leqslant b$. The length of an interval I will be denoted by $|I|$. We shall employ boxes

$$I \times J \times K$$

consisting of points (α, β, γ) with $\alpha \in I$, $\beta \in J$, $\gamma \in K$, where I, J, K are closed intervals. Given a plane Π and given any intervals I, J, K there exist signs ϱ, σ, τ such that Π is disjoint from the box

$$I^\varrho \times J^\sigma \times K^\tau;$$

for Π cannot properly intersect all the 8 cubes obtained by bisecting I, J and K.

Set

$$\omega(t) = t^{-2}\min\big(1, \psi(1), \psi(2), \ldots, \psi(t^2)\big). \tag{49}$$

Let A_0, B_0, C_0 be three closed intervals. It will suffice to prove the existence of points (α, β, γ) with the desired properties lying in the box

$$A_0 \times B_0 \times C_0.$$

We shall construct inductively rationals

$$\frac{u_0}{a_0}, \frac{v_0}{b_0}, \frac{w_0}{c_0}, \frac{u_1}{a_1}, \frac{v_1}{b_1}, \frac{w_1}{c_1}, \frac{u_2}{a_2}, \ldots \tag{50}$$

and closed intervals

$$A_1, B_1, C_1, A_2, B_2, C_2, A_3, \ldots \tag{51}$$

First, choose integers a_0, b_0, c_0 such that

$$a_0 \geqslant 9|A_0|^{-1}, \quad b_0 \geqslant \min(a_0+1, 9|B_0|^{-1}), \quad c_0 \geqslant \min(b_0+1, 9|C_0|^{-1}).$$

There are signs ϱ, σ, τ such that $A_0^\varrho \times B_0^\sigma \times C_0^\tau$ is disjoint from Π_1. Choose the integer u_0 such that the interval $|\alpha - u_0/a_0| \leqslant 1/a_0$ lies in A_0^ϱ. Similarly, choose v_0 and w_0 such that the interval $|\beta - v_0/b_0| \leqslant 1/b_0$ lies in B_0^σ and the interval $|\gamma - w_0/c_0| \leqslant 1/c_0$ lies in C_0^τ. Then the set of points (α, β, γ) having

$$\left|\alpha - \frac{u_0}{a_0}\right| \leqslant \frac{1}{a_0}, \quad \left|\beta - \frac{v_0}{b_0}\right| \leqslant \frac{1}{b_0}, \quad \left|\gamma - \frac{w_0}{c_0}\right| \leqslant \frac{1}{c_0}$$

lies in $A_0 \times B_0 \times C_0$ and is disjoint from Π_1.

Suppose now that the intervals A_m, B_m, C_m and the rationals $\frac{u_m}{a_m}, \frac{v_m}{b_m}, \frac{w_m}{c_m}$ have already been constructed in such a way that the set of points (α, β, γ) having

$$\left|\alpha - \frac{u_m}{a_m}\right| \leqslant \frac{1}{a_m}, \quad \left|\beta - \frac{v_m}{b_m}\right| \leqslant \frac{1}{b_m}, \quad \left|\gamma - \frac{w_m}{c_m}\right| \leqslant \frac{1}{c_m} \tag{52}$$

lies in $A_m \times B_m \times C_m$ and is disjoint from Π_{m+1}. Suppose $a_m < b_m < c_m$.

Let A_{m+1} be the interval

$$\left|\alpha - \frac{u_m}{a_m}\right| \leqslant \omega(c_m).$$

Choose a_{m+1} such that

$$a_{m+1} \geqslant \min(c_m + 1, 9|A_{m+1}|^{-1}).$$

Let B_{m+1} be the interval

$$\left|\beta - \frac{v_m}{b_m}\right| \leqslant \omega(a_{m+1}).$$

Choose b_{m+1} such that

$$b_{m+1} \geqslant \min(a_{m+1} + 1, 9|B_{m+1}|^{-1}).$$

Let C_{m+1} be the interval

$$\left|\gamma - \frac{w_m}{c_m}\right| \leqslant \omega(b_{m+1}),$$

and choose c_{m+1} such that

$$c_{m+1} \geqslant \min(b_{m+1} + 1, 9|C_{m+1}|^{-1}).$$

Since $\omega(t) \leqslant 1/t$ the box $A_{m+1} \times B_{m+1} \times C_{m+1}$ is contained in the box (52), and hence is contained in $A_m \times B_m \times C_m$ and is disjoint from Π_{m+1}. Since $a_{m+1}, b_{m+1}, c_{m+1}$ were chosen sufficiently large, there are integers $u_{m+1}, v_{m+1}, w_{m+1}$ such that the box

$$\left|\alpha - \frac{u_{m+1}}{a_{m+1}}\right| \leqslant \frac{1}{a_{m+1}}, \quad \left|\beta - \frac{v_{m+1}}{b_{m+1}}\right| \leqslant \frac{1}{b_{m+1}}, \quad \left|\gamma - \frac{w_{m+1}}{c_{m+1}}\right| \leqslant \frac{1}{c_{m+1}}$$

lies in $A_{m+1} \times B_{m+1} \times C_{m+1}$ and is disjoint from Π_{m+2}.

This concludes our construction of the sequences (50) and (51).

The proof of Theorem 2 is now completed as follows. There is a unique point (α, β, γ) which lies in all the boxes $A_m \times B_m \times C_m$ $(m = 1, 2, \ldots)$. This point lies on no rational plane Π.

Suppose now that t lies in some interval

$$a_m^2 \leqslant t < b_m^2 \quad (m > 0). \tag{53}$$

Since α is in A_{m+1} and γ is in C_m, we have

$$\left|\alpha - \frac{u_m}{a_m}\right| \leqslant \omega(c_m) \leqslant \omega(b_m), \qquad \left|\gamma - \frac{w_{m-1}}{c_{m-1}}\right| \leqslant \omega(b_m).$$

The number $q = c_{m-1}a_m$ satisfies $1 \leqslant q \leqslant t$ and

$$\max(\|\alpha q\|, \|\gamma q\|) \leqslant q\omega(b_m) = qb_m^{-2}\min\big(1, \psi(1), \ldots, \psi(t), \ldots, \psi(b_m^2)\big) \leqslant \psi(t).$$

Next, suppose that t lies in some interval

$$b_m^2 \leqslant t < c_m^2 \quad (m \geqslant 0). \tag{54}$$

Since α lies in A_{m+1} and β in B_{m+1}, one has

$$\left|\beta - \frac{v_m}{b_m}\right| \leqslant \omega(a_{m+1}) \leqslant \omega(c_m).$$

The number $q = a_m b_m$ satisfies $1 \leqslant q \leqslant t$ and

$$\max(\|\alpha q\|, \|\beta q\|) \leqslant q\omega(c_m) \leqslant qc_m^{-2}\psi(t) \leqslant \psi(t).$$

Finally, suppose t lies in some interval

$$c_m^2 \leqslant t < a_{m+1}^2 \quad (m \geqslant 0). \tag{55}$$

Since β is in B_{m+1} and γ is in C_{m+1} the inequalities

$$\left|\beta - \frac{v_m}{b_m}\right| \leqslant \omega(a_{m+1}), \qquad \left|\gamma - \frac{w_m}{c_m}\right| \leqslant \omega(b_{m+1}) \leqslant \omega(a_{m+1})$$

hold. The number $q = b_m c_m$ satisfies $1 \leqslant q \leqslant t$ and

$$\max(\|\beta q\|, \|\gamma q\|) \leqslant q\omega(a_{m+1}) \leqslant qa_{m+1}^{-2}\psi(t) \leqslant \psi(t).$$

Since

$$b_0 < c_0 < a_1 < b_1 < \ldots,$$

every integer $t \geqslant b_0$ is in an interval of the type (53), (54) or (55). The proof of Theorem 2 is therefore complete.

By a slight change of argument one could prove the existence of continuum many triples (α, β, γ) or in general of continuum many k-tuples $(\alpha_1, \ldots, \alpha_k)$ with the desired properties.

Note added in proof. We have observed that the inequality (2), in the first assertion of Theorem 1, can be replaced by a slightly stronger inequality of the form

$$|L(\boldsymbol{x})| \leqslant \{|\boldsymbol{x}|^{-1}\max(|P_1(\boldsymbol{x})|, \ldots, |P_m(\boldsymbol{x})|)\}^{\gamma}|\boldsymbol{x}|^{-m-1},$$

where $\gamma = \gamma(m) > 1$. Some modification of the proof is needed.

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