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Extensions of a theorem of Hardy

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The functional equation for the Riemann zeta function may be written, setting

$$f(s) = \pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s), \quad \text{as} \quad f(s) = f(1-s).$$

Since the functions defining $f(s)$ are real on the real axis, by the Schwartz reflection principle, $f(s)$ assumes complex conjugate values at complex conjugate points; this together with the functional equation shows that $f(s)$ is real valued on the critical line $\sigma = \frac{1}{2}$. Hardy has shown in [1] that the real valued function $f(\frac{1}{2}+it)$ vanishes infinitely often as $t\to\infty$, and significant quantitative results have been obtained, first by Hardy-Littlewood [2] and then by A. Selberg [4]. These zeros of $f(\frac{1}{2}+it)$ must of course be zeros of $\zeta(s)$.

The purpose of this paper is to show, by simple extensions of ideas of Hardy and Ramanujan, that given any real λ, $0<\lambda<1$, the real and imaginary parts of $f(\lambda+it)$ vanish infinitely often as $t\to\infty$. This is very far from determining whether or not $f(s)$ itself ever vanishes on any $\sigma=\lambda$, $\lambda\neq\frac{1}{2}$.

1. We begin by writing, for $0<\lambda<1$,

$$\operatorname{Re} f(\lambda+it) = \tfrac{1}{2}[f(\lambda+it)+f(\lambda-it)]$$

since $f(s)$ assumes complex conjugate values at complex conjugate points. It is clear from this relation that $\operatorname{Re} f(\lambda+it)$ is an even function of t. Using the functional equation, $f(\lambda-it) = f(1-\lambda+it)$, we obtain

$$\operatorname{Re} f(\lambda+it) = \tfrac{1}{2}[f(\lambda+it)+f(1-\lambda+it)].$$

Consider the function, for, say, positive real x,

$$\Psi_\lambda(x) = \int_0^\infty \operatorname{Re} f(\lambda+it)\cos xt\,dt.$$

Since $\cos xt$ is also an even function of t, we may write

$$\Psi_\lambda(x) = \tfrac{1}{2}\int_{-\infty}^{\infty} \operatorname{Re} f(\lambda+it)\cos xt\,dt = \tfrac{1}{2}\int_{-\infty}^{\infty} \operatorname{Re} f(\lambda+it)\,y^{it}\,dt$$

where $y = e^x$. Furthermore

$$\Psi_\lambda(x) = \frac{1}{2i\sqrt{y}} \int_{1/2-i\infty}^{1/2+i\infty} \mathrm{Re} f(\lambda - \tfrac{1}{2} + s) y^s ds$$
$$= \frac{1}{4i\sqrt{y}} \int_{1/2-i\infty}^{1/2+i\infty} [f(\lambda - \tfrac{1}{2} + s) + f(\tfrac{1}{2} - \lambda + s)] y^s ds.$$

We split this integral into a sum of integrals and evaluate separately. Let us set

$$\Omega_\lambda(x) = \frac{1}{4i\sqrt{y}} \int_{1/2-i\infty}^{1/2+i\infty} f(\lambda - \tfrac{1}{2} + s) y^s ds.$$

Then it is clear that

(1) $$\Psi_\lambda(x) = \Omega_\lambda(x) + \Omega_{1-\lambda}(x).$$

Now

$$\Omega_\lambda(x) = \frac{1}{4i\sqrt{y}} \int_{1/2-i\infty}^{1/2+i\infty} \pi^{-\frac{1}{2}(\lambda-\frac{1}{2}+s)} \Gamma\big(\tfrac{1}{2}(\lambda - \tfrac{1}{2} + s)\big) \zeta(\lambda - \tfrac{1}{2} + s) y^s ds.$$

We wish to pass to the vertical line $\sigma = 2$, and to do this, we must take into account the pole of $\zeta(\omega)$ at $\omega = 1$, or in our case, $s = 1 - \lambda + \frac{1}{2}$. Take the integral over the usual rectangle $(\frac{1}{2} + iT, 2 + iT, 2 - it, \frac{1}{2} - iT)$, and observe that the integral the horizontal lines approaches zero because of Stirling's formula for the gamma function in a fixed strip and standard bounds for the other functions.

Thus, by Cauchy's theorem,

$$\Omega_\lambda(x) = \frac{1}{4i\sqrt{y}} \Big[\int_{2-i\infty}^{2+i\infty} \pi^{-\frac{1}{2}(\lambda-\frac{1}{2}+s)} \Gamma\big(\tfrac{1}{2}(\lambda - \tfrac{1}{2} + s)\big) \zeta(\lambda - \tfrac{1}{2} + s) y^s ds - 2\pi i y^{1-\lambda+\frac{1}{2}} \Big]$$

recalling that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\mathrm{Res}_{\omega=1}\zeta(\omega) = 1$. Let us consider the integral separately. By changing variables, $2\tau = \lambda - \frac{1}{2} + s$, we get

$$\int_{2-i\infty}^{2+i\infty} \pi^{-\frac{1}{2}(\lambda-\frac{1}{2}+s)} \Gamma\big(\tfrac{1}{2}(\lambda - \tfrac{1}{2} + s)\big) \zeta(\lambda - \tfrac{1}{2} + s) y^s ds$$
$$= 2 \int_{\frac{3}{4}+\frac{\lambda}{2}-i\infty}^{\frac{3}{4}+\frac{\lambda}{2}+i\infty} \pi^{-\tau} \Gamma(\tau) \zeta(2\tau) y^{2\tau-\lambda+\frac{1}{2}} d\tau = 2y^{-\lambda+\frac{1}{2}} \int_{\frac{3}{4}+\frac{\lambda}{2}-i\infty}^{\frac{3}{4}+\frac{\lambda}{2}+i\infty} \pi^{-\tau} \Gamma(\tau) \zeta(2\tau) \left(\frac{1}{y^2}\right)^{-\tau} d\tau$$
$$= 4\pi i y^{-\lambda+\frac{1}{2}} \psi\left(\frac{1}{y^2}\right),$$

where $\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$, and this formula is an instance of Mellin inversion, which is equation (2. 15. 6) on page 34 of [5].

So we get

$$\Omega_\lambda(x) = \frac{1}{4i\sqrt{y}} \Big[4\pi i y^{-\lambda+\frac{1}{2}} \psi\left(\frac{1}{y^2}\right) - 2\pi i y^{1-\lambda+\frac{1}{2}} \Big] = \frac{\pi}{y^\lambda} \psi\left(\frac{1}{y^2}\right) - \frac{\pi}{2} y^{1-\lambda}.$$

The integral for $\Omega_{1-\lambda}(x)$ is handled in the same way, and a pole is encountered. It should be remarked at this point that a pole at 1 is encountered for *both* $\Omega_\lambda(x)$ and $\Omega_{1-\lambda}(x)$ only when $0 < \lambda < 1$, so that the proof of the theorem is not valid for vertical lines not inside the critical strip.

Recalling (1), we obtain

(2) $$\Psi_\lambda(x) = \left(\frac{\pi}{y^\lambda} + \frac{\pi}{y^{1-\lambda}}\right) \psi\left(\frac{1}{y^2}\right) - \frac{\pi}{2}(y^{1-\lambda} + y^\lambda)$$
$$= \pi(e^{-\lambda x} + e^{-(1-\lambda)x}) \psi(e^{-2x}) - \frac{\pi}{2}(e^{(1-\lambda)x} + e^{\lambda x})$$
$$= \pi(e^{-\lambda x} + e^{-(1-\lambda)x}) \big(\psi(e^{-2x}) + \tfrac{1}{2}\big) -$$
$$- \frac{\pi}{2}(e^{(1-\lambda)x} + e^{-(1-\lambda)x} + e^{\lambda x} + e^{-\lambda x}).$$

Now both the left and right hand sides of (2), originally defined for $x > 0$, are seen to be analytic functions in the half plane $\mathrm{Re}\, e^{-2x} > 0$ because the integral defining $\Psi_\lambda(x)$ is absolutely convergent by Stirling's formula in a fixed strip, and the analyticity of the right hand side is well known. Thus we may set $x = -i\alpha$, provided, say, $0 < \alpha < \pi/4$. Then (2) becomes

(3) $$\int_0^\infty \mathrm{Re} f(\lambda + it) \cosh \alpha t\, dt$$
$$= \pi(e^{\lambda\alpha i} + e^{(1-\lambda)\alpha i}) \big(\psi(e^{2\alpha i}) + \tfrac{1}{2}\big) - \frac{\pi}{2}\big(2\cos[(1-\lambda)\alpha] + 2\cos\lambda\alpha\big).$$

In the indicated range for α, since the integrand is absolutely convergent, differentiation with respect to α under the integral sign is allowed, and if we do this $2n$ times, we obtain from (3)

(4) $$\int_0^\infty t^{2n} \mathrm{Re} f(\lambda + it) \cosh \alpha t\, dt = \frac{d^{2n}}{d\alpha^{2n}} \{\pi(e^{\lambda\alpha i} + e^{(1-\lambda)\alpha i}) \big(\psi(e^{2\alpha i}) + \tfrac{1}{2}\big)\} +$$
$$+ (-1)^{n+1}\pi\big[(1-\lambda)^{2n} \cos[(1-\lambda)\alpha] + \lambda^{2n} \cos\lambda\alpha\big].$$

To finish the proof, two results are needed, and the proofs of both of these may be found in [5].

LEMMA 1. *The function* $\frac{1}{2}+\psi(x)$ *and all its derivatives tend to zero as* $x \to i$ *along any route in an angle* $|\arg(x-i)| < \frac{1}{2}\pi$.

LEMMA 2 (Fejér). *Let* n *be any positive integer. The number of changes in sign in the interval* $(0, a)$ *of a continuous function* $f(x)$ *is not less than the number of changes in sign of the sequence*

$$f(0),\ \int_0^a f(t)\,dt,\ \ldots,\ \int_0^a f(t)t^n\,dt.$$

To prove that $\mathrm{Re} f(\lambda+it)$ has infinitely many zeros as $t\to\infty$, we choose T large and a close to $\pi/4$. Then from (4), and Lemma 1, we see that the number of sign changes in the sequence

$$\int_0^T \mathrm{Re} f(\lambda+it)\cosh at\,dt,$$

$$\int_0^T t^2\,\mathrm{Re} f(\lambda+it)\cosh at\,dt,\ \ldots,\ \int_0^T t^{2n}\,\mathrm{Re} f(\lambda+it)\cosh at\,dt$$

is at least n. Now Lemma 2 shows that on the interval $(0, T)$ $\mathrm{Re} f(\lambda+it)$ changes sign at least n times, because here $\cosh at$ is of constant sign. This establishes the assertion, since we may take n arbitrarily large.

2. In order to treat the imaginary part, we write

$$i\,\mathrm{Im} f(\lambda+it) = \tfrac{1}{2}[f(\lambda+it)-f(\lambda-it)]$$

so that we see $\mathrm{Im} f(\lambda+it)$ is an odd function of t. Now $\sin xt$ is also an odd function of t, so that the product of these functions is even. Therefore we may write

$$\tilde{\Psi}_\lambda(x) = \int_0^\infty i\,\mathrm{Im} f(\lambda+it)\sin xt\,dt$$

$$= \tfrac{1}{2}\int_{-\infty}^\infty i\,\mathrm{Im} f(\lambda+it)\sin xt\,dt = \tfrac{1}{2}\int_{-\infty}^\infty \mathrm{Im} f(\lambda+it)\,y^{it}\,dt,$$

where, as before, $y = e^x$. Proceeding as in the first part, we get

$$\tilde{\Psi}_\lambda(x) = \frac{-1}{4\sqrt{y}}\int_{+1/2-i\infty}^{+1/2+i\infty} [f(\lambda-\tfrac{1}{2}+s)-f(\tfrac{1}{2}-\lambda+s)]\,y^s\,ds$$

or, in the notation of the first part,

$$i\tilde{\Psi}_\lambda(x) = \Omega_\lambda(x)-\Omega_{1-\lambda}(x).$$

Now these integrals have been evaluated, so we have

$$i\tilde{\Psi}_\lambda(x) = \pi(e^{-\lambda x}-e^{-(1-\lambda)x})\psi(e^{-2x})-\frac{\pi}{2}(e^{(1-\lambda)x}-e^{\lambda x})$$

$$= \pi(e^{-\lambda x}-e^{-(1-\lambda)x})\big(\psi(e^{-2x})+\tfrac{1}{2}\big)-\frac{\pi}{2}(e^{(1-\lambda)x}-e^{-(1-\lambda)x}-e^{\lambda x}+e^{-\lambda x}).$$

Once again we take $x = -ia$, and obtain

$$(5)\qquad i\int_0^\infty \mathrm{Im} f(\lambda+it)\sinh at\,dt$$

$$= \pi(e^{\lambda ia}-e^{(1-\lambda)ia})\big(\psi(e^{2ia})+\tfrac{1}{2}\big)-\frac{\pi i}{2}\big(2\sin[(1-\lambda)a]-2\sin\lambda a\big).$$

Let us suppose that $\lambda \neq \frac{1}{2}$, since we know already that $\mathrm{Im} f(\frac{1}{2}+it)$ is identically zero. We divide both sides of (5) by i and differentiate $2n$ times with respect to a to obtain

$$(6)\qquad \int_0^\infty t^{2n}\,\mathrm{Im} f(\lambda+it)\sinh at\,dt$$

$$= \frac{d^{2n}}{da^{2n}}\left\{\frac{\pi}{i}(e^{\lambda ia}-e^{(1-\lambda)ia})\big(\psi(e^{2ia})+\tfrac{1}{2}\big)\right\}+$$

$$+\pi(-1)^n\big[(1-\lambda)^{2n}\sin[(1-\lambda)a]-\lambda^{2n}\sin\lambda a\big].$$

If we suppose $0<1-\lambda<\frac{1}{2}<\lambda<1$ and if we take a sufficiently close to $\pi/4$, use Lemma 1, and recall that the sine is monotone increasing on $\langle 0, \pi/2\rangle$, then we see that the sign of the left hand side of (6) is that of $(-1)^{n+1}$. The proof that $\mathrm{Im} f(\lambda+it)$ has infinitely many zeroes as $t\to\infty$ is completed by using Lemma 2 in the same way it was used in the first part.

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