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On the number of integer points in the displaced circles

by

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Let a lattice of points with integer coordinates be given in the plane. Take a circle with the radius $\lambda^{1/2}$. Without loss of generality one can assume the centre of the circle to be a point $(u, v) \in G$, where G is the domain defined by the inequalities:

$$0 \leqslant u < 1, \quad 0 \leqslant v < 1.$$

Let $A(\lambda, u, v)$ denote the number of integer points inside the circle of the radius $\lambda^{1/2}$ with the centre in the point (u, v). Then it is easy to show that

$$A(\lambda, u, v) = \pi\lambda + P(\lambda, u, v),$$

where $P(\lambda, u, v) = O(\lambda^{\theta})$, $0 < \theta \leqslant 1/3$.

Kendall [4] proved that

$$\int_0^1\int_0^1 P^2(\lambda, u, v)\,du\,dv = \lambda\sum_{n=1}^{\infty}\frac{r(n)}{n}I_1^2(2\pi\sqrt{n\lambda}), \tag{1}$$

where $r(n)$ is the number of representations of the number n as the sum of two squares, $I_1(z)$ being Bessel's function.

By well known asymptotic behaviour of $I_1(z)$ it follows from (1) that

$$\int_0^1\int_0^1 P^2(\lambda, u, v)\,du\,dv = O(\lambda^{1/2}).$$

We shall show below that

$$\overline{\lim_{\lambda\to\infty}}\frac{P(\lambda, u, v)}{\lambda^{1/4}(\ln\ln\lambda)^{1/4-\varepsilon}} > c > 0, \quad \underline{\lim_{\lambda\to\infty}}\frac{P(\lambda, u, v)}{\lambda^{1/4}(\ln\ln\lambda)^{1/4-\varepsilon}} < -c < 0 \tag{2}$$

where ε is an arbitrarily small positive number, c is an absolute constant.

LEMMA 1.

$$\int_0^x P(\lambda, u, v)\,d\lambda = \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{\Phi(n)}{n} I_1(2\pi\sqrt{nx}), \tag{3}$$

where $\Phi(n) = \sum_{a^2+b^2=n} e^{2\pi i(au+bv)}$ *and the sum is taken over all the representations of the number* n *as the sum of two squares.*

The proof of Lemma 1 is analogous to the proof of a similar result in [5].

We introduce the following definitions: Let $\sigma(x)$ be some real-valued given function, and let

$$\overline{\sigma}(a, \beta) = \sup_{a\leqslant x\leqslant\beta} \sigma(x)$$

and

$$\underline{\sigma}(a, \beta) = \inf_{a\leqslant x\leqslant\beta} \sigma(x).$$

LEMMA 2. *Let* $\omega \geqslant 2\eta > 0$, $X\eta^2 \geqslant c > 1$ *and* $\sigma(t) = t^{-1/2}P(t^2, u, v)$, *then*

$$\underline{\sigma}(\omega-\eta, \omega+\eta) < -S_X(\omega)+O\left(\frac{1}{\omega}\right)+O\left(\frac{1}{\eta X^{1/4}}\right), \tag{4}$$

$$\overline{\sigma}(\omega-\eta, \omega+\eta) > -S_X(\omega)+O\left(\frac{1}{\omega}\right)+O\left(\frac{1}{\eta X^{1/4}}\right), \tag{5}$$

where

$$S_X(\omega) = \frac{1}{\pi}\sum_{1\leqslant n\leqslant X}\left(1-\frac{n^{1/2}}{X^{1/2}}\right)\frac{\Phi(n)}{n^{3/4}}\cos\left(2\pi n^{1/2}\omega+\frac{\pi}{4}\right).$$

The proof of Lemma 2 is analogous to the proof of theorem R1 from [3].

We multiply both sides of (3) by $\chi'(x)$, where $\chi(x)$ is a function having the continuous derivative on $\langle a, b\rangle$, $a < b$. Integrating both sides of (3) over the range a to b and applying (3), we get by partial integration

$$\int_a^b \chi(x)P(x, u, v)\,dx = \sum_{n=1}^{\infty}\frac{\Phi(n)}{n^{1/2}}\int_a^b x^{1/2}\chi(x) I_1(2\pi\sqrt{nx})\,dx. \tag{6}$$

Now let

$$K(y) = \left(\frac{\sin\frac{1}{2}y}{\frac{1}{2}y}\right)^2 \quad\text{and}\quad k(x) = \begin{cases} 2\pi(1-|x|) & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geqslant 1. \end{cases}$$

It is known from the theory of Fourier integral that

$$k(x) = \int_{-\infty}^{+\infty} K(y)e^{ixy}\,dy. \tag{7}$$

Choosing suitably $a, b, \chi(x)$ in (6) and replacing x by t^2 we obtain

$$\int_{\omega-\eta}^{\omega+\eta} K_U(t-\omega)\sigma(t)\,dt = \sum_{n=1}^{\infty}\frac{\Phi(n)}{n^{1/2}}\int_{\omega-\eta}^{\omega+\eta} t^{1/2}K_U(t-\omega)I_1(2\pi n^{1/2}t)\,dt, \tag{8}$$

where $K_U(\delta) = UK(U\delta)$, $U > 0$.

By the substitution $t = \omega+z/U$ in (8), we get

$$\int_{-U\eta}^{U\eta} K(z)\sigma\left(\omega+\frac{z}{U}\right)dz = \sum_{n=1}^{\infty}\frac{\Phi(n)}{n^{1/2}}\int_{-U\eta}^{U\eta}\left(\omega+\frac{z}{U}\right)^{1/2}K(z)I_1\left(2\pi n^{1/2}\left(\omega+\frac{z}{U}\right)\right)dz.$$

Using the asymptotic behaviour of Bessel's functions we obtain

$$\int_{-U\eta}^{U\eta} K(z)\sigma\left(\omega+\frac{z}{U}\right)dz = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\Phi(n)}{n^{3/4}}\int_{-U\eta}^{U\eta}K(z)\cos\left(2\pi n^{1/2}\left(\omega+\frac{z}{U}\right)+\frac{\pi}{4}\right)dz + O\left(\sum_{n=1}^{\infty}\frac{|\Phi(n)|}{n^{5/4}}\int_{-U\eta}^{U\eta}|K(z)|\left(\omega+\frac{z}{U}\right)^{-1}dz\right). \tag{9}$$

Now we study the remainder term in (9). It is of order $O(1/\omega)$ since $\omega+z/U \geqslant \omega-\eta \geqslant \frac{1}{2}\omega$ and the series $\sum_{n=1}^{\infty}|\Phi(n)|/n^{5/4}$ and the integral $\int_{-U\eta}^{U\eta}|K(y)|\,dy$ are convergent.

Hence

$$\int_{-U\eta}^{U\eta} K(z)\sigma\left(\omega+\frac{z}{U}\right)dz = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\Phi(n)}{n^{3/4}}\int_{-U\eta}^{U\eta}K(z)\cos\left(2\pi n^{1/2}\left(\omega+\frac{z}{U}\right)+\frac{\pi}{4}\right)dz+O\left(\frac{1}{\omega}\right). \tag{10}$$

We estimate the error on the right-hand side of (10) if the integral is taken over $(-\infty, \infty)$.

Denote

$$\Delta_1 = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\Phi(n)}{n^{3/4}}\int_{U\eta}^{\infty}K(z)\cos\left(2\pi n^{1/2}\left(\omega+\frac{z}{U}\right)+\frac{\pi}{4}\right)dz$$

and

$$\Delta_2 = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\Phi(n)}{n^{3/4}}\int_{-\infty}^{-U\eta} K(z)\cos\left(2\pi n^{1/2}\left(\omega+\frac{z}{U}\right)+\frac{\pi}{4}\right)dz.$$

Since Δ_1 is not essentially different from Δ_2 one can estimate only Δ_1.

The nth integral is of order $O(1/U\eta)$. Integrating by parts and using the trivial estimates

$$K(y) = O(y^{-1}), \quad K'(y) = O(y^{-2}),$$

we get another one namely $O(1/n^{1/2}\eta)$.

Therefore

$$\Delta_1 = O\left(\sum_{n=1}^{\infty}\frac{|\Phi(n)|}{n^{3/4}}\min\left(\frac{1}{n^{1/2}\eta}, \frac{1}{U\eta}\right)\right)$$

$$= O\left(\sum_{n<U^2}\frac{|\Phi(n)|}{n^{3/4}}\cdot\frac{1}{U\eta}+\frac{1}{\eta}\sum_{n\geqslant U^2}\frac{|\Phi(n)|}{n^{5/4}}\right) = O\left(\frac{1}{U^2\eta}\right).$$

Hence we obtain

$$\int_{-U\eta}^{U\eta} K(z)\,\sigma\left(\omega+\frac{z}{U}\right)dz$$

$$= -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\Phi(n)}{n^{3/4}}\int_{-\infty}^{+\infty}K(z)\cos\left(2\pi n^{1/2}\left(\omega+\frac{z}{U}\right)+\frac{\pi}{4}\right)dz+O\left(\frac{1}{\omega}\right)+O\left(\frac{1}{U^{1/2}\eta}\right).$$

In view o (7) we have

$$\int_{-\infty}^{+\infty}K(z)\cos\left(2\pi n^{1/2}\left(\omega+\frac{z}{U}\right)+\frac{\pi}{4}\right)dz = \operatorname{Re} e^{i(2\pi n^{1/2}\omega+\pi/4)}\int_{-\infty}^{+\infty}K(z)\,e^{2\pi i n^{1/2}z/U}dz$$

$$= \cos\left(2\pi n^{1/2}\omega+\frac{\pi}{4}\right)k\left(\frac{2\pi n^{1/2}}{U}\right).$$

We have

$$\int_{-U\eta}^{U\eta}K(z)\,\sigma\left(\omega+\frac{z}{U}\right)dz = -\frac{1}{\pi}\sum_{n\leqslant U^2/4\pi^2}2\pi\frac{\Phi(n)}{n^{3/4}}\left(1-\frac{n^{1/2}}{U/2\pi}\right)\cos\left(2\pi n^{1/2}\omega+\frac{\pi}{4}\right)+$$

$$+O\left(\frac{1}{\omega}\right)+O\left(\frac{1}{U^{1/2}\eta}\right).$$

Now setting $U^2/4\pi^2 = X$ we get

$$\int_{-U\eta}^{U\eta}K(z)\,\sigma\left(\omega+\frac{z}{U}\right)dz = -\frac{1}{\pi}\sum_{n\leqslant X}2\pi\left(1-\frac{n^{1/2}}{X^{1/2}}\right)\frac{\Phi(n)}{n^{3/4}}\cos\left(2\pi n^{1/2}\omega+\frac{\pi}{4}\right)+$$

$$+O\left(\frac{1}{\omega}\right)+O\left(\frac{1}{\eta X^{1/4}}\right).$$

In view of $K(y)\geqslant 0$, we have

$$\underline{\sigma}(\omega-\eta,\,\omega+\eta)\int_{-U\eta}^{U\eta}K(z)\,dz \leqslant \int_{-U\eta}^{U\eta}K(z)\,\sigma\left(\omega+\frac{z}{U}\right)dz,$$

and hence

$$\underline{\sigma}(\omega-\eta,\,\omega+\eta)\leqslant\left\{\int_{-U\eta}^{U\eta}K(z)\,\sigma\left(\omega+\frac{z}{U}\right)dz\right\}\left(\int_{-U\eta}^{U\eta}K(z)\,dz\right)^{-1}.$$

Since $k(0) = 2\pi$ we obtain

$$\left(\int_{-U\eta}^{U\eta}K(z)\,dz\right)^{-1} = \frac{1}{2\pi+2\int_{U\eta}^{\infty}K(z)\,dz} = \frac{1}{2\pi}\left(1+O\left(\frac{1}{\eta X^{1/4}}\right)\right).$$

Further we get

$$\underline{\sigma}(\omega-\eta,\,\omega+\eta)\leqslant -\frac{1}{\pi}\sum_{n\leqslant X}\left(1-\frac{n^{1/2}}{X^{1/2}}\right)\frac{\Phi(n)}{n^{3/4}}\cos\left(2\pi n^{1/2}\omega+\frac{\pi}{4}\right)+$$

$$+O\left(\frac{1}{\eta X^{1/2}}\sum_{n\leqslant X}\frac{\Phi(n)}{n^{3/4}}\right)+O\left(\frac{1}{\omega}\right)+O\left(\frac{1}{\eta X^{1/4}}\right).$$

Since

$$\frac{1}{\eta X^{1/2}}\sum_{n\leqslant X}\frac{|\Phi(n)|}{n^{3/4}} = O\left(\frac{1}{\eta X^{1/4}}\right),$$

we obtain

$$\underline{\sigma}(\omega-\eta,\,\omega+\eta)\leqslant -\frac{1}{\pi}\sum_{n\leqslant X}\left(1-\frac{n^{1/2}}{X^{1/2}}\right)\frac{\Phi(n)}{n^{3/4}}\cos\left(2\pi n^{1/2}\omega+\frac{\pi}{4}\right)+$$

$$+O\left(\frac{1}{\omega}\right)+O\left(\frac{1}{X^{1/4}\eta}\right).$$

Thus we get (4). It is clear that (5) can be obtained in the same way.

LEMMA 3. *Let $f(x)$ be defined and measurable on $\langle 0,1\rangle$ and*

$$\int_0^1 f(x)\,dx = 0, \quad \int_0^1 f^2(x)\,dx = I > 0, \quad |f(x)|\leqslant M.$$

Then

$$\Delta = \max_{x\in\langle 0,1\rangle} f(x)\geqslant I/M.$$

Proof. From the conditions of Lemma 3 there follows

(11) $$\int_{E_+}|f(x)|\,dx = \int_{E_-}|f(x)|\,dx$$

and

$$\int_{E_+} f^2(x)\,dx + \int_{E_-} f^2(x)\,dx = I \tag{12}$$

where

$$E_+ = \{x\colon f(x) > 0\}, \qquad E_- = \{x\colon f(x) < 0\},$$

$\varkappa$ is defined by the equality

$$\varkappa \int_{E_-} |f(x)|\,dx = \int_{E_-} f^2(x)\,dx. \tag{13}$$

Multiplying (11) by $\varkappa$ and adding it to (12) we get

$$\varkappa \int_{E_+} |f(x)|\,dx + \int_{E_+} f^2(x)\,dx = I$$

or

$$\varkappa\Delta\mu(E_+) + \Delta^2\mu(E_+) \geqslant I. \tag{14}$$

Let us get now the upper estimate for $\mu(E_+)$. From (12) we have

$$I = \int_0^1 f^2(x)\,dx \leqslant \Delta^2\mu(E_+) + M^2\big(1-\mu(E_+)\big)$$

or

$$\mu(E_+) \leqslant \frac{M^2-I^2}{M^2-\Delta^2}. \tag{15}$$

Then from (14) and (15) we get

$$I \leqslant \varkappa\Delta\frac{M^2-I^2}{M^2-\Delta^2} + \Delta^2\frac{M^2-I^2}{M^2-\Delta^2}.$$

As (13) implies $\varkappa \leqslant M$, we get

$$\frac{M^2-I}{M-\Delta}\Delta \geqslant I$$

or

$$\Delta \geqslant I/M.$$

Remark. The estimate of Lemma 3 cannot be improved. It is evident that

$$f(x) = \begin{cases} -M & \text{for } x \in \left\langle 0, \dfrac{I}{I+M^2}\right\rangle, \\[2ex] \dfrac{I}{M} & \text{for } x \in \left(\dfrac{I}{I+M^2}, 1\right\rangle \end{cases}$$

satisfies all conditions of Lemma 3 and

$$\max_{x\in\langle 0,1\rangle} f(x) = I/M.$$

COROLLARY. *Let $f(x)$ be measurable on $\langle 0, 1\rangle$ and*

$$\int_0^1 f(x)\,dx = 0, \qquad \int_0^1 f^2(x)\,dx = I, \qquad |f(x)| \leqslant M.$$

Then

$$\min_{x\in\langle 0,1\rangle} f(x) \leqslant -I/M.$$

LEMMA 4. *Let $|z_j| = 1$ $(j = 1, 2, \ldots, N)$ be complex numbers, let $S_k = \sum_{j=1}^{N} z_j^k$ and k_0 a prescribed integer with $N/2 < k_0 < N$. Then*

$$\max_{\substack{1\leqslant k\leqslant N\\ k\neq k_0}} |S_k| > \tfrac{1}{3}.$$

Proof. Let

$$M \overset{\text{def}}{=\!=} \max_{\substack{v=1,\ldots,N\\ v\neq k_0}} |S_v| \tag{16}$$

(we may suppose $0 < M < 1$). Let $f(z) = z^N + a_1 z^{N-1} + \ldots + a_N$ be the polynomial with z_j's as zeros. According to Waring's formula

$$a_N = \sum \frac{(-1)^{\lambda_1+\ldots+\lambda_N}}{\lambda_1!\ldots\lambda_N!}\left(\frac{S_1}{1}\right)^{\lambda_1}\left(\frac{S_2}{2}\right)^{\lambda_2}\ldots\left(\frac{S_N}{N}\right)^{\lambda_N}, \tag{17}$$

where the summation refers to the nonnegative solutions of

$$\lambda_1 + 2\lambda_2 + \ldots + N\lambda_N = N. \tag{18}$$

Owing to $N/2 < k_0 < N$ the possible values of λ_{k_0} are 0 and 1, hence (17) can be written as

$$a_N = \frac{S_{k_0}}{k_0}\sum{}' + \sum{}'' \frac{(-1)^{\lambda_1+\ldots+\lambda_N}}{\lambda_1!\ldots\lambda_{k_0-1}!\,\lambda_{k_0+1}!\ldots\lambda_N!} \times \tag{19}$$
$$\times \left\{\left(\frac{S_1}{1}\right)^{\lambda_1}\ldots\left(\frac{S_{k_0-1}}{k_0-1}\right)^{\lambda_{k_0-1}}\left(\frac{S_{k_0+1}}{k_0+1}\right)^{\lambda_{k_0+1}}\ldots\left(\frac{S_N}{N}\right)^{\lambda_N}\right\},$$

where Σ' is extended over the nonnegative solutions of

$$\lambda_1 + 2\lambda_2 + \ldots + (k_0-1)\lambda_{k_0-1} + (k_0+1)\lambda_{k_0+1} + \ldots + N\lambda_N = N - k_0 \tag{20}$$

and Σ'' over those of

$$\lambda_1 + 2\lambda_2 + \ldots + (k_0-1)\lambda_{k_0-1} + (k_0+1)\lambda_{k_0+1} + \ldots + N\lambda_N = N. \tag{21}$$

Since $|S_{k_0}| \leqslant N$, i.e. $|S_{k_0}/k_0| \leqslant N/k_0 < 2$ and $|a_N| \geqslant 1$, (16) and (19) give

$$1 \leqslant 2\sum{}' \frac{M^{\lambda_1+\ldots+\lambda_{k_0-1}+\lambda_{k_0+1}+\ldots+\lambda_N}}{\lambda_1!\ldots\lambda_{k_0-1}!\,\lambda_{k_0+1}!\,\lambda_N!\,1^{\lambda_1}\ldots(k_0-1)^{\lambda_{k_0-1}}(k_0+1)^{\lambda_{k_0+1}}\ldots N^{\lambda_N}} + \tag{22}$$
$$+ \sum{}'' \frac{M^{\lambda_1+\ldots+\lambda_{k_0-1}+\lambda_{k_0+1}+\ldots+\lambda_N}}{\lambda_1!\ldots\lambda_{k_0-1}!\,\lambda_{k_0+1}!\,\lambda_N!\,1^{\lambda_1}\ldots(k_0-1)^{\lambda_{k_0-1}}(k_0+1)^{\lambda_{k_0+1}}\ldots N^{\lambda_N}}.$$

The first sum in (22) is

$$\text{coeff } z^{N-k_0} \text{ in } \exp\left(M\left(\frac{z}{1}+\ldots+\frac{z^{k_0-1}}{k_0-1}+\frac{z^{k_0+1}}{k_0+1}+\ldots+\frac{z^N}{N}\right)\right)$$

$$= \text{coeff } z^{N-k_0} \text{ in } \exp\left(M\left(\frac{z}{1}+\frac{z^2}{2}+\ldots\right)-M\frac{z_0^{k_0}}{k_0}\right)$$

$$= \text{coeff } z^{N-k_0} \text{ in } \frac{1}{(1-z)^M}\exp\left(-M\frac{z^{k_0}}{k_0}\right).$$

Owing to $k_0 > N/2$ we have $N-k_0 < N/2$ and hence this is

$$(23) \qquad = \text{coeff } z^{N-k_0} \text{ in } \frac{1}{(1-z)^M}\left(1-\frac{M}{k_0}z^{k_0}\right)$$

$$\leqslant \text{coeff } z^{N-k_0} \text{ in } \frac{1}{(1-z)^M} = \left|\binom{-M}{N-k_0}\right|$$

$$= M\left(1-\frac{1-M}{2}\right)\left(1-\frac{1-M}{3}\right)\ldots\left(1-\frac{1-M}{N-k_0}\right) < M.$$

The second sum in (22) is

$$(24) \qquad \text{coeff } z^N \text{ in } \exp\left(M\left(\frac{z}{1}+\ldots+\frac{z^{k_0-1}}{k_0-1}+\frac{z^{k_0+1}}{k_0+1}+\ldots+\frac{z^N}{N}\right)\right)$$

$$= \text{coeff } z^N \text{ in } \frac{1}{(1-z)^M}\exp\left(-\frac{M}{k_0}z^{k_0}\right) < \text{coeff } z^N \text{ in } \frac{1}{(1-z)^M}$$

$$= \left|\binom{-M}{N}\right| = M\left(1-\frac{1-M}{2}\right)\ldots\left(1-\frac{1-M}{N}\right) < M.$$

(22), (23), (24) give

$$1 \leqslant 3M. \qquad \text{Q. e. d.}$$

LEMMA 5. *Let be given a sequence of real numbers*

$$1, \sqrt{2}, \ldots, \sqrt{\nu_m},$$

where ν_j are squarefree positive integers and let $\omega < m$. Then there exists $T^* \leqslant \exp e^{3m}$ such that

$$|\{T^*\sqrt{\nu_j}\}-\varphi_j| \leqslant \frac{1}{\omega},$$

where $\{x\}$ is the fractional part of x.

The proof is obtained by the method of Bohr and Jessen [2]. Let

$$F(t) = 1+\sum_{j=1}^{m} e^{2\pi i(\sqrt{\nu_j}t-\varphi_j)}$$

and let

$$K_n(t) = \sum_{\tau=-n}^{n}\left(1-\frac{|\tau|}{n}\right)e^{i\tau t} = \frac{1}{n}\left(\frac{\sin\frac{1}{2}nt}{\sin\frac{1}{2}t}\right)^2$$

be Fejér's kernel.

Now we construct the kernel of Bochner-Fejér

$$K_n^*(t) = \prod_{s=1}^{m} K_n\big(2\pi(\sqrt{\nu_s}t-\varphi_s)\big).$$

It follows hence

$$F(t)K_n^*(t) = 1+\frac{n-1}{n}N+S(t),$$

where $S(t)$ is almost periodic polynomial. Fourier's exponent of $S(t)$ are not 0 because $\sqrt{\nu_j}$ are linearly independent, as it was proved by Besicovitch [1].

We have

$$\frac{1}{2T}\int_{-T}^{T} F(t)K_n^*(t)\,dt = 1+\frac{n-1}{n}m+\Delta,$$

where

$$|\Delta| \leqslant \frac{1}{2T}\cdot\frac{(2n)^{m+1}}{\min\limits_{|l_j|\leqslant m}|l_0+\ldots+l_m\sqrt{\nu_m}|} = \Delta^*.$$

Let the minimum of $|l_0+\ldots+l_m\sqrt{\nu_m}|$ for $|l_j| \leqslant m$ be obtained for $l_0', \ldots, l_m'$. The number

$$a = l_0'+l_1'\sqrt{\nu_1}+\ldots+l_m'\sqrt{\nu_m}$$

belongs to the field $R(\sqrt{2}, \ldots, \sqrt{\nu_m})$ and as it is easy to show

$$|N(a)| = \prod |\varepsilon_0 l_0'+\ldots+\varepsilon_m l_m'\sqrt{\nu_m}| \geqslant 1,$$

where the product is taken over all the combinations of $\varepsilon_i = \pm 1$.

Using the inequality for the arithmetical and geometrical means, we get

$$|l_0'+l_1'\sqrt{2}+\ldots+l_m'\sqrt{\nu_m}| \geqslant \prod |\varepsilon_0 l_0'+\ldots+\varepsilon_m l_m'\sqrt{\nu_m}|^{-1}$$
$$\geqslant \{\exp(\exp(m+2\ln n))\}^{-1},$$

where the product is taken over all the combinations of ε_i.

Therefore if $n < m^{\gamma}$ we have

$$\Delta^* \leqslant \frac{e^{e^{2m}}}{T} \quad \text{for} \quad m \geqslant m_0 = 2(\gamma+1).$$

Then in view of

$$\frac{1}{2T}\int_{-T}^{T} K_n^*(t)\,dt \leqslant 1+\Delta^*,$$

we get

$$\max_{-T\leqslant t\leqslant T} |F(t)| \geqslant 1+m-\frac{m}{n}-\Delta^* m.$$

Putting here $n = 2m\omega$ and $T = \exp(\exp 3m)$ we obtain the existence of $T^* \leqslant \exp(\exp 3m)$ such that

$$|F(T^*)| \geqslant 1+m+\delta, \quad \text{where} \quad |\delta| \leqslant 1/\omega.$$

Hence the lemma follows.

THEOREM.

(25) $$\overline{\lim_{\lambda\to\infty}} \frac{A(\lambda, u, v)-\pi\lambda}{\lambda^{1/4}(\ln\ln\lambda)^{1/4-\varepsilon}} > c > 0$$

and

(26) $$\underline{\lim_{\lambda\to\infty}} \frac{A(\lambda, u, v)-\pi\lambda}{\lambda^{1/4}(\ln\ln\lambda)^{1/4-\varepsilon}} < -c < 0$$

for all $(u, v)\in G$.

Proof of (26). Using Lemma 1 and Lemma 2 we get for sufficiently large X, ω, η and a suitable $c_1 > 0$

$$\underline{\sigma}(\omega-\eta, \omega+\eta) \leqslant -c_1 \sum_{1\leqslant n\leqslant X}\left(1-\frac{n^{1/2}}{X^{1/2}}\right)\frac{\Phi(n)}{n^{3/4}}\cos\left(2\pi n^{1/2}\omega+\frac{\pi}{4}\right).$$

In the sequel we shall denote by c_s absolute constants different in general.

Let

$$Q(t) = \sum_{1\leqslant n\leqslant X}\left(1-\frac{n^{1/2}}{X^{1/2}}\right)\frac{\Phi(n)}{n^{3/4}}\cos\left(2\pi n^{1/2}t+\frac{\pi}{4}\right)$$
$$= \sum_{\nu}\sum_{q}\left(1-\frac{(\nu q^2)^{1/2}}{X^{1/2}}\right)\frac{\Phi(\nu q^2)}{(\nu q^2)^{3/4}}\cos\left(2\pi q\nu^{1/2}t+\frac{\pi}{4}\right),$$

where the external sum is taken over all the squarefree ν and the inner sum is taken over all $1\leqslant q\leqslant [\sqrt{X/\nu}]$.

Let

$$Q_q(t_\nu) = \sum_{q}\left(1-\frac{(\nu q^2)^{1/2}}{X^{1/2}}\right)\frac{\Phi(\nu q^2)}{(\nu q^2)^{3/4}}\cos\left(2\pi q t_\nu+\frac{\pi}{4}\right),$$

where $t_\nu = \sqrt{\nu} t$.

Parseval's equality gives for $Q_q(t_\nu)$

$$\int_0^1 Q_q^2(t_\nu)\,dt_\nu = \frac{1}{2}\sum_{q}\left(1-\frac{(\nu q^2)^{1/2}}{X^{1/2}}\right)^2\frac{|\Phi(\nu q^2)|^2}{(\nu q^2)^{3/2}}$$

and it is obvious that

$$|Q_q(t_\nu)| \leqslant \sum_{q}\left(1-\frac{(\nu q^2)^{1/2}}{X^{1/2}}\right)\frac{|\Phi(\nu q^2)|}{(\nu q^2)^{3/4}}.$$

In virtue of Lemma 3 there exists $0\leqslant t^*\leqslant 1$ such that

$$Q_q(t_\nu^*) \geqslant \frac{1}{2}\left(\sum_{q}\left(1-\frac{(\nu q^2)^{1/2}}{X^{1/2}}\right)^2\frac{|\Phi(\nu q^2)|^2}{(\nu q^2)^{3/2}}\right)\left(\sum_{q}\left(1-\frac{(\nu q^2)^{1/2}}{X^{1/2}}\right)\frac{|\Phi(\nu q^2)|}{(\nu q^2)^{3/4}}\right)^{-1}.$$

Using Cauchy's inequality we get

$$Q_q(t_\nu^*) \geqslant \frac{1}{2}\left(\frac{\nu}{X}\right)^{1/4}\left(\sum_{q}\left(1-\frac{(\nu q^2)^{1/2}}{X^{1/2}}\right)^2\frac{|\Phi(\nu q^2)|^2}{(\nu q^2)^{3/2}}\right)^{1/2}.$$

If $\nu = p$, p is a prime number, $p\equiv 1 \pmod 4$ then as it is well known $r(p) = 8$, where $r(p)$ is the number of representations of p as the sum of two squares. Further it is known that

$$\frac{r(n_1\cdot n_2)}{4} = \frac{r(n_1)}{4}\cdot\frac{r(n_2)}{4}$$

if $(n_1, n_2) = 1$.

Thus we get $r(k^2p) = 8$, $1\leqslant k\leqslant 8$ and $k\neq 5$ and it follows

$$\Phi(k^2p) = \sum_{a^2+b^2=p} e^{2\pi ik(au+bv)}, \quad \text{where} \quad k\neq 5,\ 1\leqslant k\leqslant 8.$$

Using Lemma 4 we get for some $1\leqslant k_\nu\leqslant 8$, $k_\nu\neq 5$ and for all $(u, v)\in G$ and $\nu = p\equiv 1 \pmod 4$, $p\leqslant X/2^{10}$

$$Q_q(t_\nu^*) \geqslant c_1\left(\frac{\nu}{X}\right)^{1/4}\frac{1}{\nu^{3/4}}.$$

Now we apply Lemma 5 to $k_\nu\sqrt{\nu}$, $\nu\leqslant X$ and $\omega = \dfrac{1}{10\sqrt{X}}$, $\varphi_j = t_j^*$.

Therefore there exists $T \leqslant \exp(\exp 3X)$ such that

$$Q(T) \geqslant \frac{c_1}{X^{1/4}} \sum_{\substack{1 \leqslant p \leqslant X/2^{10} \\ p \equiv 1 \pmod 4}} \frac{1}{p^{1/2}} \geqslant c_1 \frac{X^{1/4}}{\ln X}.$$

Hence

$$\underline{\sigma}(T-\eta, T+\eta) \leqslant -c_1 (\ln\ln T)^{1/4-\varepsilon}$$

for $T > T_0(\varepsilon)$ and $\varepsilon > 0$.

(26) is true, (25) follows by the same way, one has to use only the corollary of this lemma.

Remark. If we estimate the degree of linear independence more exactly (Lemma 5) the term $(\ln\ln\lambda)^{1/4-\varepsilon}$ in (25) and (26) can be replaced by $\left(\frac{\ln\ln\lambda}{\ln\ln\ln\lambda}\right)^{1/4}$.

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Some notes on k-th power residues

by

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Let k be a positive integer and p a rational prime satisfying $p \equiv 1 \pmod k$. We then define $n_k(p)$ to be the least positive integer which is not a kth power $(\bmod p)$. For the remaining primes we define it to be zero.

It is a long standing conjecture that the estimate $n_k(p) = O(p^\varepsilon)$ holds for any fixed value of $\varepsilon > 0$. In an average sense this result is known to be true since there is a constant c_k for which

$$\sum_{p<x} n_k(p) \sim c_k x/\log x,$$

as $x \to \infty$. For a proof of this result we refer for example to Elliott [5].

If we assume an extended form of the Riemann hypothesis then the method of N. C. Ankeny [1] shows that

$$n_k(p) = O\big((\log p)^2\big).$$

In the other direction, Chowla showed that there is a positive constant c for which $n_2(p) > c\log p$ holds infinitely often. It is our present purpose to show that a similar result holds for certain other values of k.

THEOREM 1. *If k is an odd prime there is a constant $d_k > 0$ for which*

$$n_k(p) > d_k \log p$$

holds infinitely often.

For the duration of this theorem, we assume that k is an odd prime.

We need two lemmas.

For an integer k let Q_k denote the cyclotomic field obtained by adjoining the kth roots of unity to the field of rational numbers Q. Let $\bar{Q}_k$ denote the ring of algebraic integers in this field. For any element α of $\bar{Q}_k$ we use $[\alpha]$ to denote the principal ideal generated in $\bar{Q}_k$ by α. Furthermore we take $\varrho = \exp(2\pi i/k)$ and $\lambda = 1-\varrho$ which are both algebraic integers of $\bar{Q}_k$.