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PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

ACTA ARITHMETICA
XIII (1967)

On natural numbers having unique factorization in a quadratic number field, II

by

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0. In [5] we considered the problem of distribution of natural numbers having a unique factorization (resp. whose all factorization have the same length) into irreducible factors in a quadratic number field in residue classes $(\bmod k)$ where k is a given natural number, relatively prime to the discriminant of the field. The purpose of this note is to remove the restriction on k. Note that in the announcement [4] of the results of [5] the condition on k was omitted due to an oversight. The result of this note shows nevertheless that the theorems stated in [4] are true without any restriction.

The crucial point in [5] where the assumption $(k, d) = 1$ was used (by d we shall denote the discriminant of the field in question) was Lemma 11. It turns out that this lemma fails without this assumption, however, we shall prove a substitute for it (see Lemma 3 below), which makes it possible to derive the results needed. Moreover it will be necessary to use a modification of the tauberian theorem used in [5] (Lemma 6), which can be derived from the generalization of Ikehara's theorem due to H. Delange ([1]).

1. At first we shall state the tauberian theorem on which our proof will be based:

LEMMA 1. *Let a be a real number, not equal to zero or a negative integer, let q be a non-negative integer, let $a_1, \ldots, a_q$ be complex numbers with $\operatorname{re} a_i < a$ $(i = 1, 2, \ldots, q)$ and let $\beta_0, \beta_1, \ldots, \beta_q$ be non-negative integers. Finally let $a_1, a_2, \ldots$ be a sequence of non-negative real numbers such that the series*

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

converges in the open half-plane $\operatorname{re} s > 1$ and assume that the following equality holds for $\operatorname{re} s > 1$:

$$f(s) = a_0(s)\left(\log\frac{1}{s-1}\right)^{\beta_0}\frac{1}{(s-1)^{a}} + \sum_{j=1}^{q} a_j(s)\left(\log\frac{1}{s-1}\right)^{\beta_j}\frac{1}{(s-1)^{a_j}} + b(s)$$

with $a_0(s), \ldots, a_q(s), b(s)$ *regular in the closed half-plane* $\operatorname{re} s \geqslant 1$ *and* $a_0(1) \neq 0$.

Then for x *tending to infinity*

$$\sum_{n \leqslant x} a_n \sim a_0(1) \Gamma(a)^{-1} x (\log x)^{a-1} (\log\log x)^{\beta_0}.$$

This lemma can be deduced from the tauberian theorem of H. Delange ([1], th. I) in the same way as theorem IV of that paper was deduced there. One has to observe only, that the function $\beta(t)$ occurring in th. I of [1] needs not to be real, as assumed there, which fact can be easily seen from the proof of the said theorem.

2. Now we shall prove the result which will replace in our arguments the Lemma 11 of [5]. First we shall introduce some definitions and notations: For any set P of rational primes we shall denote by $\omega_P(n)$ the number of distinct primes from P dividing n, and by $\Omega_P(n)$ the number of primes from P dividing n, each prime counted according to its multiplicity. The Galois group C_2 of $K = Q(d^{1/2})$ acts in an obvious way on the classgroup H of K and the orbit defined by $X \in H$ is clearly (X, X^{-1}). Let $\varepsilon(X) = 2$ if $X^2 = E$, the unit class, and let $\varepsilon(X) = 1$ otherwise.

Let k be a given natural number. We shall divide all ideals relatively prime to (k) (the ideal generated by k) into classes as follows: *two ideals belong to the same class if and only if they belong to the same absolute class and their norms are congruent* $(\operatorname{mod} k)$. The classes so defined form a group under multiplication and moreover each such class contains the same number, say $C(k)$, of ideal classes $(\operatorname{mod} (k))$. (Cf. [5], p. 12). Let $J_1, \ldots, J_N$ be the classes so defined.

For any absolute class X let $N_k(X)$ be the set of residues $(\operatorname{mod} k)$, relatively prime to k, which can be represented by norms of ideals from the class X. By $|N_k(X)|$ we shall denote the number of elements in $N_k(X)$. It is easy to see that this number is determined by k and the field in question, and does not depend on X. Let $h((k))$ be the number of classes $(\operatorname{mod} (k))$. Observe that the absolute unit class E contains $|N_k(E)|$ classes J_r each containing $C(k)$ classes $(\operatorname{mod} (k))$. This shows that

$$h((k)) = C(k)|N_k(E)|h. \tag{1}$$

We shall also use the following classical result:

LEMMA 2. (See e.g. [2], [3]). *Let* $\mathfrak{f}$ *be an ideal in* K *and* W *let be an ideal class* $(\operatorname{mod} \mathfrak{f})$.

Then the following equality holds for $\operatorname{re} s > 1$:

$$\sum_{\mathfrak{p} \in W} (N\mathfrak{p})^{-s} = \frac{1}{h(\mathfrak{f})} \log \frac{1}{s-1} + g(s)$$

where the sum is taken over all prime ideals in W, $h(\mathfrak{f})$ *is the number of classes* $(\operatorname{mod} \mathfrak{f})$, *and* $g(s)$ *is regular for* $\operatorname{re} s \geqslant 1$.

Now let X be an arbitrary absolute class and let χ be a non-principal character $(\operatorname{mod} k)$. Define

$$A(X, \chi) = \varepsilon^{-1}(X) h^{-1} |N_k(E)|^{-1} \sum_{g \in N_k(X)} \chi(g). \tag{2}$$

We prove the following

LEMMA 3. *Let* K *be a quadratic number field. Let* $X_1, \ldots, X_t$ *be ideal classes belonging to disjoint orbits and none of them equal to the unit class* E. *Let for* $i = 1, 2, \ldots, t$, P_i *be the set of all rational primes whose decomposition into prime ideals in* K *is of the form* $p = \mathfrak{p}_1\mathfrak{p}_2$ *with* $\mathfrak{p}_1 \in X_i$ *and* $\mathfrak{p}_2 \in X_i^{-1}$, *and let* $A_1, \ldots, A_t$ *be given non-negative integers, not all of them equal to zero. Finally let* k *be a given natural number and* χ *a non-principal character* $(\operatorname{mod} k)$. *If* f *stands for one of the symbols* Ω, ω *then the series*

$$\sum_n \chi(n) n^{-s}$$

(*where the sum is taken over all* n *which satisfy* $f_{P_i}(n) = A_i$ *for* $i = 1, 2, \ldots, t$) *defines in the half-plane* $\operatorname{re} s > 1$ *a regular function* $H_f(s, \chi)$ *which satisfies the following equality*:

$$H_f(s, \chi) = (s-1)^{-a} \sum_{j=0}^{r} g_j(s) \left(\log \frac{1}{s-1}\right)^j + g(s)$$

where $a = -\sum_{i=1}^{t} A(X_i, \chi)$, $r = A_1 + \ldots + A_t$ *and the functions* $g_0(s), \ldots, g_r(s), g(s)$ *are regular in the closed half-plane* $\operatorname{re} s \geqslant 1$. (*By the symbol* $g(s)$ *with or without indices we shall here and in the sequel denote functions regular for* $s \geqslant 1$, *not always the same*).

Proof. For every absolute class X we have, using Lemma 2

$$\sum_{\substack{p = N\mathfrak{p} \\ \mathfrak{p} \in X}} \chi(p) p^{-s} = \sum_{g \in N_k(X)} \chi(g) \sum_{\substack{p = N\mathfrak{p} \equiv g \,(\operatorname{mod} k) \\ \mathfrak{p} \in X}} p^{-s}$$

$$= \varepsilon^{-1}(X) \sum_{g \in N_k(X)} \chi(g) \sum_{\substack{\mathfrak{p} \in X \\ N\mathfrak{p} \equiv g \,(\operatorname{mod} k)}} N\mathfrak{p}^{-s}$$

$$= \varepsilon^{-1}(X) \sum_{g \in N_k(X)} \chi(g) \left\{ \frac{C(k)}{h((k))} \log \frac{1}{s-1} + g(s) \right\}$$

$$= \left\{ \frac{C(k)}{h((k))} \varepsilon^{-1}(X) \sum_{g \in N_k(X)} \chi(g) \right\} \log \frac{1}{s-1} + g(s)$$

$$= \left\{ \varepsilon^{-1}(X) h^{-1} |N_k(E)|^{-1} \sum_{g \in N_k(X)} \chi(g) \right\} \log \frac{1}{s-1} + g(s)$$

$$= A(X, \chi) \log \frac{1}{s-1} + g(s).$$

The obtained equality leads us, after a short computation similar to that done in [5], p. 12-13, to the following equality (where we write for shortness $\Omega_i(n)$ in place of $\Omega_{Q_i}(n)$):

$$\sum_{n=1}^{\infty} \frac{z_1^{\Omega_1(n)} \ldots z_t^{\Omega_t(n)} \chi(n)}{n^s}$$

$$= (s-1)^{-a} \prod_{j=1}^{t} \left(\sum_{n=0}^{\infty} z_j^n \left\{ \sum_{i=0}^{n} \frac{1}{i!} g_{n-i}^{(j)}(s) A(X_j, \chi)^i \log^i \frac{1}{s-1} \right\} \right)$$

$$(|z_i| < 1, i = 1, \ldots, t)$$

from which the statement of Lemma 3 for $f = \Omega$ follows immediately by equating coefficients. The proof for $f = \omega$ follows the same line. (Cf. [5], p. 13).

The behaviour of the series corresponding to the principal character χ_0 is described by

LEMMA 4. *Under the assumptions of Lemma 3 we have*

$$H_f(s, \chi_0) = \sum_n \chi_0(n) n^{-s} = (s-1)^{-\varrho} \sum_{j=0}^{r} g_j(s) \log^j \frac{1}{s-1} + g(s)$$

where the sum is extended over all n's with $f_{Q_i}(n) = A_i$ $(i = 1, 2, \ldots, t)$, $\varrho = 1 - h^{-1} \sum_{i=1}^{t} \varepsilon^{-1}(X_i)$, $r = A_1 + \ldots + A_t$, $g, g_1, \ldots, g_r$ are regular for $\operatorname{re} s \geqslant 1$ and $g_r(1) \neq 0$. Moreover the function $g_r(s)$ does not depend on f, i.e. is the same for $f = \Omega$ and $f = \omega$.

This result is contained in the proof of Lemma 8 in [5]. Cf. in particular the equality (10) there.

LEMMA 5. *Under the assumptions of Lemma 3 we have for $(k, j) = 1$ the following equality, holding for $\operatorname{re} s > 1$:*

$$\sum_{\substack{n \equiv j \,(\mathrm{mod}\, k) \\ f_{Q_i}(n) = A_i \,(i=1,\ldots,t)}} n^{-s}$$

$$= \frac{g_r(s)}{\varphi(k)} (s-1)^{-\varrho} \log^r \frac{1}{s-1} + \sum_{j=1}^{q} h_j(s) \log^{\beta_j} \frac{1}{s-1} (s-1)^{-\alpha_j} + g(s).$$

Here r, ϱ and $g_r(s)$ are the same as in Lemma 4, $q, \beta_1, \ldots, \beta_q$ are non-negative integers, $\alpha_1, \ldots, \alpha_q$ are complex numbers satisfying $\operatorname{re} \alpha_j < \varrho$ and $h_1, \ldots, h_q, g$ are regular for $\operatorname{re} s \geqslant 1$.

Proof. As evidently

$$\sum_{\substack{n \equiv j \,(\mathrm{mod}\, k) \\ f_{Q_i}(n) = A_i \,(i=1,2,\ldots,t)}} n^{-s} = \varphi^{-1}(k) \sum_{\chi} H_f(\chi, s)$$

the equality asserted in the statement of the lemma follows from Lemmas 3 and 4. The only thing which has to be proved is the inequality

$$(3) \qquad \operatorname{re}\Big(-\sum_{i=1}^{t} A(X_i, \chi)\Big) < 1 - h^{-1} \sum_{i=1}^{t} \varepsilon^{-1}(X_i) \quad \text{for} \quad \chi \neq \chi_0.$$

Let $\delta \in N_k(X_i)$ be fixed and consider the mapping ψ_i: $N_k(X_i) \to N_k(E)$ defined by $\psi_i(g) = g\delta^{-1}$. (The inverse here is to be understood as the inverse in the group of residue classes $(\mathrm{mod}\, k)$, relatively prime to k). Clearly ψ_i is one-to-one and onto, hence we get

$$\sum_{g \in N_k(X_i)} \chi(g) = \sum_{g \in N_k(X_i)} \chi(g\delta_i^{-1}) \chi(\delta_i) = \chi(\delta_i) \sum_{g \in N_k(E)} \chi(g)$$

which implies

$$\sum_{i=1}^{t} A(X_i, \chi) = |N_k(E)|^{-1} h^{-1} \sum_{i=1}^{t} \chi(\delta_i) \varepsilon^{-1}(X_i) \sum_{g \in N_k(E)} \chi(g).$$

If the character χ is not principal on $N_k(E)$, then (as $N_k(E)$ is a group), the right hand side of the last equality vanishes, and so (3) follows trivially. Assume thus that the character χ is principal on $N_k(E)$. In this case we get from the least equality

$$\sum_{i=1}^{t} A(X_i, \chi) = h^{-1} \sum_{i=1}^{t} \chi(\delta_i) \varepsilon^{-1}(X_i).$$

Observe now that this implies

$$\operatorname{re}\Big(-\sum_{i=1}^{t} A(X_i, \chi)\Big) \leqslant h^{-1}\sum_{i=1}^{t} \varepsilon^{-1}(X_i).$$

If (3) were not true, then we would have

$$h^{-1}\sum_{i=1}^{t} \varepsilon^{-1}(X_i) \geqslant 1 - h^{-1}\sum_{i=1}^{t} \varepsilon^{-1}(X_i),$$

i.e.

$$\sum_{i=1}^{t} \varepsilon^{-1}(X_i) \geqslant h/2. \tag{4}$$

But if the class-group H has s elements $X \neq E$ with $X^2 = E$, then we have $s+(h-s-1)/2$ orbits $\neq (E, E)$ and the sum of $\varepsilon^{-1}(X)$ (where X ranges over all classes $\neq E$ belonging to disjoint orbits) is equal to $s/2 + (h-s-1)/2 = (h-1)/2 < h/2$, whence (4) is impossible for any choice of the orbits $X_1, \ldots, X_t$. The lemma is thus proved.

The last lemma together with Lemma 1 implies that for $(j, k) = 1$

$$N\big(n \leqslant x \mid\ n \equiv j \,(\operatorname{mod} k),\ f_{Q_i}(n) = A_i\ (i = 1, 2, \ldots, t)\big)$$
$$\sim \varphi^{-1}(k) N\big(n \leqslant x \mid\ (n, k) = 1, f_{Q_i}(n) = A_i\ (i = 1, 2, \ldots, t)\big),$$

a result, which under the restriction $(k, d) = 1$ is contained in Lemma 12 of [5]. Now we can use the same arguments as presented in [5] (pp. 15-22) to get the desired conclusions. We state now the final results which can be obtained in this way:

THEOREM I. *Let K be a quadratic number field with the class-number $h \neq 1$, and let k and j be natural numbers with $(k, j) = 1$. If $F_{kj}(x)$ is the number of natural numbers not exceeding x, which are congruent to $j \,(\operatorname{mod} k)$ and have in K a unique factorization, then $F_{kj}(x)$ is asymptotically equal to*

$$C(k, K)\, x (\log\log x)^M (\log x)^{(1-h)/2h},$$

where M is a non-negative integer and $C(k, K)$ is a positive constant, whose precise values are described in [4], [5].

THEOREM II. *Let K be a quadratic field with $h \neq 1, 2$* (¹) *and let k and j are natural numbers with $(k, j) = D$. If $G_{kj}(x)$ is the number of natural numbers not exceeding x, which are congruent to $j \,(\operatorname{mod} k)$ and whose all factorizations in K have the same length, then either $G_{kj}(x)$ is zero (in the case when D has factorizations of different lengths) or $G_{kj}(x)$ is asymptotically equal to*

$$C_1(k, K, D)\, x (\log\log x)^N (\log x)^{(1+S(D)-h)/2h},$$

where N is a non-negative integer, $C_1(k, K, D)$ is a positive constant and $S(D)$ is a natural number, not exceeding the number g of even invariants of the class-group, whose precise values are described in [4], [5]. *In the case $D = 1$, $S(D)$ is equal to g.*

(¹) The cases $h = 1, 2$ are not interesting, as in those cases all factorizations of a given number in K have the same length.

We want to use this opportunity to note that in the statement of Theorem II in [5], the factor h^{-M} was omitted by an oversight.

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Reçu par la Rédaction le 28. 11. 1966