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On oscillations of number-theoretic functions

by

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1. Introduction. In recent years S. Knapowski, P. Turán and W. Staś obtained important results concerning localized omega-estimations of effective type of many number-theoretical functions. The proofs of their theorems are based on Turán's results in the theory of diophantine approximation. The author obtained some results of a similar type which were partially published in [2]. The following estimation is a typical one: if $T > c_1$, then

(1.1) $$\max_{T\leqslant x\leqslant T^{\varkappa}} M(x)x^{-1/2} > \delta, \quad \min_{T\leqslant x\leqslant T^{\varkappa}} M(x)x^{-1/2} < -\delta,$$

where $c_1 > 0$, $\delta > 0$ are explicitly calculable numerical constants and $\varkappa = (2+\sqrt{3})^2 = 14, \ldots$; $M(x) = \sum_{n\leqslant x}\mu(n)$, $\mu(n)$ the Möbius function. We also proved in [2], that (1.1) are fulfilled with $\varkappa = 1+\varepsilon$ (ε arbitrary positive constant) if $T > T_0(\varepsilon)$. But here we cannot estimate the value of $T_0(\varepsilon)$. It is natural to try to obtain an effective inequality of type (1.1) with better localization.

The aim of this paper is to refine the localization in (1.1) and in similar inequalities of some other functions.

In our estimations we shall use the numerical results of Rosser and Schoenfeld concerning the position of the roots of zeta-function [10].

2. Formulation of the results.

2.1. Let us introduce the following notations:

(2.1.1) $$M(x) = \sum_{n\leqslant x}\mu(n);$$

(2.1.2) $$M_0(x) = \sum_{n\leqslant x}\frac{\mu(n)}{n};$$

(2.1.3) $$S(\beta) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-(\beta/n)^2};$$

$$T(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)!\,\zeta(2k)}; \tag{2.1.4}$$

$$m(x) = \sum_{n=1}^{\infty} \mu(n) e^{-n/x}. \tag{2.1.5}$$

Let $\varrho_k(n)$ be the indicator function of k-free numbers, i.e.

$$\varrho_k(n) = \begin{cases} 1 & \text{if } n \text{ is a } k\text{-free number}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1.6}$$

$$P_k(x) = \sum_{n \leqslant x} \varrho_k(n) - \frac{x}{\zeta(k)}. \tag{2.1.7}$$

Let B $(\geqslant 2)$ be a positive number,

$$r = r(B) = \frac{1{,}01}{2\log 2B}, \tag{2.1.8}$$

$$\varkappa = \frac{1}{2} - r \log \frac{2e}{r}. \tag{2.1.9}$$

In the following $c_1, c_2, \ldots c_1', c_2', \ldots$ denote explicitly calculable numerical constants, and as usual $\log_1 x = \log x$, $\log_{\nu+1} x = \log(\log_\nu x)$ $(\nu = 1, 2, \ldots)$; $\exp x = e^x$.

THEOREM 1. *Suppose that the zeta-function of Riemann* ($\zeta(s) = \zeta(\sigma+it)$) *is non-vanishing in the domain*

$$\sigma > \tfrac{1}{2}, \quad |t| \leqslant B + 20. \tag{2.1.10}$$

Then for every $T > c_1 B^{c_1'}$ *we have*

$$\max_{T^\varkappa \leqslant x \leqslant T} M(x) x^{-1/2} > \delta, \quad \min_{T^\varkappa \leqslant x \leqslant T} M(x) x^{-1/2} < -\delta, \tag{2.1.11}$$

$$\max_{T^\varkappa \leqslant x \leqslant T} M_0(x) x^{1/2} > \delta, \quad \min_{T^\varkappa \leqslant x \leqslant T} M_0(x) x^{1/2} < -\delta, \tag{2.1.12}$$

$$\max_{T^\varkappa \leqslant x \leqslant T} S(x) x^{1/2} > \delta, \quad \min_{T^\varkappa \leqslant x \leqslant T} S(x) x^{1/2} < -\delta, \tag{2.1.13}$$

$$\max_{T^\varkappa \leqslant x \leqslant T} T(x) x^{-1/2} > \delta, \quad \min_{T^\varkappa \leqslant x \leqslant T} T(x) x^{-1/2} < -\delta, \tag{2.1.14}$$

$$\max_{T^\varkappa \leqslant x \leqslant T} m(x) x^{-1/2} > \delta, \quad \min_{T^\varkappa \leqslant x \leqslant T} m(x) x^{-1/2} < -\delta, \tag{2.1.15}$$

and for $T > c_2 \exp(c_2' k \log k \cdot \log B)$

$$\max_{T^\varkappa \leqslant x \leqslant T} P_k(x) x^{-1/2k} > \delta/k, \quad \min_{T^\varkappa \leqslant x \leqslant T} P_k(x) x^{-1/2k} < -\delta/k, \tag{2.1.16}$$

where δ *is a suitable, explicitly calculable positive numerical constant;* $\varkappa$ *as in* (2.1.9).

Using numerical computations of Rosser and Schoenfeld we can choose $\varkappa = 0{,}36$.

THEOREM 2. *Let us suppose that the zeta-function is non-vanishing in the domain* (2.1.10). *Then for every* $T > c_2 B^{c_2'}$ *we have*

$$\int_1^T \frac{|M(x)|}{x}\, dx > \delta T^{\varkappa/2}, \tag{2.1.17}$$

$$\int_1^T |M_0(x)|\, dx > \delta T^{\varkappa/2}, \tag{2.1.18}$$

$$\int_1^T |S(x)|\, dx > \delta T^{\varkappa/2}, \tag{2.1.19}$$

$$\int_1^T \frac{|T(x)|}{x}\, dx > \delta T^{\varkappa/2}, \tag{2.1.20}$$

$$\int_1^T \frac{|m(x)|}{x}\, dx > \delta T^{\varkappa/2}, \tag{2.1.21}$$

and for $T > c_2 \exp(c_2' k \log k \cdot \log B)$

$$\int_1^T \frac{|P_k(x)|}{x}\, dx > \frac{\delta}{k} T^{\varkappa/2k}, \tag{2.1.22}$$

where δ *is a suitable explicitly calculable positive numerical constant and* $\varkappa$ *as in* (2.1.9).

Using numerical computations [10] we can choose $\varkappa = 0{,}36$.

Remarks. The first investigations concerning the oscillatory properties of $M(x)$ (in an effective sense) were made by S. Knapowski ([6], [7], [8]). In his paper [7] he proved the following: assuming that in the rectangle $0 \leqslant \sigma \leqslant 1$, $|t| \leqslant \omega$ all the roots of the zeta-function lie on the line $\sigma = \frac{1}{2}$, we have for all T in the interval $c_1 \leqslant T \leqslant \exp(\omega^{10})$

$$\max_{1 \leqslant x \leqslant T} M(x) \geqslant T^{1/2} \lambda(T), \quad \min_{1 \leqslant x \leqslant T} M(x) \leqslant -T^{1/2} \lambda(T),$$

$$\lambda(T) = \exp\left(-15 \frac{\log T \log_3 T}{\log_2 T}\right).$$

It is evident that the changing of the sign infinitely many times follows from this assertion only if $\omega = \infty$.

In this case (i.e. if the Riemann conjecture is true) the author obtained a slightly better estimation [3], namely if the Riemann conjecture is true, then

$$\max_{1\leqslant x\leqslant T} M(x) \geqslant T^{1/2}\exp\left(-c_2(\log_2 T)^2\right), \quad \min_{1\leqslant x\leqslant T} M(x) \leqslant -T^{1/2}\exp\left(-c_2(\log_2 T)^2\right),$$

for every $T > c_2$.

M. Riesz stated that the estimation $T(x) = O(x^{1/2+\varepsilon})$ and the Riemann conjecture are equivalent [9]. Later, using the ideas of M. Riesz, Hardy and Littlewood proved the equivalence of the Riemann conjecture and of the estimation (see [1])

$$\sum_{k=1}^{\infty} \frac{(-x)^k}{k!\,\zeta(2k+1)} = O(x^{-\frac{1}{4}+\varepsilon}). \tag{2.1.23}$$

Using the substitution $x = \beta^2$ we can write (2.1.23) in the form

$$S(\beta) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-(\beta/n)^2} = O(\beta^{-\frac{1}{2}+\varepsilon}).$$

From the theorem of Hardy and Littlewood it follows immediately that if the Riemann conjecture does not hold we have $S(\beta) = \Omega(\beta^{-1/2})$. W. Staś dealt with the localized Ω-estimation of $S(\beta)$ ([11], [12], [13]), and in [13] he proved the following theorem: *if the Riemann conjecture is true, then for $T > c_1$*

$$\max_{T^{1-o(1)}\leqslant\beta\leqslant T} |S(\beta)| > T^{-\frac{1}{2}-o(1)}.$$

I can prove the following one-sided theorem: *if the Riemann conjecture is true, then for $T > c_1$ we have*

$$\max_{T_1\leqslant x\leqslant T} S(x) \geqslant T^{-1/2}\psi(T), \qquad \min_{T_1\leqslant x\leqslant T} S(x) \leqslant -T^{-1/2}\psi(T),$$

$$\max_{T_1\leqslant x\leqslant T} T(x) \geqslant T^{1/2}\psi(T), \qquad \min_{T_1\leqslant x\leqslant T} T(x) \leqslant -T^{1/2}\psi(T),$$

where

$$T_1 = T\exp\left(-\frac{\log T\log_3 T}{(\log_2 T)^2}\right), \qquad \psi(T) = \exp\left(-c_2(\log_2 T)^2\right).$$

The proofs of these results are not published.

In my dissertation [4] and in [2] I obtained results for these functions without any conjectures. Here the inequalities (2.1.11)-(2.1.16) were proved with $\varkappa = (2-\sqrt{3})^2$. In the proofs of those theorems I used an idea of Rodossky.

Concerning our Theorem 2 we note that S. Knapowski obtained conditional results for the lower estimation of $\int_1^T |M(x)|x^{-1}dx$. In my opinion the inequality (2.1.17) is the first unconditional result for the integral mean value of $|M(x)|$.

We shall deduce our Theorems from two general theorems concerning the Dirichlet integrals.

2.2. Let $A(x)$ be a real function on the interval $1 \leqslant x < \infty$, $A(1) = 0$, and

$$f(s) = \int_1^{\infty} \frac{dA(x)}{x^s}. \tag{2.2.1}$$

We assume that

$$\int_1^x |dA(u)| \leqslant c_1 x^{\theta_1}, \tag{2.2.2}$$

where $0 < \theta_1 \leqslant 1$.

Further, let $f(s)$ be analytically continuable into the half-plane $\sigma > (1-\varepsilon)\theta_2$ $(s = \sigma + it)$, where $0 < \varepsilon < 1$.

Let $f(s)$ have a simple pole at the point

$$\varrho = \theta_2 + i\gamma \qquad (\gamma > 0) \tag{2.2.3}$$

with residue b,

$$b = \operatorname*{Res}_{s=\varrho} f(s), \tag{2.2.4}$$

and let $\gamma < c_2$, where c_2 is a constant.

Further, let $f(s)$ be regular in the domain D except for the point ϱ, where D is defined as

$$D = \{s = \sigma + it;\ \sigma \geqslant \theta_2(1-\varepsilon),\ -\varepsilon_1\theta_2 \leqslant t < \gamma + \varepsilon_1\theta_2\}, \quad 0 < \varepsilon_1 \leqslant \varepsilon. \tag{2.2.5}$$

Let

$$M = \max_{s\in\Delta} \max\left(|f(s)|, |f(s+i\gamma)|\right), \tag{2.2.6}$$

where Δ is a broken-line with vertices

$$\theta_1 - \varepsilon_1\theta_2 i, \quad \theta_2(1-\varepsilon_1) - \varepsilon_1\theta_2 i, \quad \theta_2(1-\varepsilon_1) + \varepsilon_1\theta_2 i, \quad \theta_1 + \varepsilon_1\theta_2 i. \tag{2.2.7}$$

Let $f(s)$ be regular in the strip $\sigma > \theta_2$, $|t| \leqslant \theta_2(B+2)+\gamma$ and let

(2.2.8) $$|f(\sigma+it)| \leqslant (|t|+\theta_2)^{-c_3\log(\sigma-1)} \quad \text{in} \quad \sigma > \theta_2,\ |t| \leqslant \theta_2(B+1)+\gamma.$$

Suppose that for every real T there exists a t in the interval $T \leqslant t \leqslant T+\theta_2$ such that

(2.2.9) $$|f(\sigma+it)| \leqslant (|t|+\theta_2)^{c_4} \quad \text{if} \quad \sigma \geqslant \theta_2.$$

Finally let us assume that there exist a positive ε_2 such that

(2.2.10) $$\frac{\theta_1-\theta_2}{\log B}(1+2\varepsilon_2) < \frac{\theta_2\varepsilon_1}{\log(1/(1-\varepsilon_1))}.$$

If these conditions are satisfied, then the following assertion holds:

THEOREM A. *For* $T > c_5$ *we have*

(2.2.11) $$\max_{T^\beta \leqslant x \leqslant T} A(x)x^{-\theta_2} > \delta, \qquad \min_{T^\beta \leqslant x \leqslant T} A(x)x^{-\theta_2} < -\delta,$$

where

(2.2.12) $$\delta = \frac{|b|}{20(\theta_2+\gamma)},$$

(2.2.13) $$r = (1+\varepsilon_2)\frac{\theta_1-\theta_2}{\log B},$$

(2.2.14) $$\beta = \frac{\theta_2 - r\log(\theta_2 e/r)}{\theta_1},$$

$\varepsilon_2 > 0$ *is an arbitrary constant satisfying* (2.2.10) *and* c_5 *is a numerically calculable function of* $c_1, \ldots, c_4, \varepsilon_2, M, B, b$.

Suppose that the conditions of the Theorem A are satisfied, except the assumption that $\gamma > 0$ in (2.2.3).

Introduce the notation

(2.2.15) $$K(x) = \int_1^x \frac{|A(u)|}{u}\,du.$$

The following assertion holds:

THEOREM B. *For* $T > c_6$ *we have*

(2.2.16) $$K(T) > \frac{0{,}4|b|}{\theta_2^2} T^{\beta\theta_2},$$

where β *is defined by* (2.2.14) *and* c_6 *is a numerically calculable function of* $c_1, \ldots, c_4, \varepsilon_2, M, B, b$.

3. Deduction of the Theorems 1 and 2 from the Theorems A and B.

3.1. Lemmas. We shall need the following lemmas:

LEMMA 1. *If the function* $\zeta(s)$ *is non-vanishing in the domain*

$$\sigma > \tfrac{1}{2}, \quad |t| \leqslant B+3 \quad (s = \sigma+it),$$

then

(3.1.1) $$|\zeta^{-1}(\sigma+it)| \leqslant (|t|+1)^{-c_1\log(\sigma-\frac{1}{2})},$$

in $\frac{1}{2} < \sigma \leqslant \frac{5}{4}$, $|t| \leqslant B$.

For the proof see [5], Lemma 2.

LEMMA 2 ([14]). *For every real* T *we can find a* t *in* $T \leqslant t \leqslant T+1$, *such that*

(3.1.2) $$|\zeta^{-1}(\sigma+it)| \leqslant (|t|+1)^{c_2}$$

in the interval $-1 \leqslant \sigma \leqslant 2$.

It is known that the function $\zeta(s)$ has a simple root at the point $\varrho_0 = \frac{1}{2}+i\gamma_0$ ($\gamma_0 = 14{,}13\ldots$), further, that in $\sigma \geqslant 0$, $|t| \leqslant 20$ all the roots of $\zeta(s)$ are ϱ and $\bar{\varrho}$.

3.2. Since

$$\frac{1}{\zeta(s)} = \int_1^\infty \frac{dM(x)}{x^s}, \quad M(1) = 0, \quad |M(x)| \leqslant x,$$

we can apply Theorem A with $\theta_1 = 1$, $\theta_2 = \frac{1}{2}$, $\varrho = \varrho_0$ (see Lemmas 1, 2). Hence (2.1.11) follows.

For the proof of (2.1.12) let

$$f(s) = \int_1^\infty x^{-s}\,dxM_0(x) = \int_1^\infty \frac{M_0(x)}{x^s}\,dx = \frac{1}{(s-1)}\cdot\frac{1}{\zeta(s)}, \quad A(x) = xM_0(x).$$

It is known that $f(s)$ is regular in $s = 1$. Let $\theta_1 = 1$, $\theta_2 = \frac{1}{2}$, $\varrho = \varrho_0$. Using Theorem A we obtain (2.1.12).

For the proof of (2.1.15) we choose $A(x) = m(x) - m(1)$,

$$f(s) = \int_1^\infty \frac{d\big(m(x)-m(1)\big)}{x^s} = m(1) + s\int_0^1 m\left(\frac{1}{y}\right)y^{s-1}dy$$
$$= m(1) + s\int_0^\infty m\left(\frac{1}{y}\right)y^{s-1}dy - \varphi(s),$$

where

$$\varphi(s) = \int_1^\infty m\left(\frac{1}{y}\right)y^{s-1}dy$$

is an integral function and

$$|\varphi(\sigma+it)| < \int_1^\infty e^{-y} y^{\sigma-1} dy < \Gamma(\sigma) \quad \text{for} \quad \sigma > 0.$$

Using the relation

$$\int_0^\infty y^{s-1} \sum_{n=1}^\infty \mu(n) e^{-ny} dy = \frac{\Gamma(s)}{\zeta(s)},$$

we obtain

$$f(s) = s\frac{\Gamma(s)}{\zeta(s)} + m(1) - \varphi(s).$$

In the domain $\frac{1}{4} \leqslant \sigma \leqslant 2$ we have $|\Gamma(\sigma+it)| < c_3$. Hence easily follows (2.1.15).

It is known that

(3.2.1) $$\beta S(\beta) = \frac{1}{2\pi i} \int_{(\frac{1}{2}+\varepsilon)} \beta^{2s} \frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)} ds \quad (\varepsilon > 0).$$

From this, by the Mellin formula, we obtain

$$\int_0^\infty \beta^{-s} \frac{S(\sqrt{\beta})}{\sqrt{\beta}} d\beta = \frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}.$$

Using the prime-number theorem in the form $\sum \mu(n)/n = 0$, we have

$$S(\beta) = \sum_{n=1}^\infty \frac{\mu(n)}{n} (e^{-(\beta/n)^2} - 1) = O(\beta^2),$$

if $0 \leqslant \beta \leqslant 1$.

So the function $\varphi(s)$ defined by

$$\varphi(s) = \int_0^1 \beta^{-s} \frac{S(\sqrt{\beta})}{\sqrt{\beta}} d\beta$$

is regular and $\varphi(s) = O\left(\frac{1}{\frac{3}{2}-\sigma}\right)$ on the half-plane $\sigma < \frac{3}{2}$. By partial integration

$$h(s) \stackrel{\text{def}}{=} \int_1^\infty \beta^{-s} d\big(\beta^{1/2} S(\sqrt{\beta})\big) = s\left(\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)} - \varphi(s)\right) + S(1).$$

Transforming the integration line in (3.2.1) onto $\operatorname{Re} s = \frac{1}{2}$ ($\Gamma(\frac{1}{2}-s)/\zeta(2s)$ being regular on $\sigma = \frac{1}{2}$), we get

(3.2.2) $$|\beta S(\beta)| \leqslant \frac{\beta}{2\pi} \int_{-\infty}^\infty \frac{|\Gamma(0-it)|}{|\zeta(1+2it)|} dt.$$

Now let $f(s) = h(s)$, $A(\beta) = \beta^{1/2} S(\sqrt{\beta}) - S(1)$, $\theta_1 = \frac{1}{2}$, $\theta_2 = \frac{1}{4}$. From (3.2.2) follows $|A(\beta)| < c_5\sqrt{\beta}$. Applying Theorem A we obtain (2.1.13).

The inequalities concerning $T(x)$ can be proved similarly.

It is known that $|P_k(x)| < cx^{1/k}$. Now let us choose

$$f(s) = \int_1^\infty x^{-s} dP_k(x) = \sum_{n=1}^\infty \frac{\varrho_k(n)}{n^s} - \frac{1}{\zeta(k)(s-1)} = \frac{\zeta(s)}{\zeta(ks)} - \frac{1}{\zeta(k)(s-1)}.$$

The function $f(s)$ is regular in $s = 1$. From the assumption (2.1.10) it follows that $f(\sigma+it)$ is regular in $\sigma > 1/k$, $|t| \leqslant (B+20)/k$, and we can apply Theorem A with $\theta_1 = 1/k$, $\theta_2 = 1/2k$, $\varrho = \varrho_0/k$. Further,

$$\operatorname*{Res}_{s=\varrho_0/k} f(s) = \zeta\left(\frac{1}{2k} + i\frac{\gamma_0}{k}\right) \frac{1}{k} \operatorname*{Res}_{s=\varrho_0} \frac{1}{\zeta(s)}.$$

The deduction of the inequalities (2.1.17)-(2.1.21) is quite similar.

4. Proofs of Theorems A and B.

4.1. For the proof we need the following

LEMMA 3. *If*

(4.1.1) $$f(s) = \int_1^\infty \frac{dA(x)}{x^s},$$

$A(x)$ is a real function and

(4.1.2) $$\int_1^x |dA(u)| < c_1 x^{\theta_1}, \quad A(1) = 0,$$

then for every integer $k \geqslant 1$ and for every real τ the integral

(4.1.3) $$I_k(\tau) = \frac{1}{2\pi i} \int_{(\sigma)} f(w+it) \frac{x^w}{w^{k+1}} dw$$

exists if $\operatorname{Re} w = \sigma > \theta_1$ and is equal to

(4.1.4) $$I_k(\tau) = \frac{1}{k!} \int_1^x \left(\log\frac{x}{u}\right)^k e^{-i\tau \log u} dA(u).$$

Proof. The existence of the integral in (4.1.3) is evident from (4.1.2). From the definition of $f(s)$ we have

$$\frac{1}{2\pi i}\int_{(\sigma)} f(w+i\tau)\frac{x^w}{w^{k+1}}\,dw = \int_1^\infty e^{-i\tau\log u}\left\{\frac{1}{2\pi i}\int_{(\sigma)}\left(\frac{x}{u}\right)^w\frac{1}{w^{k+1}}\,dw\right\}dA(u)$$

and, changing the order of integration, we obtain

$$= \frac{1}{k!}\int_1^\infty \left(\log\frac{x}{u}\right)^k e^{-i\tau\log u}\,dA(u)$$

by using the well-known relation

$$\frac{1}{2\pi i}\int_{(\sigma)}\left(\frac{x}{u}\right)^w\frac{dw}{w^{k+1}} = \begin{cases}\dfrac{1}{k!}\left(\log\dfrac{x}{u}\right)^k, & \text{if}\quad u \leqslant x,\\ 0, & \text{if}\quad u > x.\end{cases}$$

4.2. In this section we shall estimate the integrals $I_k(\gamma)$, $I_k(0)$. Let us take the points $A_1, \ldots, A_{10}, A_2', A_3', A_8', A_9'$, on the complex plane defined by

$$A_1\colon\ \theta_1+\frac{1}{\log x}-i\infty;\qquad A_2\colon\ \theta_1+\frac{1}{\log x}-iB_1;\qquad A_3\colon\ \theta_2+\frac{1}{\log x}-iB_1;$$

$$A_4\colon\ \theta_2+\frac{1}{\log x}-i\varepsilon_1\theta_2;\qquad A_5\colon\ \theta_2(1-\varepsilon_1)-i\varepsilon_1\theta_2;\qquad A_6\colon\ \theta_2(1-\varepsilon_1)+i\varepsilon_1\theta_2;$$

$$A_7\colon\ \theta_2+\frac{1}{\log x}+i\varepsilon_1\theta_2;\qquad A_8\colon\ \theta_2+\frac{1}{\log x}+iB_2;\qquad A_9\colon\ \theta_1+\frac{1}{\log x}+iB_2;$$

$$A_{10}\colon\ \theta_1+\frac{1}{\log x}+i\infty;\qquad A_2'\colon\ \theta_1+\frac{1}{\log x}-iB_1'\colon\qquad A_3'\colon\ \theta_2+\frac{1}{\log x}-iB_1';$$

$$A_8'\colon\ \theta_2+\frac{1}{\log x}+iB_2';\qquad A_9'\colon\ \theta_1+\frac{1}{\log x}+iB_2',$$

where B_1, B_2, B_1', B_2' are chosen so that

$$\theta_2 B \leqslant B_1, B_2, B_1', B_2' \leqslant \theta_2(B+1),$$

and

$$|f(\sigma-iB_1)|,\ |f(\sigma+iB_2)|,\ |f(\sigma+i\gamma-iB_1')|,\ |f(\sigma+i\gamma+iB_2')| \leqslant [\theta_2(B+1)]^{c_4}. \tag{4.2.1}$$

(The existence of B's follows from (2.2.9).)

Transform the integration line of $I_k(0)$ and $I_k(\gamma)$ into the broken line $A_1, \ldots, A_{10}$ and $A_1, A_2', A_3', A_4, \ldots, A_7, A_8', A_9', A_{10}$ respectively.

Let us denote by $K_i(\tau)$ ($\tau = 0$ or γ) the part of $I_k(\tau)$ on the ith segment.

From the condition (2.2.2) it follows immediately that

$$|f(\sigma+it)| < \frac{c_1}{\sigma-1} \tag{4.2.2}$$

in the half-plane $\sigma > \theta_1$.

Now let

$$k = [r\log x],\quad r \text{ as in (2.2.13)}. \tag{4.2.3}$$

From (4.2.2) and (4.2.3) it follows that any of the integrals $|K_1(\tau)|$, $|K_9(\tau)|$ ($\tau = 0$ or γ) is smaller than

$$\frac{c_1 e x^{\theta_1\log x}}{2\pi}\int_{B\theta_2}^{\infty}\frac{dt}{|t|^{k+1}} < \frac{|b|}{100}\cdot\frac{x^{\theta_2}}{\theta_2^{k+1}}, \tag{4.2.4}$$

if $x > c_6$, where

$$c_6 = \exp\left([\varepsilon_2(\theta_1-\theta_2)]^{-1}\log\frac{|b|c}{(\theta_1-\theta_2)\varepsilon_2}\right),\qquad c \text{ absolute constant}.$$

Similarly, from (2.2.9) it follows that

$$|K_2(\tau)|,\ |K_8(\tau)| < \frac{c_1 e(\theta_1-\theta_2)x^{\theta_1}}{2\pi(B\theta_2)^{k+1}} < \frac{|b|}{100}\cdot\frac{x^{\theta_2}}{\theta_2^{k+1}}, \tag{4.2.5}$$

if $x > c_6$.

Further, using (2.2.8), we have

$$|K_3(\tau)|,\ |K_7(\tau)| < e x^{\theta_2}\int_{\varepsilon_1\theta_2}^{(B+1)\theta_2}\frac{(|t|+\theta_2)^{c_4\log\log x}}{(\theta_2^2+t^2)^{k/2}}\,dt \leqslant \frac{|b|}{100}\cdot\frac{x^{\theta_2}}{\theta_2^{k+1}} \tag{4.2.6}$$

if $x > c_7$, where

$$c_7 = \exp\left(2\frac{c_4^{\log 2\theta_2}}{r\log(1+\varepsilon_1^2)}\right) + \exp\left(\frac{2\log(100/|b|)}{r\log(1+\varepsilon_1^2)}\right).$$

Further,

$$|K_4(\tau)|,\ |K_6(\tau)| \leqslant \frac{Mex^{\theta_2}}{2\pi\,\theta_2^k(1+\varepsilon_1^2)^{k/2}} < \frac{|b|}{100}\cdot\frac{x^{\theta_2}}{\theta_2^{k+1}}, \tag{4.2.6}$$

if $x > c_9$, where

$$c_9 = \exp\left(\frac{2\log(100Me/|b|)}{\varepsilon_1^2 r}\right).$$

Further

$$|K_5(\tau)| \leqslant M\frac{x^{\theta_2}}{\theta_2^{k+1}}\cdot\frac{x^{-\theta_2\varepsilon_1}}{(1-\varepsilon_1)^k}. \tag{4.2.7}$$

Using the inequality (2.2.10), we have

$$\frac{x^{-\theta_2\varepsilon_1}}{(1-\varepsilon_1)^k} = x^{-\theta_2\varepsilon_1+r\log(1/1-\varepsilon_1)} < \exp\left(-\varepsilon_2\frac{\theta_1-\theta_2}{\log B}\log\frac{1}{1-\varepsilon_1}\log x\right) < \frac{|b|}{100M}$$

for $x > c_{10}$, where

$$c_{10} = \exp\left(\frac{\log(100M/|b|)}{\varepsilon_2 r\log(1/1-\varepsilon_1)}\right),$$

and so

$$|K_5(\tau)| < \frac{|b|}{100}\cdot\frac{x^{\theta_2}}{\theta_2^{k+1}}. \tag{4.2.8}$$

From the inequalities (4.2.4)-(4.2.8) we obtain

$$|I(0)| < 0{,}1\frac{|b|x^{\theta_2}}{\theta_2^{k+1}}, \tag{4.2.9}$$

for $x > \max(c_6, c_7, c_8, c_9, c_{10}) = c_{11}$.

Since

$$I(\gamma) = \frac{bx^{\theta_2}}{\theta_2^{k+1}} + \sum_{j=1}^{9} K_j(\gamma),$$

we have

$$|I(\gamma)| > 0{,}9\,|b|\frac{x^{\theta_2}}{\theta_2^{k+1}} \quad \text{for} \quad x > c_{11}. \tag{4.2.10}$$

4.3. We obtain from (4.1.4) by partial integration

$$I(\tau) = \frac{1}{k!}\int_1^x e^{-i\tau\log u}\left(\log\frac{x}{u}\right)^{k-1}\left(k+i\tau\log\frac{x}{u}\right)\frac{A(u)}{u}\,du. \tag{4.3.1}$$

Introduce for the sake of brevity the following notation:

$$B_\tau(x, y, k) = \frac{1}{k!}\int_y^x e^{-i\tau\log u}\left(\log\frac{x}{u}\right)^{k-1}\left(k+i\tau\log\frac{x}{u}\right)\frac{A(u)}{u}\,du, \tag{4.3.2}$$

$$I(x, y, k) = \frac{1}{(k-1)!}\int_y^x u^{\theta_2-1}\left(\log\frac{x}{u}\right)^{k-1}du. \tag{4.3.3}$$

Now let $\delta > 0$ be constant and suppose that at least one of the functions $A(u)\pm\delta u^{\theta_2}$ has a constant sign on the interval $y \leqslant u \leqslant x$.

From this assumption we deduce the inequality

$$|I_k(\tau)| \leqslant 2c_1y^{\theta_1}\left\{\frac{(\log x)^k}{k!}+\frac{(\log x)^{k+1}|\tau|}{(k+1)!}\right\}+ \\ +2\delta\big(I(x, y, k)+I(x, y, k+1)\,|\tau|\big)+|I_k(0)|+|\tau|\,|I_{k+1}(0)|. \tag{4.3.4}$$

Using our assumption we have

$$|B_\tau(x, y, k)| \leqslant \frac{1}{k!}\int_y^x\left(\log\frac{x}{u}\right)^{k-1}\left|k+|\tau|\log\frac{x}{u}\right|\left\{\frac{|A(u)\pm\delta u^{\theta_2}|+\delta u^{\theta_2}}{u}\right\}du$$

$$\leqslant \frac{1}{k!}\left|\int_y^x\left(\log\frac{x}{u}\right)^{k-1}\left(k+|\tau|\log\frac{x}{u}\right)\frac{A(u)}{u}\,du\right|+$$

$$+\frac{2\delta}{k!}\int_y^x\left(\log\frac{x}{u}\right)^{k-1}\left(k+|\tau|\log\frac{x}{u}\right)u^{\theta_2-1}du$$

$$\leqslant B_0(x, y, k)+|\tau|\,B_0(x, y, k+1)+$$

$$+2\delta\big(I(x, y, k)+|\tau|\,I(x, y, k+1)\big).$$

Further, for arbitrary real τ we have

$$|I_k(\tau)-B_\tau(x, y, k)| \leqslant \max_{1\leqslant u\leqslant y}|A(u)|\cdot\frac{1}{k!}\int_1^y\left(\log\frac{x}{u}\right)^{k-1}\left(k+|\tau|\log\frac{x}{u}\right)\frac{du}{u}$$

$$\leqslant 2c_1y^{\theta_1}\frac{(\log x)^k}{k!}\left(1+\frac{|\tau|\log x}{k+1}\right),$$

because

$$\max_{1\leqslant u\leqslant y}|A(u)| \leqslant \int_1^y|dA(u)| < c_1y^{\theta_1}$$

(see (2.2.2)). From these the inequality (4.3.4) follows.

Further,

$$I(x, y, k) \tag{4.3.5}$$

$$< \frac{1}{k!}\int_1^x u^{\theta_2}\left[\left(\log\frac{x}{u}\right)^k\right]'du = \frac{x^{\theta_2}}{\theta_2^k\cdot k!}\int_1^x e^{-\theta_2\log\frac{x}{u}}\,d\left(\theta_2\log\frac{x}{u}\right)^k$$

$$< \frac{x^{\theta_2}}{\theta_2^k\cdot k!}\int_0^\infty e^{-v}kv^{k-1}dv = \frac{x^{\theta_2}}{\theta_2^k}$$

holds.

Now let $\tau = \gamma$ and take into account the inequalities (4.2.9), (4.2.10), (4.3.4), (4.3.5). Then we have

$$0{,}9\frac{|b|\,x^{\theta_2}}{\theta_2^{k+1}} \leqslant 2c_1y^{\theta_1}\frac{(\log x)^k}{k!}\left(1+\frac{\gamma\log x}{k+1}\right)+\frac{2\delta x^{\theta_2}}{\theta_2^{k+1}}(\theta_2+\gamma). \tag{4.3.6}$$

Let $y = x^\beta$, where

$$\beta = \frac{\theta_2-r\log(\theta_2e/r)}{\theta_1}.$$

Using the Stirling-formula for $k!$ $(k! = (k/e)^k\sqrt{2\pi k}\,(1+o(1)))$, we obtain

$$2c_1 y^{\theta_1}\frac{(\log x)^k}{k!}\left(1+\frac{\gamma\log x}{k+1}\right) < \frac{|b|}{10}\cdot\frac{x^{\theta_2}}{\theta_2^{k+1}},$$

if $x > c_{12}$.

Now choose

$$\delta \leqslant \frac{|b|}{20(\theta_2+\gamma)}.$$

Then from (4.3.6) it follows that $\frac{9}{10}|b| < \frac{5}{10}|b|$, which is a contradiction. So

$$\max_{T^\beta\leqslant x\leqslant T}\frac{A(x)}{x^{\theta_2}} > \delta, \qquad \min_{T^\beta\leqslant x\leqslant T}\frac{A(x)}{x^{\theta_2}} < -\delta,$$

if $T > c_1$, and Theorem A is proved.

4.4. For the proof of Theorem B we start with the integral

$$I_k(\tau) = \frac{1}{k!}\int_1^x\left(\log\frac{x}{u}\right)^k e^{-i\tau\log u}\,dA(u).$$

Let us introduce the following notation:

$$D_\tau(x) = \int_1^x e^{-i\tau\log u}\frac{A(u)}{u}\,du. \tag{4.4.1}$$

Then partial integration gives

$$I_k(\tau) = \frac{1}{k!}\int_1^x\left(\log\frac{x}{u}\right)^{k-1}\left(k+i\tau\log\frac{x}{u}\right)dD_\tau(u)$$

$$= \frac{1}{k!}\int_1^x\frac{D_\tau(u)}{u}\left(\log\frac{x}{u}\right)^{k-2}\left\{(k-1)\left(k+i\tau\log\frac{x}{u}\right)+i\tau\right\}du.$$

Now let $y = x^\beta$. Then

(4.4.2) $|I_k(\tau)|$

$$\leqslant \max_{y\leqslant u\leqslant x}\frac{|D_\tau(u)|}{u^{\theta_2}}\cdot\frac{1}{k!}\int_y^x u^{\theta_2-1}\left(\log\frac{x}{u}\right)^{k-2}\left\{k(k-1)+|\tau|+|\tau|\log\frac{x}{u}\right\}du+$$

$$+\max_{1\leqslant u\leqslant y}|D_\tau(u)|\cdot\frac{1}{k!}\int_1^y\left(k(k-1)+|\tau|+|\tau|\log\frac{x}{u}\right)\left(\log\frac{x}{u}\right)^{k-2}\frac{du}{u}$$

holds.

Let us denote by $L_1(\tau)$ and $L_2(\tau)$ the first and the second integral of the right-hand side of (4.4.2).

It is evident that

$$L_1(\tau) = \left(1+\frac{|\tau|}{k(k-1)}\right)I(x,y,k-1)+\frac{|\tau|}{k}I(x,y,k).$$

So from (4.3.5) it follows that

$$L_1(\tau) \leqslant \frac{x^{\theta_2}}{\theta_2^k}\left\{\frac{|\tau|}{k}+\theta_2\left(1+\frac{|\tau|}{k(k-1)}\right)\right\}.$$

Let $\tau = \gamma$, k as in (4.2.3). Then we have

$$L_1(\gamma) \leqslant 2\frac{x^{\theta_2}}{\theta_2^{k-1}}, \tag{4.4.3}$$

if $x > c_{12}$, and we have

$$L_2(\gamma) \leqslant \frac{k(k-1)+\gamma}{k!}\int_1^y\left(\log\frac{x}{u}\right)^{k-2}\frac{du}{u}+\frac{\gamma}{k!}\int_1^y\left(\log\frac{x}{u}\right)^{k-1}\frac{du}{u} \tag{4.4.4}$$

$$< \frac{k(k-1)+\gamma}{k!}\cdot\frac{(\log x)^{k-1}}{(k-1)!}+\frac{\gamma(\log x)^k}{k\cdot k!}.$$

Further,

$$\max_{1\leqslant u\leqslant y}|D_\gamma(u)| \leqslant \int_1^y\frac{|A(u)|}{u}du < c_1\int_1^y u^{\theta_1-1}du \leqslant c_1\frac{y^{\theta_1}}{\theta_1}.$$

Using the Stirling formula for $k!$ we infer that the second term on the right-hand side of (4.4.2) is

$$\max_{1\leqslant u\leqslant y}|D_\gamma(u)|\,I_2(\gamma) < \frac{|b|}{10}\cdot\frac{x^{\theta_2}}{\theta_2^{k+1}},$$

if $x > c_{13}$. Further, from the inequality (4.2.10) it follows that

$$0{,}9\,|b|\frac{x^{\theta_2}}{\theta_2^{k+1}} \leqslant 2\max_{y\leqslant u\leqslant x}\frac{|D_\gamma(u)|}{u^{\theta_2}}\cdot\frac{x^{\theta_2}}{\theta_2^{k-1}}+\frac{|b|}{10}\cdot\frac{x^{\theta_2}}{\theta_2^{k+1}},$$

whence

$$\max_{y\leqslant u\leqslant x}\frac{K(u)}{u^{\theta_2}} \geqslant \max_{y\leqslant u\leqslant x}\frac{|D_\tau(u)|}{u^{\theta_2}} \geqslant \frac{0{,}4\,|b|}{\theta_2^2},$$

if $x > c_{13}$.

Now let $T > c_{13}$; then

$$K(T) \geqslant \left(\max_{T^\beta\leqslant u\leqslant T}\frac{K(u)}{u^{\theta_2}}\right)T^{\theta_2\beta} \geqslant \frac{0{,}4\,|b|}{\theta_2^2}T^{\beta\theta_2},$$

and Theorem B is proved.

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