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On the twin-prime problem, II

by

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To the 100th anniversary of B. Riemann's death

1. The asymptotical determination of the number of twin primes not exceeding x amounts essentially to the investigation of the sum

$$S_2(x) = \sum_{n \leqslant x} \Lambda(n)\Lambda(n+2)$$

or more generally for even D to

$$S_D(x) = \sum_{\substack{n \leqslant x \\ (n,D)=1}} \Lambda(n)\Lambda(n+D).$$

Several interesting results "of approximative character" have been achieved since Brun; but the only method which could produce at least a plausible heuristical asymptotical formula for $S_D(x)$ itself was Hardy-Littlewood's circle method. Their formula (see [1], p. 42, conjecture B) asserted that for fixed even D and $x \to \infty$ the relation

$$S_D(x) \sim B_1 x \prod_{\substack{p|D \\ p>2}} \frac{p-1}{p-2} \tag{1.1}$$

holds with (¹)

$$B_1 = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right). \tag{1.2}$$

(¹) Or equivalently denoting by $\pi_D(x)$ the number of prime pairs p_1, p_2 with $p_1 - p_2 = D$ not exceeding x,

$$\pi_D(x) \sim B_1 \frac{x}{\log^2 x} \prod_{\substack{p|D \\ p>2}} \frac{p-1}{p-2}.$$

Recently I observed that a much shorter heuristical deduction of (1.1) can be given which gives rise to new-type sieve methods. Starting namely from the formula

$$(1.3)\qquad \Lambda(m) = -\sum_{k|m}{}^{(k)} \mu(k)\log k \qquad \left(= \sum_{k|m}{}^{(k)} \mu(k)\log\frac{x}{k} \ \text{ for } m>1\right)$$

we get at once

$$S_D(x) = -\sum_{\substack{n\le x\\(n,D)=1}}{}^{(n)} \Lambda(n) \sum_{k|(n+D)}{}^{(k)} \mu(k)\log k = -\sum_{\substack{k\le x+D\\(k,D)=1}}{}^{(k)} \mu(k)\log k \sum_{\substack{n\le x\\ n\equiv -D \bmod k\\(n,D)=1}}{}^{(n)} \Lambda(n)$$

$$\sim -\sum_{\substack{(k,D)=1\\k\le x+D}}{}^{(k)} \mu(k)\log k \sum_{\substack{n\le x\\ n\equiv -D \bmod k}}{}^{(n)} \Lambda(n).$$

Replacing the inner sum heuristically by $x/\varphi(k)$ one gets heuristically

$$S_D(x) \sim x\left\{-\sum_{\substack{k\le x+D\\(k,D)=1}}{}^{(k)} \frac{\mu(k)\log k}{\varphi(k)}\right\}.$$

But as the proof of Lemma III will show rigorously the sum in brackets

$$\sim B_1 \prod_{\substack{p|D\\p>2}}{}^{(p)} \frac{p-1}{p-2}$$

which gives the formula (1.1) of Hardy-Littlewood at once. In principle the argument could be remedied by using for $\sum_{\substack{n\le x\\ n\equiv -D \bmod k}} \Lambda(n)$ instead of $x/\varphi(k)$ an "exact" prime-formula of Riemann-type but this seems hopelessly complicated. Instead of doing so I realised in the first paper of this series (see [3]) that "sieving on the generating Dirichlet series" leads at once to the formula ($s=\sigma+it$, $\sigma>1$)

$$(1.4)\qquad \sum_{\substack{n\le N-2\\ n\ \mathrm{odd}}}{}^{(n)} \frac{\Lambda(n)\Lambda(n+2)}{n^s} + \sum_{4\le 2^\lambda\le N-2}{}^{(\lambda)} \frac{\log 2\cdot\Lambda(2^{\lambda-1}+1)}{2^{\lambda s}} + \sum_{n>N-2}{}^{(n)} \frac{b_n}{n^s}$$

$$= \sum_{\substack{k<N\\ k\ \mathrm{odd}}}{}^{(k)} \frac{\mu(k)\log k}{\varphi(k)} \sum_{\chi \bmod k}{}^{(\chi)} \bar{\chi}(-2,k)\frac{L'}{L}(s,k,\chi)$$

and this offers several possibilities for a more elegant treatment and various conclusions. So I used the formula (1.4) to show that the infiniteness of the twin primes depends only upon "small" zeros of the Dirichlet $L(s,k,\chi)$-functions; it turned out on this way recently that among these zeros only those "near to the line $\sigma=\frac{1}{2}$" have a significance for the problem. As I remarked in my paper [4], this "function-theoretical sieve" method can be applied to a large class of "indefinite" problems of the additive prime number theory a typical of them being the twin prime problem. To formulate it in a general form (which is far from being most general) let the integers $x_1,\ldots,x_l$ be restricted to the sets $A_1, A_2, \ldots, A_l$ respectively and let $f(x_1,\ldots,x_l)=f(x)$ be any integer-valued function for which the number $h(\nu)$ of the solutions of the equation

$$(1.5)\qquad f(x)=\nu,\quad x_j\in A_j,\quad j=1,2,\ldots,l,$$

is finite, even the estimation [2]

$$(1.6)\qquad h(\nu)<\nu^{B_2}$$

holds. Let $h(\nu)$ be in addition such that for all positive integer B's the function-theoretical behaviour of the functions

$$(1.7)\qquad G_B(s,k,\chi) \overset{\mathrm{def}}{=} \sum_{(\nu,B)=1}{}^{(\nu)} \frac{h(\nu)\chi(\nu,k)}{\nu^s}$$

is at our disposal; this is the case e. g. when $h(\nu)$ — apart perhaps from a numerical factor — is multiplicative, an example being $f=x_1^2+x_2^2$, the sets A_j being identical to all integers. Let us consider with a positive integer $N>B$ the function

$$(1.8)\qquad K_N(s,B) \overset{\mathrm{def}}{=} \sum_{\substack{k\le N\\(k,B)=1}}{}^{(k)} \frac{\mu(k)\log(N/k)}{\varphi(k)} \sum_{\chi \bmod k}{}^{(\chi)} \bar{\chi}(-B,k)G_B(s,k,\chi)$$

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[2] c will stand throughout for positive constants (not necessarily the same) whose numerical values are irrelevant, $B_1, B_2, \ldots$ stand for positive constants whose numerical values matter. The characters belonging to modulus k will be denoted (as in (1.4)) by $\chi(n,k)$, the corresponding L-function by $L(s,k,\chi)$, their non-trivial zeros by $\varrho=\varrho(k,\chi)=\beta+i\gamma$. p is always a prime, the O-and o-signs refer to $N\to\infty$. The empty sum means 0, empty product 1. $\tau(l)$ stands for the number of positive divisors of l. $\sum_{\chi \bmod k}^{*}$ means that the summation is to be extended to primitive characters mod$k$ only. $\exp x$ stands for e^x.

for $\sigma > B_2+1$. The series (1.7) can be inserted here and this gives

(1.9)
$$K_N(s, B) = \sum_{(\nu,B)=1}^{(\nu)} \frac{h(\nu)}{\nu^s} \sum_{\substack{k\leqslant N\\(k,B)=1}}^{(k)} \mu(k)\log\frac{N}{k}\left\{\frac{1}{\varphi(k)}\sum_{\chi \bmod k}^{(\chi)} \chi(\nu, k)\bar{\chi}(-B, k)\right\}$$
$$= \sum_{(\nu,B)=1}^{(\nu)} \frac{h(\nu)}{\nu^s} \sum_{\substack{k\leqslant N, k|(\nu+B)\\(k,\nu)=(k,B)=1}}^{(k)} \mu(k)\log\frac{N}{k}.$$

We may observe that the restrictions

$$(k, \nu) = (k, B) = 1$$

are automatically satisfied; e.g. $p|k$ and $p|\nu$ would imply $p|B$, i.e. $(\nu, B) > 1$, which is excluded. Hence the inner sum is for $\nu \leqslant N-B$ owing to (1.3) equal to $\Lambda(\nu+B)$ and hence for $\sigma > B_2+1$

$$K_N(s, B) = \sum_{\substack{\nu\leqslant N-B\\(\nu,B)=1}}^{(\nu)} \frac{h(\nu)\Lambda(\nu+B)}{\nu^s} + \sum_{\nu>N-B}^{(\nu)} \frac{c_\nu}{\nu^s}.$$

Hence, *any of the coefficient formulae for general Dirichlet series applied to coefficients with* $\nu \leqslant N-B$ *eliminates the* c_ν*'s at once* and gives an integral representation of

$$\sum_{\substack{\nu\leqslant N-B\\(\nu,B)=1}}^{(\nu)} h(\nu)\Lambda(\nu+B)$$

which, in very general cases, for $N\to\infty$ is

$$\sim \sum_{B<p\leqslant N}^{(p)} \log p\cdot h(p-B) \sim \log N \sum_{(p,x)} 1,$$

where the last sum represents obviously the number of solutions of

(1.10) $$p = f(x_1, \ldots, x_l)+B, \quad p\leqslant N.$$

Since—if we use appropriate coefficient formulae—the singularities of the integrand are among those of the $G_B(s, k, \chi)$-functions which are in turn, in very general cases, meromorphic on the whole plane, the contour-integration technique is fully applicable. This description of our general sieving process does not contain (1.4) literally, but it would be very easy to modify it so as to include (1.4) in it. We shall apply also this method systematically in later papers of this series; here we shall confine ourselves to prove the following

THEOREM I. *For the number* $\pi_D(N)$ *of prime pairs* (p_1, p_2) *with*

$$p_1-p_2 = D, \quad p_2\leqslant N, \quad (D \text{ even})$$

the representation

(1.11)
$$\pi_D(N) = \{1+O(\varepsilon(N))\}B_1\frac{N}{\log^2 N}\prod_{\substack{p|B\\p>2}}^{(p)}\frac{p-1}{p-2} -$$
$$-\frac{1+O(1/\sqrt{\log N})}{\log^2 N}\sum_{\substack{k\leqslant N+D\\(k,D)=1}}^{(k)}\frac{\mu(k)\log(N/k)}{\varphi(k)}\times$$
$$\times\sum_{\chi\bmod k}^{(\chi)}\bar{\chi}(-D, k)\sum_{|\gamma|<\log N\cdot\exp(3/2\varepsilon(N))}^{(\varrho(\chi))}\left\{\frac{N^\varrho-N^{\varrho/100}}{\varrho(1+\varrho/\log N)^{[\varepsilon(N)\log N]+1}}\right\}$$

holds.

Here B_1, the Hardy-Littlewood constant in (1.2), $D\leqslant\dfrac{N}{\log^{10}N}$, $\varepsilon(x)$ is positive and tends to 0 with $1/x$ arbitrarily slowly but so that

$$\varepsilon(x)\sqrt{\log x}\to\infty.$$

The O-sign refers to $N\to\infty$ uniformly in D.

This explicit representation holds of course without conjectures. In the main term one can recognise at once the heuristical Hardy-Littlewood formula in (1.1). One could prove also here (somewhat less strongly than before) that one could drop from the critical sum in (1.11) everything except the contribution of the zeros "near to $\sigma = \frac{1}{2}$". Theorem I of this paper (and the theorem in paper [3]) seem to be the first ones of this type in the literature. Another thing which gives significance to such explicit formulae "of Riemann type" is the fact that several problems can be reduced to averaging of (1.11) with respect to D and owing to the structure of the formula (1.11) this leads to character-sums non-trivially estimable. Since after the relation (1.4) (and some facts from the proof of Theorem II) the proof of Theorem I is easy, we shall postpone it to an appendix.

2. As I observed recently (see [4]), another function-theoretical sieve-method can be devised to investigate "definite" problems of the additive prime number theory typical of them being the binary Goldbach problem or the representability of the integer N in the form $p+x_1^2+x_2^2$. As given in [4] *it is possible to construct a function which can be developed into Dirichlet series in two ways, both convergent (even absolutely) in a half-plane* $\sigma > c$ *the corresponding coefficients being thus equal. In one series the coefficient of* N^{-s} *is "essentially" the number of different representations* *of N in the required form, in the other series the corresponding coefficient is an asymptotical representation of it.*

In particular, as regards the binary Goldbach problem, I stated l.c. without proof a theorem which in a slightly more general form runs as follows

THEOREM II. *Let* $M/2 \leqslant N \leqslant M$ *(N even) and* $\nu_2(N)$ *stand for the number of binary Goldbach decompositions of the even N. Then the "explicit" formula*

$$(2.1)\qquad \nu_2(N) = \{1+O(\varepsilon(N))\}\frac{N}{\log^2 N}B_1\prod_{\substack{p|N\\ p>2}}{}^{(p)}\frac{p-1}{p-2} - \\ -\frac{1+O(1/\sqrt{\log N})}{\log^2 M}\sum_{\substack{k\leqslant M\\ (k,N)=1}}{}^{(k)}\frac{\mu(k)\log(M/k)}{\varphi(k)}\times \\ \times\sum_{\chi \bmod k}{}^{(\chi)}\bar{\chi}(N,k)\sum_{|\gamma|<\log M\exp(3/2\varepsilon(M))}{}^{(\varrho(\chi))}\frac{N^{\varrho}-N^{\varrho/100}}{\varrho(1+\varrho/\log M)^{[\varepsilon(M)\log M]+1}}$$

holds (³).

Here again B_1 is the Hardy-Littlewood constant in (1.2) and $\varepsilon(x)$ is positive and tends to 0 with $1/x$ arbitrarily slowly but so that

$$\varepsilon(x)\sqrt{\log x}\to\infty.$$

The O-sign refers of course to $M\to\infty$.

In the main term one recognises again the heuristic formula for the number of the binary Goldbach decompositions given by Hardy and Littlewood in their paper [1] (in particular in p. 32, Conjecture A). It is perhaps not uninteresting to note in advance that "the singular factors"

$$B_1\prod_{p|D,p>2}{}^{(p)}\frac{p-1}{p-2}\quad\text{resp.}\quad B_1\prod_{p|N,p>2}{}^{(p)}\frac{p-1}{p-2}$$

in our theorems do not arise from singular series in Hardy-Littlewood's sense; no dissection of the line of integration occurs at all. Also such Riemann type explicit formula for the Goldbach problem seems to me unnoticed so far. Here the assertion that this formula gives a clearer insight into the essence of the problem is even more plausible than before since in several important problems "the averaging with respect to N" means for our critical sum estimation of character-sums, sometimes with consecutive integer arguments, for "long" intervals which can be performed

(³) The theorem could have been stated replacing everywhere M by N; however for "averaging" purposes this form is more suitable.

non-trivially. We shall give here a detailed proof of Theorem II without referring to the general frame of the method in [4], and return to its applications in subsequent papers of this series.

In accordance to what was said on the twin prime problem also here can be asserted that the truth of the (binary) Goldbach conjecture depends exclusively on the "small" zeros of $L(s,k,\chi)$ functions notably on those "near" to the line $\sigma=\frac{1}{2}$. This follows from a suitable modification of Theorem II and we shall return to it in a later paper of this series.

3. For the proofs of our theorems we shall need a number of auxiliary propositions. Let

$$(3.1)\qquad a>1,\quad x>0,\quad \omega\geqslant 3,\quad \delta\geqslant c\ (>14)\ \text{and integer}$$

and

$$(3.2)\qquad \Phi(x,\delta,\omega)=\Phi(x)\overset{\text{def}}{=}\frac{1}{2\pi i}\int_{(a)}\frac{x^w}{w(1+w/\omega)^{\delta+1}}\,dw.$$

Besides the evident relation

$$(3.3)\qquad \Phi(x)=0\quad\text{for}\quad 0\leqslant x\leqslant 1$$

we need

LEMMA I. *For all* $x\geqslant 0$ $\Phi(x)$ *is non-decreasing and*

(a) $0\leqslant\Phi(x)<1$,

(b) *for* $x\geqslant\exp(2\delta/\omega)$ *we have*

$$1-\exp(-\delta/5)\leqslant\Phi(x)\quad(<1),$$

(c) *supposing for integer* $N>c$ *also* $\omega\leqslant\log 2N$ *we have*

$$\Phi\left(\frac{N}{N-1}\right)<N^{-2}.$$

For the proof we remark that for $x\geqslant 1$ we have

$$(3.4)\qquad \Phi(x)=1+\frac{\omega^{\delta+1}}{\delta!}(x^w w^{-1})^{(\delta)}_{w=-\omega}=1-\exp(-\omega\log x)\sum_{\nu=0}^{\delta}\frac{(\omega\log x)^\nu}{\nu!}.$$

Since we have for all real r's

$$1-e^{-r}\sum_{\nu=0}^{\delta}\frac{r^\nu}{\nu!}=\frac{1}{\delta!}\int_0^r e^{-\lambda}\lambda^\delta\,d\lambda,$$

(3.4) gives for $x\geqslant 1$

$$(3.5)\qquad \Phi(x)=\frac{1}{\delta!}\int_0^{\omega\log x}e^{-\lambda}\lambda^\delta\,d\lambda,$$

which, using also the relation $\Phi(x) = 0$ for $0 \leqslant x \leqslant 1$, puts the monotonicity and (a) into evidence. In case (b) we have

$$\omega \log x \geqslant 2\delta$$

and thus

$$(0 <)\ 1-\Phi(x) = \frac{1}{\delta!}\int_{\omega\log x}^{\infty} e^{-\lambda}\lambda^{\delta}\,d\lambda$$

$$\leqslant \frac{1}{\delta!}\int_{2\delta}^{\infty} e^{-\lambda}\lambda^{\delta}\,d\lambda = \frac{\delta^{\delta+1}}{\delta!\,e^{\delta}}\int_{1}^{\infty}\{(1+r)e^{-r}\}^{\delta}\,dr.$$

Since for $r \geqslant 1$ we have

$$\log(1+r) \leqslant r\log 2, \qquad (1+r)e^{-r} \leqslant e^{-r/4}$$

we get

$$1-\Phi(x) \leqslant 4\frac{\delta^{\delta}}{\delta!}\exp\left(-\frac{5}{4}\delta\right) < 2\exp\left(-\frac{\delta}{4}\right) < \exp\left(-\frac{\delta}{5}\right)$$

which proves (b). Finally from (3.5) (roughly) indeed

$$\Phi\left(\frac{N}{N-1}\right) < \delta!^{-1}\int_{0}^{\frac{\log 2N}{N-2}} \lambda^{\delta}\,d\lambda < N^{\frac{-\delta+1}{2}} < N^{-2}.$$

4. Let $N > c$, further

(4.1) $$1 \geqslant \lambda \geqslant 1-\frac{1}{\log\log N}$$

and

(4.2) $$f(m,\lambda) \overset{\text{def}}{=} \sum_{p|m}{}^{(p)} p^{-\lambda},$$

the summation being extended to the different prime factors of m only.

Then we need the simple

LEMMA II. *The inequality*

$$\max_{1\leqslant m\leqslant N} f(m,\lambda) < 6\log\log\log N$$

holds.

For the proof we remark first that the maximum in question is obviously attained also for a square-free $m = m_0$ and further this m_0 has evidently the form

$$m_0 = 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot\ldots\cdot p_r$$

where r is defined by

$$2\cdot 3\cdot\ldots\cdot p_r \leqslant N < 2\cdot 3\cdot\ldots\cdot p_{r+1}.$$

Since for $N > c$ we have as well-known

$$\frac{1}{2}\log N \leqslant p_r \leqslant 2\log N, \qquad \frac{1}{2}\cdot\frac{\log N}{\log\log N} \leqslant r \leqslant 2\frac{\log N}{\log\log N}$$

and for $x > c$

$$\pi(x) < \frac{2x}{\log x}$$

we get for $N > c$

$$\max_{1\leqslant m\leqslant N} f(m,\lambda) = \sum_{\nu=1}^{r} p_\nu^{-\lambda} = \int_{3/2}^{p_r} x^{-\lambda}\,d\pi(x) = rp_r^{-\lambda} + \lambda\int_{3/2}^{p_r}\pi(x)x^{-\lambda-1}\,dx$$

$$< 4\frac{\log^{1-\lambda} N}{\log\log N} + 2\int_{3/2}^{p_r}\frac{x^{1-\lambda}}{x\log x}\,dx < O(1) + 2e\log\log\log N$$

$$< 6\log\log\log N$$

indeed.

Putting with the above λ

(4.3) $$f_0(m,\lambda) \overset{\text{def}}{=} \sum_{p|m}{}^{(p)} \frac{\log p}{p^{\lambda}-1}$$

(summation extended again only over the different prime factors) we get for $N > c$ by Lemma II,

(4.4) $$\max_{1\leqslant m\leqslant N} f_0(m,\lambda) \leqslant 3\log N \max_{1\leqslant m\leqslant N} f(m,\lambda) < \log^2 N.$$

Defining further for $z = x+iy$ the function $D(m,z)$ by

(4.5) $$D(m,z) \overset{\text{def}}{=} \prod_{p|m}\frac{1}{1-1/p^{z}}$$

(product extended only to different prime factors of m) we get from Lemma II for $N > c$ and

(4.6) $$1-\frac{1}{\log\log N} \leqslant x \leqslant 2$$

the inequality

$$(4.7)\quad \max_{1\leqslant m\leqslant N}|D(m,z)|\leqslant \max_{1\leqslant m\leqslant N} D(m,x)\leqslant c\max_{1\leqslant m\leqslant N}\exp f(m,x)<c(\log\log N)^6.$$

Further we have

$$\left|\frac{dD(m,z)}{dz}\right| = |D(m,z)|\left|\sum_{p|m}\frac{\log p}{p^z-1}\right|\leqslant |D(m,z)|f_0(m,x)$$

and hence, in the range (4.6), using (4.4) and (4.7), we get for $N>c$

$$(4.8)\quad \max_{1\leqslant m\leqslant N}\left|\frac{dD(m,z)}{dz}\right|\leqslant \log^3 N.$$

5. We shall need further

LEMMA III. *For $1\leqslant m\leqslant M$ integer, m even, we have* [4] *for $M>c$ the inequality* (B_1 *in* (1.2))

$$\left|\sum_{\substack{1\leqslant k\leqslant M\\(k,m)=1}}^{(k)}\frac{\mu(k)\log(M/k)}{\varphi(k)} - B_1\prod_{\substack{p|m\\p>2}}^{(p)}\frac{p-1}{p-2}\right| < c\log^{-16}M.$$

For the proof we consider the function $\varphi_m(z)$ defined for $x>0$ by

$$(5.1)\quad \sum_{(k,m)=1}^{(k)}\frac{\mu(k)}{\varphi(k)}k^{-z}.$$

This can be written as

$$\prod_{(p,m)=1}^{(p)}\left\{1-\frac{p^{-z}}{p-1}\right\}$$

and hence

(5.2)

$$\varphi_m(z) = \frac{1}{L(z+1,m,\chi_0)}\prod_{(p,m)=1}^{(p)}\left\{1-\frac{1/(p-1)p^{z+1}}{1-1/p^{z+1}}\right\} = \frac{1}{L(z+1,m,\chi_0)}U_1(z)$$

where $U_1(z)$ (and later $U_2(z),\ldots$) is regular for, say, $x\geqslant -\frac14$, and here the inequality

$$(5.3)\quad \frac{1}{c}\leqslant |U_\nu(z)|\leqslant c$$

[4] If m is odd the product must be replaced by 0, as is clear from the proof; we shall not need it. We put no stress in obtaining a possibly good error-term.

holds. (5.1) and (5.2) give for $x>0$

$$(5.4)\quad -\sum_{(k,m)=1}^{(k)}\frac{\mu(k)\log k}{\varphi(k)}k^{-z}$$
$$= \varphi_m'(z) = \varphi_m(z)\big(\log\varphi_m(z)\big)'$$
$$= \frac{U_1(z)}{L(z+1,m,\chi_0)}\left\{U_2(z)-\frac{L'}{L}(z+1,m,\chi_0)\right\}$$
$$= U_1(z)\left(\frac{1}{L(z+1,m,\chi_0)}\right)' + \frac{U_3(z)}{L(z+1,m,\chi_0)}$$
$$= U_1(z)\left(\frac{1}{\zeta(z+1)}\prod_{p|m}^{(p)}\frac{1}{1-1/p^{z+1}}\right)' + \frac{U_3(z)}{\zeta(z+1)}\prod_{p|m}^{(p)}\frac{1}{1-1/p^{z+1}}$$
$$= -U_1(z)\frac{\zeta'(z+1)}{\zeta(z+1)^2}D(m,z+1) + \frac{U_1(z)}{\zeta(z+1)}\cdot\frac{dD(m,z+1)}{dz} +$$
$$+\frac{U_3(z)}{\zeta(z+1)}D(m,z+1)\overset{\text{def}}{=} g(z).$$

We remark further that (m being even)

(5.5)

$$U_1(0)\prod_{p|m}^{(p)}\frac{1}{1-1/p} = \prod_{(p,m)=1}^{(p)}\left\{1-\frac{1}{(p-1)^2}\right\}2\prod_{\substack{p|m\\p>2}}\frac{p}{p-1} = B_1\prod_{\substack{p|m\\p>2}}\frac{p-1}{p-2}.$$

Starting from the integral

$$(5.6)\quad J_1 = \frac{1}{2\pi i}\int\limits_{1/\log M-i\log^{20}M}^{1/\log M+i\log^{20}M}\frac{(M+\frac12)^z}{z}g(z)\,dz,$$

the series representation of $g(z)$ in (5.4) gives

$$J_1 = -\sum_{(k,m)=1}^{(k)}\frac{\mu(k)\log k}{\varphi(k)}\cdot\frac{1}{2\pi i}\int\limits_{1/\log M-i\log^{20}M}^{1/\log M+i\log^{20}M}\frac{1}{z}\left(\frac{M+\frac12}{k}\right)^z dz$$
$$= -\sum_{k\leqslant M,\,(k,m)=1}^{(k)}\frac{\mu(k)\log k}{\varphi(k)} + O(\log^{-20}M)\sum^{(k)}\frac{\log k}{\varphi(k)}\cdot\frac{k^{-1/\log M}}{\left|\log\frac{M+\frac12}{k}\right|}$$

and hence routine reasoning gives

(5.7) $$J_1 = -\sum_{\substack{k\leqslant M\\(k,m)=1}}^{(k)} \frac{\mu(k)\log k}{\varphi(k)} + O(\log^{-17} M).$$

Let further be

(5.8) $$J_2 = \frac{\log M}{2\pi i} \int\limits_{1/\log M - i\log^{20} M}^{1/\log M + i\log^{20} M} \frac{(M+\frac{1}{2})^z}{z} \varphi_m(z)\,dz.$$

(5.1) gives again as before

$$J_2 = \sum_{\substack{k\leqslant M\\(k,m)=1}}^{(k)} \frac{\mu(k)}{\varphi(k)} \log M + O(\log^{-16} M)$$

and hence

(5.9) $$J_1 + J_2 = \sum_{\substack{k\leqslant M\\(k,m)=1}}^{(k)} \frac{\mu(k)\log(M/k)}{\varphi(k)} + O(\log^{-16} M).$$

Using the representations (5.2) resp. (5.4), we can apply Cauchy's theorem to the parallelogramm with the vertices

$$\frac{1}{\log M} \pm i\log^{20} M, \qquad -\frac{1}{\log\log M} \pm i\log^{20} M.$$

Since this parallelogramm lies for $M > c$ in the zerofree domain of $L(z+1, m, \chi_0)$, the only singularity in the inside is the pole of first order of $g(z)$ at $z = 0$ with the residuum

$$g(0) = U_1(0) \prod_{p|m}^{(p)} \frac{1}{1-1/p} = B_1 \prod_{\substack{p|m\\p>2}}^{(p)} \frac{p-1}{p-2}$$

using (5.5). As is well known, on the periphery of our parallelogramm we have for $M > c$

$$\left|\frac{\zeta'(z+1)}{\zeta(z+1)^2}\right| < (\log\log M)^c, \qquad \frac{1}{|\zeta(z+1)|} < (\log\log M)^c;$$

these, combined with (4.5), (4.6), (4.7) and (4.8), give for the remaining parts of the contourintegral the upper bound

$$O(\log^{-20} M)\log^4 M + O(M^{-1/\log\log M})\log^5 M$$

which completes the proof of Lemma III.

We shall further need the

LEMMA IV (5). *For square-free odd k the estimation*

$$Z(m, k) \overset{\text{def}}{=} \Big| \sum_{\chi \bmod k}^{*} \chi(m, k) \Big| \leqslant (m-1, k)$$

holds.

For the proof we remark that if

$$k = p_1 p_2 \ldots p_l, \qquad p_1 < p_2 < \ldots < p_l,$$

g_j is a primitive root mod p_j and, further, if

$$m \equiv g_j^{\xi_j} \bmod p_j, \qquad 0 \leqslant \xi_j \leqslant p_j - 2, \qquad j = 1, \ldots, l$$

then for fixed ν_j's with

$$0 \leqslant \nu_j \leqslant p_j - 2, \qquad j = 1, 2, \ldots, l$$

the characters belonging to mod k are given by

$$\chi(m, k) = \exp\left\{2\pi i \left(\frac{\nu_1 \xi_1}{p_1 - 1} + \ldots + \frac{\nu_l \xi_l}{p_l - 1}\right)\right\}.$$

Since they are primitive characters if and only if

$$1 \leqslant \nu_j \leqslant p_j - 2, \qquad j = 1, 2, \ldots, l,$$

we indeed have

$$Z(m, k) = \prod_{j=1}^{l} Z(m, p_j) = \prod_{j=1}^{l} \left| \sum_{\nu_j=1}^{p_j-2} \exp\frac{2\pi i \nu_j \xi_j}{p_j - 1} \right|$$
$$= \prod_{\xi_j=0}^{(j)} (p_j - 2) = \prod_{p_j|(m-1)}^{(j)} (p_j - 2) < \prod_{p_j|(m-1)}^{(j)} p_j = |(m-1, k)|.$$

6. Now we turn to the proof of Theorem II. Let $M > c$, further, let ω, ω_1 and δ be functions of M to be determined later; here we restrict them only by

(6.1) $$3 \leqslant \omega \leqslant \log M,$$

(6.2) $$3\log\log M \leqslant \omega_1 \leqslant 0{,}99 \log M,$$

(6.3) $$\delta \geqslant \sqrt{\log M} \text{ and integer},$$

(6.4) $$\delta/\omega = o(1)$$

and for the even N

(6.5) $$M/2 \leqslant N \leqslant M.$$

(5) This lemma was also needed in [4]; we include it for the reader's convenience.

Let $s=\sigma+it$, $\sigma>2$, further, let η be such that

$$1<\eta<\sigma-1; \tag{6.6}$$

we start with the function

$$F_M(s) = -\sum_{k\leqslant M}^{(k)} \frac{\mu(k)\log(M/k)}{\varphi(k)} \times \tag{6.7}$$

$$\times\Big\{\sum_{\chi \bmod k}^{(\chi)} \frac{1}{2\pi i}\int_{(\eta)} L(s-w,k,\bar{\chi})\frac{L'}{L}(w,k,\chi)\frac{1-e^{-\omega_1 w}}{w(1+w/\omega)^{\delta+1}}\,dw\Big\}.$$

Owing to (6.6) we can replace both $L(s-w,k,\bar{\chi})$ and $\frac{L'}{L}(w,k,\chi)$ by their respective Dirichlet series and integrate termwise. This gives

$$F_M(s) = \sum_{k\leqslant M}^{(k)} \mu(k)\log(M/k)\sum_{(n,k)=(\nu,k)=1}^{(n)}\sum^{(\nu)}\frac{\Lambda(\nu)}{n^s}\times \tag{6.8}$$

$$\times\Big\{\frac{1}{\varphi(k)}\sum_{\chi \bmod k}^{(\chi)}\chi(\nu,k)\bar{\chi}(n,k)\Big\}\frac{1}{2\pi i}\int_{(\eta)}\frac{(n/\nu)^w-(ne^{-\omega_1}/\nu)^w}{w(1+w/\omega)^{\delta+1}}\,dw.$$

The last integral can be evaluated by using Lemma I; what we actually need is only that it is

(6.9)
- (a) $=0$ for $\nu\geqslant n$,
- (b) between 0 and 1 for $\nu<n$,
- (c) between $1-\exp(-\frac{1}{5}\delta)$ and 1 for $ne^{-\omega_1}\leqslant\nu\leqslant n\exp(-2\delta/\omega)$.

Denoting the value of the integral in (6.8) by $b(n,\nu)$, we get from (6.8)

$$F_M(s) = \sum^{(n)}\frac{a_n}{n^s}, \tag{6.10}$$

where

$$a_n = \sum_{\substack{k\leqslant M\\(k,n)=1}}^{(k)}\mu(k)\log(M/k)\sum_{\substack{\nu<n\\(\nu,k)=1\\n\equiv\nu \bmod k}}^{(\nu)}\Lambda(\nu)b(n,\nu). \tag{6.11}$$

This representation shows at once that the series (6.10) converges (even absolutely) in a half-plane $\sigma>c$.

7. We investigate in particular a_N. (6.11) gives, taking also in account Lemma I (c)

$$a_N = \sum_{\nu\leqslant N-2}^{(\nu)}\Lambda(\nu)b(N,\nu)\sum_{\substack{(k,N)=(k,\nu)=1\\k|(N-\nu)}}^{(k)}\mu(k)\log(M/k)+O(1/M), \tag{7.1}$$

which can be written as $(a_N'+a_N'')$, where

$$a_N' = \sum_{\substack{\nu\leqslant N-2\\(\nu,N)=1}}^{(\nu)}\Lambda(\nu)b(N,\nu)\sum_{\substack{(k,N)=(k,\nu)=1\\k|(N-\nu)}}^{(k)}\mu(k)\log(M/k), \tag{7.2}$$

$$a_N'' = \sum_{\substack{\nu<N\\(\nu,N)>1}}^{(\nu)}\Lambda(\nu)b(N,\nu)\sum_{\substack{(k,N)=(k,\nu)=1\\k|(N-\nu)}}^{(k)}\mu(k)\log(M/k)+O(1/M). \tag{7.3}$$

We consider the inner sum in (7.2). Since owing to $(\nu,N)=1$ and $k|(N-\nu)$ the restrictions $(k,N)=(k,\nu)=1$ are automatically satisfied, this is owing to $N-\nu\geqslant 2$ equal to

$$\sum_{k|(N-\nu)}^{(k)}\mu(k)\log(M/k) = \Lambda(N-\nu)$$

and hence

$$a_N' = \sum_{\substack{\nu\leqslant N-2\\(\nu,N)=1}}^{(\nu)}\Lambda(\nu)\Lambda(N-\nu)b(N,\nu). \tag{7.4}$$

As to a_N'' in (7.3), the inner sum cannot exceed absolutely (roughly)

$$\log M\cdot\tau(N-\nu)$$

and hence, using also (6.9)

$$|a_N''| < 2\log M\max_{1\leqslant l\leqslant M}\tau(l)\sum_{\substack{\nu\leqslant M\\(\nu,N)>1}}\Lambda(\nu).$$

But this last sum is equal to

$$\sum_{p|N}^{(p)}\log p\sum_{p^\lambda\leqslant M}^{(\lambda)}1 = \sum_{p|N}^{(p)}\log p\cdot\left[\frac{\log M}{\log p}\right] < \log M\cdot\tau(N)$$

and hence for $M>c$

$$|a_N''| < 2\log^2 M\max_{l\leqslant M}\tau(l)^2 < M^{1/10}.$$

This and (7.4) give

$$a_N = \sum_{\substack{\nu<N\\(\nu,N)=1}}^{(\nu)}\Lambda(\nu)\Lambda(N-\nu)b(N,\nu)+O(M^{1/10});$$

since the contribution of prime-powers p^α with $\alpha\geqslant 2$ cannot exceed $2\log^2 M\cdot O(\sqrt{M}\log M)$, we get

$$a_N = \sum_{\substack{p_1+p_2=N\\(p_j,N)=1}}^{(p_1,p_2)}\log p_1\log p_2\cdot b(N,p_1)+O(\sqrt{M}\log^3 M)$$

and also

$$a_N = \sum_{p_1+p_2=N}^{(p_1,p_2)}\log p_1\log p_2\cdot b(N,p_1)+O(\sqrt{M}\log^3 M). \tag{7.5}$$

8. In order to get a more elegant form for a_N, we remark first that (6.2) and (6.9)(b) give at once

$$(8.1)\qquad \sum_{\substack{p_1+p_2=N \\ \min p_j \leqslant Ne^{-\omega_1}}}^{(p_1,p_2)} \log p_1 \log p_2 \cdot b(N, p_1) < \log^2 N \cdot Ne^{-\omega_1} < \frac{M}{\log M}.$$

Further, according to Schnirelman's theorem [6] (and (6.9)(b) and (6.4))

$$(8.2)\qquad \sum_{\substack{p_1+p_2=N \\ \max p_j > N\exp\left(-2\frac{\delta}{\omega}\right)}}^{(p_1,p_2)} \log p_1 \log p_2 \cdot b(N, p_1) < \log^2 N \sum_{\substack{p_1+p_2=N \\ \max p_j > N\exp\left(-2\frac{\delta}{\omega}\right)}}^{(p_1,p_2)} 1$$

$$< cN\frac{\delta}{\omega}\prod_{p|N}^{(p)}\left(1+\frac{1}{p}\right) < cM\frac{\delta}{\omega}\prod_{\substack{p|N \\ p>2}}\frac{p-1}{p-2}.$$

Replacing further in the remaining part of the sum in (7.5) $b(N, p_1)$ by 1 the error is, owing to (6.9)(c) and (6.3), easily

$$(8.3)\qquad e^{-\delta/5}\log^2 N \cdot c\frac{N}{\log N} < c\frac{M}{\log^{10} M}\prod_{\substack{p|N \\ p>2}}\frac{p-1}{p-2}.$$

Collecting all these we get

$$(8.4)\qquad a_N = \sum_{p_1+p_2=N}^{(p_1,p_2)} \log p_1 \log p_2 + O(M)\left(\frac{1}{\log M}+\frac{\delta}{\omega}\right)\prod_{\substack{p|N \\ p>2}}\frac{p-1}{p-2}.$$

9. Next we proceed to obtain another Dirichlet series for $F_M(s)$' convergent (even absolutely) in some half-plane $\sigma > c$. For a character $\chi(n, k)$, let k^* be its conductor and $\chi^*(n, k^*)$ the corresponding primitive character; then for a square free k, as is well known,

$$L(w, k, \chi) = L(w, k^*, \chi^*)\prod_{p \mid \frac{k}{k^*}}\left(1-\frac{\chi^*(p, k^*)}{p^w}\right)$$

and hence

$$(9.1)\qquad \frac{L'}{L}(w, k, \chi) = \frac{L'}{L}(w, k^*, \chi^*) + \sum_{p \mid \frac{k}{k^*}}^{(p)} \frac{\chi^*(p, k^*)\log p}{p^w - \chi^*(p, k^*)}.$$

---

[6] See e.g. K. Prachar [2], p. 51, Satz 4.4. Actually we have used here a slightly stronger form of the theorem — one could prove it mutatis mutandis — namely that the number of binary Goldbach decompositions of N with $\max p_j > N-y$ cannot exceed $c\frac{y}{\log^2 N}\prod_{|N}\left(1+\frac{1}{p}\right)$ if $N/2 \geqslant y \geqslant N/\sqrt{\log N}$.

Then we write

$$(9.2)\qquad F_M(s) = F_{M1}(s) + F_{M2}(s),$$

where by (6.7) (with the notation of p. 63)

$$(9.3)\qquad F_{M1}(s) = -\sum_{k\leqslant M}^{(k)} \frac{\mu(k)\log(M/k)}{\varphi(k)} \sum_{k^*|k}^{(k^*)} \sideset{}{^*}\sum_{\chi^* \bmod k^*}^{(\chi^*)} \frac{1}{2\pi i}\times$$

$$\times \int_{(\eta)} L(s-w, k, \bar{\chi})\frac{L'}{L}(w, k^*, \chi^*)\frac{1-e^{-\omega_1 w}}{w(1+w/\omega)^{\delta+1}}\,dw$$

and

$$(9.4)\qquad F_{M2}(s) = -\sum_{k\leqslant M}^{(k)} \frac{\mu(k)\log(M/k)}{\varphi(k)} \sum_{k^*|k}^{(k^*)} \sideset{}{^*}\sum_{\chi^* \bmod k^*}^{(\chi^*)} \frac{1}{2\pi i}\times$$

$$\times \int_{(\eta)} L(s-w, k, \bar{\chi})\frac{1-e^{-\omega_1 w}}{w(1+w/\omega)^{\delta+1}}\left\{\sum_{p \mid \frac{k}{k^*}}^{(p)} \frac{\chi^*(p, k^*)\log p}{p^w - \chi^*(p, k^*)}\right\}dw.$$

10. Let us consider $F_{M2}(s)$ first. Writing the inner sum in brackets in the form

$$\sum_{p \mid \frac{k}{k^*}}^{(p)} \log p \sum_{\lambda=1}^{\infty}{}^{(\lambda)} \frac{\chi^*(p^\lambda, k^*)}{p^{\lambda w}}$$

and putting it into (9.4), we get

$$F_{M2}(s) = -\sum_{k\leqslant M}^{(k)} \frac{\mu(k)\log(M/k)}{\varphi(k)} \sum_{k^*|k}^{(k^*)} \sideset{}{^*}\sum_{\chi^* \bmod k^*}^{(\chi^*)} \sum_{p \mid \frac{k}{k^*}}^{(p)} \log p \sum_{\lambda=1}^{\infty}{}^{(\lambda)} \chi^*(p^\lambda, k^*)\times$$

$$\times \sum_{(n,k)=1}^{(n)} \frac{\bar{\chi}(n, k)}{n^s}\cdot\frac{1}{2\pi i}\int_{(\eta)} \frac{(n/p^\lambda)^w(1-e^{-\omega_1 w})}{w(1+w/\omega)^{\delta+1}}\,dw.$$

This gives using (6.9)(a)

$$(10.1)\qquad F_{M2}(s) = \sum^{(n)} \frac{e_n^{(2)}}{n^s},$$

where

$$(10.2)\qquad e_n^{(2)} =$$

$$= -\sum_{\substack{k\leqslant M \\ (k,n)=1}}^{(k)} \frac{\mu(k)\log(M/k)}{\varphi(k)} \sum_{k^*|k}^{(k^*)} \sum_{\substack{p<n \\ p \mid \frac{k}{k^*}}}^{(p)} \log p \sum_{p^\lambda<n}^{(\lambda)} b(n, p^\lambda) \sideset{}{^*}\sum_{\chi^* \bmod k^*}^{(\chi^*)} \bar{\chi}(np^{-\lambda}, k^*).$$

This representation shows at once that the series in (10.1) converges (absolutely) in a half-plane $\sigma > c$. For the estimation of $|e_N^{(2)}|$ we apply Lemma IV. Owing to $(p, k^*) = 1$ we have for all q primes dividing k^*

$$np^{-\lambda} - 1 \equiv 0 \bmod q \rightleftarrows n - p^\lambda \equiv 0 \bmod q$$

and hence

$$(10.3)\qquad |e_N^{(2)}| \leqslant \log^2 M \sum_{\substack{k \leqslant M \\ (k,N)=1}}^{(k)} \frac{|\mu(k)|}{\varphi(k)} \sum_{k^*|k}^{(k^*)} \sum_{\substack{p<N \\ p \mid \frac{k}{k^*}}}^{(p)} \sum_{p^\lambda < N}^{(\lambda)} (N - p^\lambda, k^*).$$

Introducing k_1 and k_2 as new summation variables by

$$k^* = k_1, \qquad k/k^* = k_2,$$

we have

$$k = k_1 k_2, \qquad (k_1, k_2) = 1$$

and hence from (10.3)

$$(10.4)\qquad |e_N^{(2)}| \leqslant \log^2 M \sum_{\substack{k_1 k_2 \leqslant M \\ (k_1 k_2, N)=1}}^{(k_1,k_2)} \frac{|\mu(k_1)|\,|\mu(k_2)|}{\varphi(k_1)\varphi(k_2)} \sum_{\substack{p<N \\ p|k_2}}^{(p)} \sum_{p^\lambda<N}^{(\lambda)} (N-p^\lambda, k_1)$$

$$\leqslant \log^2 M \sum_{k_1 \leqslant M}^{(k_1)} \frac{|\mu(k_1)|}{\varphi(k_1)} \sum_{p<N}^{(p)} \sum_{p^\lambda<N}^{(\lambda)} (N-p^\lambda, k_1) \sum_{\substack{k_2 \leqslant M \\ k_2 \equiv 0 \bmod p}}^{(k_2)} \frac{|\mu(k_2)|}{\varphi(k_2)}.$$

The last sum is owing to $(p, k_2/p) = 1$

$$= \frac{1}{p-1} \sum_{\substack{k_3 \leqslant M/p \\ (k_3,p)=1}}^{(k_3)} \frac{|\mu(k_3)|}{\varphi(k_3)} < \frac{c}{p} \sum_{k_3 \leqslant M}^{(k_3)} \frac{\log\log M}{k_3} < c\,\frac{\log^{3/2} M}{p}$$

and hence from (10.4)

$$|e_N^{(2)}| \leqslant c \log^4 M \sum_{k_1 \leqslant M}^{(k_1)} \frac{1}{k_1} \sum_{p<N}^{(p)} \sum_{p^\lambda<N}^{(\lambda)} \frac{(N-p^\lambda, k_1)}{p}$$

$$= c \log^4 M \sum_{p<N}^{(p)} \frac{1}{p} \sum_{p^\lambda<N}^{(\lambda)} \sum_{k_1 \leqslant M}^{(k_1)} \frac{(N-p^\lambda, k_1)}{k_1}.$$

Again the last sum cannot exceed

$$\sum_{d|(N-p^\lambda)}^{(d)} d \sum_{k_4 \leqslant M/d}^{(k_4)} \frac{1}{dk_4} < \log M \cdot \tau(N-p^\lambda) < \exp\left(c\,\frac{\log M}{\log\log M}\right)$$

and thus

$$(10.5)\qquad |e_N^{(2)}| \leqslant \exp\left(c\,\frac{\log M}{\log\log M}\right) \sum_{p<N}^{(p)} \frac{1}{p}\left[\frac{\log N}{\log p}\right] < \exp\left(c\,\frac{\log M}{\log\log M}\right).$$

11. Next we investigate the function $F_{M1}(s)$ in (9.3). Shifting the line of integration to the line $\mathrm{Re}\,w = -0{,}99$, we get

$$(11.1)\qquad F_{M1}(s) = F_{M3}(s) + F_{M4}(s),$$

where $F_{M3}(s)$ consists of the residual terms and

$$(11.2)\qquad F_{M4}(s) = -\sum_{k \leqslant M}^{(k)} \frac{\mu(k)\log(M/k)}{\varphi(k)} \times$$

$$\times \sum_{k^*|k}^{(k^*)} \sum_{\chi^* \bmod k^*}^{(\chi^*)}{}^{*} \frac{1}{2\pi i} \int_{(-0{,}99)} L(s-w, k, \bar\chi) \frac{L'}{L}(w, k^*, \chi^*) \frac{1-e^{-\omega_1 w}}{w(1+w/\omega)^{\delta+1}}\,dw.$$

Replacing $L(s-w, k, \bar\chi)$ by its Dirichlet series, we get

$$(11.3)\qquad F_{M4}(s) = \sum^{(n)} \frac{e_n^{(4)}}{n^s},$$

where

$$(11.4)\qquad e_n^{(4)} = \sum_{\substack{k \leqslant M \\ (k,n)=1}}^{(k)} \frac{\mu(k)\log(M/k)}{\varphi(k)} \times$$

$$\times \sum_{k^*|k}^{(k^*)} \sum_{\chi^* \bmod k^*}^{(\chi^*)}{}^{*} \bar\chi(n, k) \frac{1}{2\pi i} \int_{(-0{,}99)} \frac{L'}{L}(w, k^*, \chi^*) \frac{n^w(1-e^{-\omega_1 w})}{w(1+w/\omega)^{\delta+1}}\,dw.$$

Trivial estimation shows again that the series (11.3) converges (absolutely) in a half-plane $\sigma > c$. In particular (putting $w = -0{,}99 + iv$), we get

$$(11.5)\qquad |e_N^{(4)}| \leqslant 2\left(\frac{e^{\omega_1}}{N}\right)^{0{,}99} \log M \sum_{k \leqslant M}^{(k)} \frac{1}{\varphi(k)} \times$$

$$\times \sum_{k^*|k}^{(k^*)} \sum_{\chi^* \bmod k^*}^{(\chi^*)}{}^{*} \frac{1}{\pi} \int_0^\infty \left|\frac{L'}{L}(-0{,}99+iv, k^*, \chi^*)\right| \frac{dv}{|\frac{1}{2}+iv|\left|1-\frac{1}{\omega}+i\frac{v}{\omega}\right|^{\delta+1}}.$$

Splitting up the range of the integral into

$$\int_0^1 + \int_1^{\omega^2} + \int_{\omega^2}^{\infty}$$

we obtain respectively

$$\int_0^1 < c\log M,$$

$$\int_1^{\omega^2} < c\int_1^{\omega^2} \frac{\log k^*(2+v)}{v}\,dv < c\log M\cdot\log^2\omega < c\log^3 M$$

and

$$\int_{\omega^2}^{\infty} < c\int_{\omega^2}^{\infty}\frac{\log k^* v}{v^2}\,dv < c\log M;$$

hence, using also (6.2) and (6.1),

$$|e_N^{(4)}| \leqslant cM^{0,9999}\log^4 M < c\frac{M}{\log M}. \tag{11.6}$$

12. Now we consider $F_{M3}(s)$ from (11.1). The residual terms are due to poles at

$$w = 1, \quad w = 0, \quad w = \varrho$$

(see the footnote on p. 63). The contribution of poles at $w = 1$ is obviously from (9.3)

$$\frac{1-e^{-\omega_1}}{(1+1/\omega)^{\delta+1}}\sum_{k\leqslant M}^{(k)}\frac{\mu(k)\log(M/k)}{\varphi(k)}L(s-1,k,\chi_0)\overset{\text{def}}{=}F_{M3}^{(1)}(s). \tag{12.1}$$

This can be written as

$$\sum_{(n)}\frac{e_n^{(3,1)}}{n^s} \tag{12.2}$$

with

$$e_n^{(3,1)} = \frac{1-e^{-\omega_1}}{(1+1/\omega)^{\delta+1}}\,n\sum_{\substack{k\leqslant M\\(k,n)=1}}^{(k)}\frac{\mu(k)\log(M/k)}{\varphi(k)}, \tag{12.3}$$

which shows at once that the series in (12.2) converges (absolutely) for $\sigma > c$. In particular, for $n = N$ Lemma III gives

$$e_N^{(3,1)} = B_1 N\frac{1-e^{-\omega_1}}{(1+1/\omega)^{\delta+1}}\prod_{\substack{p|N\\p>2}}^{(p)}\frac{p-2}{p-1}+O\left(\frac{M}{\log M}\right)$$

and thus from (6.2) and (6.4)

$$e_N^{(3,1)} = \left\{1+O\left(\frac{\delta}{\omega}\right)+O\left(\frac{1}{\log M}\right)\right\}B_1 N\prod_{\substack{p|N\\p>2}}^{(p)}\frac{p-1}{p-2}. \tag{12.4}$$

13. Next we consider the contribution of the poles at $w = 0$. They occur only if $k^* > 1$ and $\chi^*(-1, k^*) = +1$; hence the contribution of these poles is from (9.3)

$$-\omega_1\sum_{k\leqslant M}^{(k)}\frac{\mu(k)\log(M/k)}{\varphi(k)}\sum_{\substack{k^*|k\\k^*>1}}^{(k^*)}\ \sideset{}{^*}\sum_{\chi^*(-1,k^*)=+1}^{(\chi^*)} L(s,k,\bar\chi)\overset{\text{def}}{=}F_{M3}^{(2)}(s), \tag{13.1}$$

which can be written as

$$\sum_{(n)}\frac{e_n^{(3,2)}}{n^s} \tag{13.2}$$

with

$$e_n^{(3,2)} = -\omega_1\sum_{\substack{k\leqslant M\\(k,n)=1}}^{(k)}\frac{\mu(k)\log(M/k)}{\varphi(k)}\sum_{\substack{k^*|k\\k^*>1}}^{(k^*)}\ \sideset{}{^*}\sum_{\substack{\chi^*\bmod k^*\\\chi^*(-1,k^*)=+1}}^{(\chi^*)}\overline{\chi^*}(n,k^*). \tag{13.3}$$

This again shows at once that series (13.2) converges (absolutely) in a half-plane $\sigma > c$. Considering again $e_N^{(3,2)}$, the inner sum in (13.3) can be written as [7]

$$\sideset{}{^*}\sum_{\chi^*\bmod k^*}^{(\chi^*)}\chi^*(N,k^*)\frac{1+\chi^*(-1,k^*)}{2},$$

and hence its absolute value cannot exceed (in the notation of Lemma IV)

$$\tfrac{1}{2}\big(Z(N,k^*)+Z(-N,k^*)\big).$$

This shows—also by (6.2)—

$$|e_N^{(3,2)}| \leqslant \log^2 M\sum_{k\leqslant M}^{(k)}\frac{|\mu(k)|}{\varphi(k)}\sum_{k^*|k}^{(k^*)}\{(N+1,k^*)+(N-1,k^*)\}.$$

But since

$$\sum_{k^*|k}^{(k^*)}(N\pm 1,k^*) \leqslant (N\pm 1,k)\tau(k),$$

we get

$$|e_N^{(3,2)}| \leqslant \exp\left(c\frac{\log M}{\log\log M}\right)\sum_{k\leqslant M}^{(k)}\frac{|\mu(k)|}{\varphi(k)}\{(N+1,k)+(N-1,k)\}.$$

Since

$$\sum_{k\leqslant M}^{(k)}\frac{|\mu(k)|}{\varphi(k)}(N\pm 1,k) \leqslant \sum_{d|(N\pm1)}^{(d)}\frac{d}{\varphi(d)}\sum_{r\leqslant M/d}^{(r)}\frac{|\mu(r)|}{\varphi(r)}$$
$$< c(\log\log M)^2\sum_{d|(N\pm1)}^{(d)}\sum_{r\leqslant M}^{(r)}\frac{1}{r} < \exp\left(c\frac{\log M}{\log\log M}\right),$$

[7] Simplified by Mr. I. Kátai and Mr. I. Környei.

we get

$$(13.4)\qquad |e_N^{(3,2)}| \leqslant \exp\left(c\frac{\log M}{\log\log M}\right).$$

14. Next we consider the contribution of the ϱ's to $F_{M3}(s)$. This is obviously

$$-\sum_{k\leqslant M}{}^{(k)}\frac{\mu(k)\log(M/k)}{\varphi(k)}\sum_{\chi \bmod k}{}^{(\chi)}\sum{}^{(\varrho(\chi))}L(s-\varrho, k, \bar\chi)\frac{1-e^{-\omega_1\varrho}}{\varrho(1+\varrho/\omega)^{\delta+1}} \overset{\text{def}}{=} F_{M3}^{(3)}(s)$$

and can obviously be written as

$$(14.1)\qquad \sum{}^{(n)}\frac{e_n^{(3,3)}}{n^s}$$

with

(14.2)

$$e_n^{(3,3)} = -\sum_{\substack{k\leqslant M\\(k,n)=1}}{}^{(k)}\frac{\mu(k)\log(M/k)}{\varphi(k)}\sum_{\chi\bmod k}{}^{(\chi)}\bar\chi(n,k)\sum{}^{\varrho(\chi)} n^\varrho\frac{1-e^{-\omega_1\varrho}}{\varrho(1+\varrho/\omega)^{\delta+1}}.$$

This shows again that the series in (14.1) converges (absolutely) in a half-plane $\sigma > c$. This gives for $n = N$

(14.3)

$$e_N^{(3,3)} = -\sum_{\substack{k\leqslant M\\(k,N)=1}}{}^{(k)}\frac{\mu(k)\log(M/k)}{\varphi(k)}\sum_{\chi\bmod k}{}^{(\chi)}\bar\chi(N,k)\sum{}^{(\varrho(\chi))} N^\varrho\frac{1-e^{-\omega_1\varrho}}{\varrho(1+\varrho/\omega)^{\delta+1}}.$$

Next we estimate trivially the contribution of ϱ's with

$$(14.4)\qquad |\gamma| > \omega M^{3/2\delta}.$$

This cannot exceed absolutely the quantity

$$4M\log M\sum_{k\leqslant M}{}^{(k)}\frac{1}{\varphi(k)}\sum_{\chi\bmod k}{}^{(\chi)}\sum_{\gamma>\omega M^{3/2\delta}}{}^{(\varrho)}\frac{1}{\gamma(1+\gamma^2/\omega^2)^{(\delta+1)/2}}$$
$$< cM^2\log^2 M\sum_{\nu>\omega M^{3/2\delta}}{}^{(\nu)}\frac{\log\nu}{\nu(1+\nu^2/\omega^2)^{(\delta+1)/2}}$$
$$< cM^2\log^2 M\cdot\omega^{\delta+1}\sum_{\nu>\omega M^{3/2\delta}}{}^{(\nu)}\nu^{-\delta-2}\log\nu$$
$$< cM^2\log^3 M\log\omega\cdot M^{-\frac{3}{2\delta}(\delta+1)} < c\sqrt{M}\log^4 M$$

using (6.3) and (6.1). Hence

$$(14.5)\qquad e_N^{(3,3)} = -\sum_{\substack{k\leqslant M\\(k,N)=1}}{}^{(k)}\frac{\mu(k)\log(M/k)}{\varphi(k)}\sum_{\chi\bmod k}{}^{(\chi)}\bar\chi(N,k)\times$$
$$\times\sum_{|\gamma|\leqslant\omega M^{3/2\delta}}{}^{(\varrho(\chi))} N^\varrho\frac{1-e^{-\omega_1\varrho}}{\varrho(1+\varrho/\omega)^{\delta+1}} + O(\sqrt{M}\log^4 M).$$

Now we have in a half-plane $\sigma > c$ the identity

$$F_M(s) = F_{M2}(s)+F_{M4}(s)+F_{M3}^{(1)}(s)+F_{M3}^{(2)}(s)+F_{M3}^{(3)}(s),$$

and all these developable into Dirichlet series convergent here.

Hence by the uniqueness theorem we have

$$a_N = e_N^{(2)}+e_N^{(4)}+e_N^{(3,1)}+e_N^{(3,2)}+e_N^{(3,3)}.$$

Collecting the corresponding expressions, we get

$$(14.6)\qquad \sum_{p_1+p_2=N}{}^{(p_1,p_2)}\log p_1\log p_2 = B_1N\prod_{\substack{p|N\\p>2}}{}^{(p)}\frac{p-1}{p-2}\left\{1+O\left(\frac{\delta}{\omega}\right)+O\left(\frac{1}{\log M}\right)\right\}-$$
$$-\sum_{\substack{k\leqslant M\\(k,N)=1}}{}^{(k)}\frac{\mu(k)\log(M/k)}{\varphi(k)}\sum_{\chi\bmod k}{}^{(\chi)}\bar\chi(N,k)\sum_{|\gamma|\leqslant\omega M^{3/2\delta}}{}^{(\varrho(\chi))}N^\varrho\frac{1-e^{-\omega_1\varrho}}{\varrho(1+\varrho/\omega)^{\delta+1}}.$$

As to the left side we have evidently

$$\sum_{p_1+p_2=N}{}^{(p_1,p_2)}\log p_1\log p_2 \leqslant \nu_2(N)\log^2 N.$$

On the other hand we have

$$\sum_{p_1+p_2=N}{}^{(p_1,p_2)}\log p_1\log p_2 \geqslant \sum_{\substack{p_1+p_2=N\\ \min p_j\geqslant N/\log^2 N}}{}^{(p_1,p_2)}\log p_1\log p_2$$
$$\geqslant \log^2 N\left(1-8\frac{\log\log N}{\log N}\right)\Big\{\nu_2(N)-\sum_{\substack{p_1+p_2=N\\ \min p_j<N/\log^2 N}}{}^{(p_1,p_2)}1\Big\}$$
$$\geqslant \nu_2(N)\log^2 N\left(1-9\frac{\log\log M}{\log M}\right\}-c\frac{M}{\log M}.$$

Hence

$$\sum_{p_1+p_2=N}{}^{(p_1,p_2)}\log p_1\log p_2 = \nu_2(N)\log^2 N\left\{1+O\left(\frac{\log\log M}{\log M}\right)\right\}+O\left(\frac{M}{\log M}\right);$$

putting it into (14.6) we get

$$(14.7)\qquad \nu_2(N) = B_1\frac{N}{\log^2 N}\prod_{\substack{p|N\\ p>2}}{}^{(p)}\frac{p-1}{p-2}\left\{1+O\left(\frac{\delta}{\omega}\right)+O\left(\frac{\log\log M}{\log M}\right)\right\}-$$

$$-\frac{1+o(1)}{\log^2 M}\sum_{\substack{k\leqslant M\\ (k,N)=1}}\frac{\mu(k)\log(M/k)}{\varphi(k)}\sum_{\chi \bmod k}{}^{(\chi)}\bar{\chi}(N,k)\sum_{|\gamma|<\omega M^{3/2\delta}}{}^{(\varrho(\chi))}N^{\varrho}\frac{1-e^{-\omega_1\varrho}}{\varrho(1+\varrho/\omega)^{\delta+1}}.$$

Until now δ, ω and ω_1 were restricted only by (6.1)-(6.2)-(6.3)-(6.4). Let $\varepsilon(x)>0$ and tending for $x\to\infty$ monotonically to 0 arbitrarily slowly but so that

$$(14.8)\qquad \lim_{x\to\infty}\varepsilon(x)\sqrt{\log x} = +\infty.$$

Then if we choose

$$(14.9)\qquad \omega = \log M,\quad \omega_1 = 0{,}99\log N,\quad \delta = [\varepsilon(M)\log M]$$

(14.7) takes the form

$$\nu_2(N) = B_1\frac{N}{\log^2 N}\prod_{\substack{p|N\\ p>2}}{}^{(p)}\frac{p-1}{p-2}\{1+O(\varepsilon(M))\}-$$

$$-\frac{1+o(1)}{\log^2 M}\sum_{\substack{k\leqslant M\\ (k,N)=1}}{}^{(k)}\frac{\mu(k)\log(M/k)}{\varphi(k)}\times$$

$$\times\sum_{\chi \bmod k}{}^{(\chi)}\bar{\chi}(N,k)\sum_{|\gamma|\leqslant\log M\exp(3/2\varepsilon(M))}{}^{(\varrho)}\frac{N^{\varrho}-N^{\varrho/100}}{\varrho(1+\varrho/\log M)^{[\varepsilon(M)\log M]+1}}$$

which proves Theorem II.

Appendix

15. For the proof of Theorem I we start from the function for $\sigma>1$ (D positive even)

$$(15.1)\qquad H_N(s) \stackrel{\text{def}}{=} -\sum_{\substack{k\leqslant N+D\\ (k,D)=1}}{}^{(k)}\frac{\mu(k)\log((N+D)/k)}{\varphi(k)}\sum_{\chi \bmod k}{}^{(\chi)}\bar{\chi}(-D,k)\frac{L'}{L}(s,k,\chi).$$

Substituting the Dirichlet series we get as previously

$$(15.2)\qquad H_N(s) = \sum_{\substack{n\leqslant N\\ (n,D)=1}}{}^{(n)}\frac{\Lambda(n)\Lambda(n+D)}{n^s}+$$

$$+\sum_{\substack{n\leqslant N\\ (n,D)>1}}{}^{(n)}\frac{\Lambda(n)}{n^s}\sum_{\substack{k\leqslant N+D\\ (k,n)=(k,D)=1\\ k|(n+D)}}{}^{(k)}\mu(k)\log\frac{N+D}{k}+\sum_{n>N}\frac{d_n}{n^s}.$$

Writing $H_N(s)$ in the form

$$(15.3)\qquad H_N(s) = -\sum_{\substack{k\leqslant N+D\\ (k,D)=1}}{}^{(k)}\frac{\mu(k)\log((N+D)/k)}{\varphi(k)}\times$$

$$\times\sum_{k^*|k}{}^{(k^*)}\sum_{\chi^* \bmod k^*}{}^{*(\chi^*)}\bar{\chi}(-D,k)\frac{L'}{L}(s,k^*,\chi^*)-\sum_{\substack{k\leqslant N+D\\ (k,D)=1}}{}^{(k)}\frac{\mu(k)\log((N+D)/k)}{\varphi(k)}\times$$

$$\times\sum_{k^*|k}{}^{(k^*)}\sum_{\chi^* \bmod k^*}{}^{*(\chi^*)}\bar{\chi}^*(-D,k^*)\sum_{p|\frac{k}{k^*}}{}^{(p)}\frac{\chi^*(p,k^*)\log p}{p^s-\chi^*(p,k^*)}\stackrel{\text{def}}{=}H_{N1}(s)+H_{N2}(s)$$

we start from the integral

$$(15.4)\qquad I = I_1+I_2,$$

where, with $a>1$,

$$(15.5)\qquad I_j = \frac{1}{2\pi i}\int_{(a)}\frac{N^s(1-e^{-\omega_3 s})}{s(1+s/\omega_2)^{\delta_1+1}}H_{Nj}(s)\,ds,\qquad j=1,2,$$

and δ_1, ω_2, ω_3 are restricted at present only by

$$(15.6)\qquad 3\leqslant\omega_2\leqslant\log N,$$

$$(15.7)\qquad \delta_1>3\log\log N\quad\text{and integer},$$

$$(15.8)\qquad \frac{\delta_1}{\omega_2}\to 0,$$

$$(15.9)\qquad 3\log\log N\leqslant\omega_3\leqslant 0{,}99\log N,$$

$$(15.10)\qquad 2\leqslant D\leqslant\frac{N}{\log^{10}N}$$

and they will be determined exactly later.

16. Using the Dirichlet series representation of $H_N(s)$ in (15.2) and also Lemma I, we get (with the notation (6.8) but with δ_1 instead of δ

and similarly with the ω's)

$$(16.1)\qquad I = \sum_{\substack{n\leqslant N\\(n,D)=1}}^{(n)} \Lambda(n)\Lambda(n+D)b(N,n)+$$

$$+\sum_{\substack{n\leqslant N\\(n,D)>1}}^{(n)} \Lambda(n)b(N,n)\sum_{\substack{k\leqslant N+D\\(k,n)=(k,D)=1\\k|(n+D)}}^{(k)} \mu(k)\log\frac{N+D}{k}.$$

The second sum in (16.1) is absolutely, as is easy to see,

$$\leqslant c\log^2(N+D)\max_{1\leqslant l\leqslant N+D}\tau(l)\sum_{p|D}^{(p)}\sum_{p^a\leqslant N}^{(a)}1 < cN^{1/10}.$$

In the first sum of (16.1) Schnirelman's theorem implies, together with (15.8)

$$\sum_{\substack{N\exp\left(-2\frac{\delta_1}{\omega_2}\right)\leqslant n\leqslant N\\(n,D)=1}}^{(n)} \Lambda(n)\Lambda(n+D)b(N,n) < cN\frac{\delta_1}{\omega_2}\prod_{\substack{p>2\\p|D}}\frac{p-1}{p-2}$$

and evidently from (15.9) and (15.10)

$$\sum_{n\leqslant Ne^{-\omega_3}}^{(n)} \Lambda(n)\Lambda(n+D)b(N,n) < c\log^2 N\cdot Ne^{-\omega_3} < c\frac{N}{\log N}.$$

Hence we got from (16.1)

$$I = \sum_{\substack{p_1=p_2+D\\Ne^{-\omega_3}\leqslant p_2\leqslant N\exp\left(-2\frac{\delta_1}{\omega_2}\right)}}^{(p_1,p_2)} \log p_1\log p_2\cdot b(N,p_2)+O(N)\left\{\frac{\delta_1}{\omega_2}+\frac{1}{\log N}\right\}\prod_{\substack{p|D\\p>2}}\frac{p-1}{p-2}$$

and on using (15.7) and (6.9)(c)

$$(16.2)\qquad I = \sum_{\substack{p_2\leqslant N\\p_1=p_2+D}}^{(p_1,p_2)} \log p_1\log p_2 + O(N)\prod_{\substack{p|D\\p>2}}\frac{p-1}{p-2}\left(\frac{\delta_1}{\omega_2}+\frac{1}{\log N}\right).$$

17. Next we investigate I_2 in (15.4). This gives as in 10

$$I_2 = -\sum_{\substack{k\leqslant N+D\\(k,D)=1}}^{(k)} \frac{\mu(k)\log\big((N+D)/k\big)}{\varphi(k)}\sum_{k^*|k}^{(k^*)}\sum_{\chi^*\bmod k^*}^{(\chi^*)}{}^{*}\,\overline{\chi^*}(-D,k^*)\times$$

$$\times\sum_{p\,\left|\,\frac{k}{k^*}\right.}^{(p)}\log p\sum_{\lambda=1}^{\infty}{}^{(\lambda)}\chi^*(p^\lambda,k^*)b(N,p^\lambda) = \sum_{\substack{k\leqslant N+D\\(k,D)=1}}^{(k)}\frac{\mu(k)\log\big((N+D)/k\big)}{\varphi(k)}\times$$

$$\times\sum_{k^*|k}^{(k^*)}\sum_{p\,\left|\,\frac{k}{k^*}\right.}^{(p)}\log p\sum_{p^\lambda\leqslant N+D^*}^{(\lambda)} b(N,p^\lambda)\Big\{\sum_{\chi^*\bmod k^*}^{(\chi^*)}{}^{*}\chi^*\big(p^\lambda(-D)^{-1},k^*\big)\Big\}.$$

Using Lemma IV and (6.9)(b), we get

$$|I_2| \leqslant \log^2(N+D)\sum_{\substack{k\leqslant N+D\\(k,D)=1}}^{(k)}\frac{|\mu(k)|}{\varphi(k)}\sum_{k^*|k}^{(k^*)}\sum_{p\,\left|\,\frac{k}{k^*}\right.}^{(p)}\sum_{p^\lambda\leqslant N+D}^{(\lambda)}(p^\lambda+D,k^*).$$

The next steps are quite analogous to those in 10 (after (10.3)) and result

$$(17.1)\qquad |I_2| < \exp\left(c\frac{\log N}{\log\log N}\right).$$

Next we consider I_1. Shifting the line of integration to the line $\operatorname{Re}s = -0{,}99$ let the contribution of the residua resp. of the integral be I_3 resp. I_4. For I_4 we get from (15.1) and (15.5)

$$(17.2)\qquad |I_4|$$

$$\leqslant c\left(\frac{e^{\omega_3}}{N}\right)^{0{,}99}\log(N+D)\sum_{k\leqslant N+D}^{(k)}\frac{1}{\varphi(k)}\sum_{\chi\bmod k}^{(\chi)}\int_0^\infty\frac{\log\{(N+D)(2+v)\}\,dv}{(1+v)\{(1-1/\omega_2)^2+v^2/\omega_2^2\}^{(\delta_1+1)/2}}$$

$$< cN\log N\left\{\log^2 N+\frac{1}{N}\int_N^\infty\frac{\log Nv}{v^{\delta_1+1}}\,\omega_2^{\delta_1+1}dv\right\}\left(\frac{e^{\omega_3}}{N}\right)^{0{,}99}$$

$$< cN\log^3 N\left(\frac{e^{\omega_3}}{N}\right)^{0{,}99} = O\left(\frac{N}{\log N}\right)$$

owing to (15.9) and (15.6). Collecting (16.2), (17.1) and (17.2), we get

$$(17.3)\qquad \sum_{p_2\leqslant N,\,p_1=p_2+D}^{(p_1,p_2)}\log p_1\log p_2 = I_3+O(N)\left\{\frac{\delta_1}{\omega_2}+\frac{1}{\log N}\right\}\prod_{\substack{p|D\\p>2}}^{(p)}\frac{p-1}{p-2}.$$

18. Finally we consider the residua. The contribution of the poles at $s = 1$ is

$$(18.1)\qquad N\frac{1-e^{-\omega_3}}{(1+1/\omega_2)^{\delta_1+1}}\sum_{\substack{k\leqslant N+D\\(k,D)=1}}^{(k)}\frac{\mu(k)\log\big((N+D)/k\big)}{\varphi(k)}$$

$$= B_1N\prod_{\substack{p>2\\p|D}}^{(p)}\frac{p-1}{p-2}\left\{1+O\left(\frac{\delta_1}{\omega_2}\right)+O\left(\frac{1}{\log N}\right)\right\}$$

owing to Lemma III, (15.8) and (15.9).

The contribution of residua at $s = 0$ is

$$-\sum_{\substack{k\leqslant N+D\\(k,D)=1}}^{(k)}\frac{\mu(k)\log\big((N+D)/k\big)}{\varphi(k)}\sum_{\substack{k^*|k\\k^*>1}}^{(k^*)}\sum_{\substack{\chi^*\bmod k^*\\\chi^*(-1,k^*)=+1}}^{(\chi^*)}{}^{*}\chi^*(-D,k^*);$$

the reasoning in 13 gives mutatis mutandis for the absolute value of this expression the upper bound

$$c\log^2 N \sum_{k\leqslant N+D}^{(k)} \frac{|\mu(k)|}{\varphi(k)} \sum_{k^*|k}^{(k^*)} \{(D+1, k^*)+(D-1, k^*)\},$$

i.e.

$$\exp\left(c\frac{\log N}{\log\log N}\right),$$

using also (15.10). Finally the contribution of the nontrivial zeros is

$$-\sum_{\substack{k\leqslant N+D\\(k,D)=1}}^{(k)} \frac{\mu(k)\log\big((N+k)/D\big)}{\varphi(k)} \sum_{\chi \bmod k}^{(\chi)} \bar{\chi}(-D, k) \sum^{(\varrho(\chi))} \frac{N^{\varrho}(1-e^{-\omega_3\varrho})}{\varrho(1+\varrho/\omega_2)^{\delta_1+1}}.$$

The reasoning in 14 gives mutatis mutandis that the contribution of ϱ's with

$$|\gamma| > \omega_2(N+D)^{3/2\delta_1}$$

is $O(N^{3/4})$.

19. Let $\varepsilon(x)$ be positive tending to 0 for $x\to+\infty$ monotonically arbitrarily slowly but so that

(19.1) $$\varepsilon(x)\sqrt{\log x}\to+\infty.$$

Then choosing

(19.2) $$\omega_3 = 0{,}99\log N,\quad \delta_1=[\varepsilon(N)\log N],\quad \omega_2=\log N,\quad N>\max(c, D\log^{10}D)$$

the requirements (15.6)-(15.10) are fulfilled and

$$\sum_{\substack{p_2\leqslant N\\p_1=p_2+D}}^{(p_2)} \log p_1\log p_2 = \{1+O(\varepsilon(N))\}B_1 N\prod_{\substack{p|D\\p>2}}\frac{p-1}{p-2} - \sum_{\substack{k\leqslant N+D\\(k,D)=1}}^{(k)} \frac{\mu(k)\log\big((N+D)/k\big)}{\varphi(k)}\times$$

$$\times\sum_{\chi \bmod k}^{(\chi)}\bar{\chi}(-D,k)\sum_{|\gamma|\leqslant\log N\exp(3/2\varepsilon(N))}^{(\varrho(\chi))}\frac{N^{\varrho}-N^{\varrho/100}}{\varrho(1+\varrho/\log N)^{[\varepsilon(N)\log N]+1}}.$$

Then the proof can be completed as in Theorem II.

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