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A new estimate for the sum $M(x) = \sum_{n \leqslant x} \mu(n)$

by

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1. Introduction. Let $\mu(n)$ be the ordinary Möbius function, so that if $n = 1$, $\mu(n) = 1$, and if $n = p_1^{a_1} \ldots p_k^{a_k}$, then

$$\mu(n) = \begin{cases} 0, & \text{if any } a_i > 1,\ 1 \leqslant i \leqslant k, \\ (-1)^k, & \text{otherwise}, \end{cases}$$

and define $M(x)$ by

$$M(x) = \sum_{n \leqslant x} \mu(n). \tag{1}$$

R. D. von Sterneck [6] showed that

$$|M(x)| < \tfrac{1}{9}x + 8 \tag{2}$$

and R. Hackel [2] improved this to

$$|M(x)| < \tfrac{1}{26}x + 155 \quad \text{for all } x. \tag{3}$$

Our object here will be to obtain the result

$$|M(x)| < \frac{x}{80} \quad \text{for } x \geqslant 1119 \tag{4}$$

(if we wished a result valid for all x, we could say

$$|M(x)| < \frac{x}{80} + 5, \quad \text{for all } x). \tag{5}$$

We observe that it is well known (see e.g. R. Ayoub [1], p. 111) that the result $M(x) = o(x)$ is equivalent to the prime number theorem.

G. Neubauer [4] has shown that

$$|M(x)| < \tfrac{1}{2}\sqrt{x} \quad \text{for} \quad 200 < x \leqslant 10^8 \tag{6}$$

(and that this fails to be true for a number of larger x). Using this we can prove (4) for $1119 \leqslant x \leqslant 10^8$. For since $\frac{1}{2}\sqrt{x} < x/80$ for $x > 1600$, (4) follows for $1600 \leqslant x \leqslant 10^8$, and one obtains by simple checking that (4) also holds for $1119 \leqslant x \leqslant 1600$; thus, (4) remains to be verified for $x > 10^8$.

2. The method. We outline the method to be used, which is a refinement of that of von Sterneck. Consider the function

$$f(x) = [x]-\left[\frac{x}{2}\right]-\left[\frac{x}{3}\right]-\left[\frac{x}{5}\right]+\left[\frac{x}{30}\right].$$

Since for $x \geqslant 1$

$$\sum_{d\leqslant x}\mu(d)\left[\frac{x}{d}\right] = 1,$$

we have

$$\sum_{d\leqslant x}\mu(d)\left[\frac{x}{md}\right] = \sum_{d\leqslant x/m}\mu(d)\left[\frac{x}{md}\right]+\sum_{x/m<d\leqslant x}\mu(d)\left[\frac{x}{md}\right] = 1+0 = 1, \quad \text{for} \quad x \geqslant m,$$

and thus

$$\sum_{d\leqslant x}\mu(d)f\left(\frac{x}{d}\right) = -1, \quad \text{for} \quad x \geqslant 30.$$

Since $f(x) = 1$ for $1 \leqslant x < 6$ and 0 or 1 for $x \geqslant 6$, we have $f(x/d) = 1$ for $d > x/6$, so that

$$\left|\sum_{d\leqslant x}\mu(d)\left(1-f\left(\frac{x}{d}\right)\right)\right| \leqslant \sum_{d\leqslant x/6}|\mu(d)| = Q\left(\frac{x}{6}\right).$$

Thus,

(7) $$|M(x)+1| \leqslant Q\left(\frac{x}{6}\right), \quad \text{for} \quad x \geqslant 30.$$

Moser and the author observed in [3] that

(8) $$\left|Q(x)-\frac{6}{\pi^2}x\right| < \frac{1}{2}\sqrt{x}, \quad \text{for} \quad x \geqslant 8,$$

which follows readily using the methods therein and the result

(9) $$|M(x)| < \tfrac{1}{25}x \quad \text{for} \quad x > 200,$$

a combination of the Hackel and Neubauer results. It follows that

(10) $$0.600x < Q(x) < 0.615x \quad \text{for} \quad x > 5000,$$

and one readily checks that (10) holds for $x \geqslant 475$. Similarly,

(11) $$Q(x) < 0.635x \quad \text{for} \quad x \geqslant 75.$$

Using (10) in (7) we obtain

(12) $$|M(x)+1| \leqslant 0.103x \quad \text{for} \quad x \geqslant 2950$$

and, by (6), for $x > 200$. If we further observe that $f(x) = 1$ for $7 \leqslant x < 10$, we have

(13) $$|M(x)+1| \leqslant Q\left(\frac{x}{6}\right)-Q\left(\frac{x}{7}\right)+Q\left(\frac{x}{10}\right) < 0.079x, \quad \text{for} \quad x > 200.$$

Using the function

$$f_1(x) = [x]-\left[\frac{x}{2}\right]-\left[\frac{x}{3}\right]-\left[\frac{x}{5}\right]+\left[\frac{x}{15}\right]-\left[\frac{x}{30}\right]$$

and similar refinements to that used in deriving (13), one can obtain

(14) $$|M(x)+2| < 0.04x, \quad \text{for} \quad x > 200,$$

which is the same as (9). It seems difficult to get a fairly simple function like $f_1(x)$ which will substantially improve (14).

If we examine the characteristics of a "good" function f, we see that what we would like is a function which takes the value 1 for $1 \leqslant x \leqslant n$ for fairly large n, and then does not differ too widely from 1 thereafter. We shall employ the techniques of E. Waage ([7] and [8]) to obtain such a function.

3. Main result. In line with Waage, we define

$$u_k(x) = \left[\frac{x}{k}\right]-\left[\frac{x}{k+1}\right]-\left[\frac{x}{k(k+1)}\right]$$

and use the symbol

$$(n_1, n_2, \ldots, n_m; l_1, l_2, \ldots, l_t)$$

to stand for the function

$$\left[\frac{x}{n_1}\right]+\left[\frac{x}{n_2}\right]+\ldots+\left[\frac{x}{n_m}\right]-\left[\frac{x}{l_1}\right]-\left[\frac{x}{l_2}\right]-\ldots-\left[\frac{x}{l_t}\right].$$

Let

$$U_2(x) = u_1(x) = (1; 2, 2),$$
$$U_5(x) = u_1(x)+u_2(x) = (1; 2, 3, 6),$$
$$U_6(x) = U_5(x)-u_5(x) = (1, 30; 2, 3, 5) \quad (\text{this is our } f(x)),$$
$$U_{10}(x) = U_6(x)+u_6(x) = (1, 6, 30; 2, 3, 5, 7, 42).$$

Define $U_s(x)$, $U_f(x)$, $u(x)$, and $u'(x)$, respectively, as follows:

$$U_s(x) = U_{10}(x) - u_5\left(\frac{x}{6}\right) - u_6\left(\frac{x}{6}\right) + u_2\left(\frac{x}{35}\right) - u_1\left(\frac{x}{105}\right) - u_6\left(\frac{x}{30}\right) - u_5\left(\frac{x}{42}\right)$$

$$= (1, 6, 70; 2, 3, 5, 7, 210),$$

$$U_f(x) = U_s(x) + u_{10}(x) - u_2\left(\frac{x}{35}\right) - u_{21}\left(\frac{x}{5}\right) = (1, 6, 10, 2310; 2, 3, 5, 7, 11),$$

$$u(x) = U_s(x) + u_1\left(\frac{x}{10}\right) + u_2\left(\frac{x}{10}\right) + u_1\left(\frac{x}{30}\right) + u_1\left(\frac{x}{14}\right) + 2u_1\left(\frac{x}{28}\right)$$

$$+ 4u_4\left(\frac{x}{14}\right) - u_1\left(\frac{x}{70}\right) - 2u_1\left(\frac{x}{140}\right) + u_1\left(\frac{x}{21}\right) + u_1\left(\frac{x}{35}\right) + u_2\left(\frac{x}{35}\right)$$

$$= (1, 6, 10, 14, 21, 35; 2, 3, 5, 7, 30, 30, 30, 42, 42, 70, 70, 70, 70, 70, 105, 210, 210),$$

$$u'(x) = u_2\left(\frac{x}{5}\right) + u_6\left(\frac{x}{5}\right) - u_{14}(x) = (10; 14, 35).$$

Let R_1, R_2, R_3 and R_4 be respectively the sets

$\{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$,
$\{18, 70, 90, 90, 118, 134, 142, 146, 162, 177, 183, 213\}$,
$\{113, 131, 139, 154, 170, 173, 191\}$,
$\{18, 18, 54, 90, 90, 105, 107, 108, 109, 700, 700, 700, 700\}$.

Then the function $e(x)$ which we shall use is defined by the formula

$$e(x) = u(x) - \sum_{r \in R_1} U_f\left(\frac{x}{r}\right) + u'\left(\frac{x}{7}\right) + 2u'\left(\frac{x}{14}\right) + u'\left(\frac{x}{17}\right) + U_5\left(\frac{x}{42}\right)$$

$$- U_5\left(\frac{x}{79}\right) - U_5\left(\frac{x}{101}\right) - U_5\left(\frac{x}{103}\right) - U_5\left(\frac{x}{137}\right) - U_5\left(\frac{x}{163}\right) - U_5\left(\frac{x}{167}\right)$$

$$+ 2U_6\left(\frac{x}{30}\right) + \sum_{r \in R_2} u_1\left(\frac{x}{r}\right) - \sum_{r \in R_3} u_1\left(\frac{x}{r}\right) + \sum_{r \in R_4} u_2\left(\frac{x}{r}\right) - 2u_2\left(\frac{x}{1400}\right)$$

$$+ u_3\left(\frac{x}{18}\right) - u_3\left(\frac{x}{80}\right) + u_3\left(\frac{x}{108}\right) + 3u_4\left(\frac{x}{108}\right) + u_5\left(\frac{x}{3}\right) - 3u_8\left(\frac{x}{200}\right)$$

$$+ 6u_9\left(\frac{x}{60}\right) - 3u_9\left(\frac{x}{115}\right) + u_{10}\left(\frac{x}{21}\right) - 3u_{10}\left(\frac{x}{46}\right) + 3u_{10}\left(\frac{x}{92}\right)$$

$$- 3u_{10}\left(\frac{x}{103}\right) - 2u_{14}\left(\frac{x}{20}\right) + u_{18}\left(\frac{x}{18}\right) - 2u_{19}\left(\frac{x}{23}\right) + u_{21}\left(\frac{x}{14}\right) - 2u_{23}\left(\frac{x}{10}\right)$$

$$- u_{26}\left(\frac{x}{12}\right) + u_{27}\left(\frac{x}{11}\right) + 3u_{33}\left(\frac{x}{10}\right) - u_{34}\left(\frac{x}{2}\right) + u_{44}\left(\frac{x}{5}\right) - 2u_{48}\left(\frac{x}{5}\right)$$

$$+ u_{49}\left(\frac{x}{8}\right) + u_{50}\left(\frac{x}{5}\right) - u_{50}\left(\frac{x}{10}\right) - u_{52}\left(\frac{x}{10}\right) - u_{53}(x) - 2u_{58}\left(\frac{x}{100}\right)$$

$$- u_{59}(x) + u_{60}(x) + 3u_{60}\left(\frac{x}{10}\right) - u_{67}(x) + 3u_{67}\left(\frac{x}{100}\right) + u_{68}\left(\frac{x}{5}\right) - u_{68}\left(\frac{x}{10}\right)$$

$$+ u_{70}(x) + u_{72}(x) - u_{81}\left(\frac{x}{20}\right) - u_{83}(x) - u_{89}(x) + u_{90}\left(\frac{x}{2}\right) + u_{94}\left(\frac{x}{10}\right)$$

$$- u_{96}\left(\frac{x}{3}\right) - u_{97}(x) + u_{106}(x) + u_{108}(x) + 2u_{109}\left(\frac{x}{3}\right) - u_{114}\left(\frac{x}{2}\right)$$

$$- u_{121}(x) + u_{126}(x) - u_{139}\left(\frac{x}{2}\right) - u_{143}\left(\frac{x}{2}\right) - u_{144}\left(\frac{x}{20}\right) - u_{149}(x) + u_{150}(x)$$

$$- u_{157}(x) - u_{158}(x) - u_{165}(x) + u_{178}(x) + u_{180}(x) - u_{193}(x) - u_{195}(x) + u_{196}(x)$$

$$- u_{199}(x) - u_{200}(x) + u_{210}(x) - u_{263}(x) - u_{264}(x) + u_{264}\left(\frac{x}{10}\right) - u_{266}(x)$$

$$+ u_{270}(x) + 3u_{270}\left(\frac{x}{20}\right) - u_{283}(x) - u_{320}(x)$$

$$= \sum_{n=1}^{218} \mu(n)\left[\frac{x}{n}\right] + \sum_{p \in P}\left[\frac{x}{p}\right] - \sum_{q \in Q}\left[\frac{x}{q}\right],$$

where

(15) $P = \{220, 226, 226, 235, 237, 245, 250, 253, 259, 262, 262, 265, 267, 274, 278, 287, 291, 294, 297, 300, 300, 301, 303, 309, 319, 326, 327, 329, 330, 334, 340, 341, 346, 346, 382, 382, 392, 407, 411, 451, 473, 474, 489, 501, 506, 506, 506, 510, 517, 530, 540, 540, 540, 540, 606, 618, 690, 690, 690, 720, 720, 720, 800, 800, 822, 920, 920, 920, 940, 960, 978, 1002, 1133, 1133, 1133, 1150, 1150, 1150, 1200, 1200, 1400, 1400, 1400, 1400, 1640, 1800, 1800, 1800, 2380, 2640, 2862, 2900, 3540, 4556, 5060, 5060, 5060, 5520, 5520, 5900, 5900, 6700, 6700, 6700, 6972, 8010, 8400, 8400, 8424, 8740, 8740, 9506, 10350, 10350, 10350, 11330, 11330, 11330, 11760, 11760, 13200, 13200, 14400, 14400, 14400, 14762, 15180, 15180, 16002, 22350, 24806, 25122, 25500, 26220, 27390, 27560, 27936, 37442, 38220, 38920, 39800, 40200, 41184, 46920, 66420, 69432, 69960, 71022, 80372, 102720, 342200, 342200, 417600\}$

and

$$Q = \{222, 225, 228, 230, 230, 231, 236, 236, 238, 240, 246, 252, 255, 258, 263, 266, 268, 268, 270, 271, 280, 282, 283, 284, 286, 290, 292, 292, 310, 312, 315, 324, 342, 345, 345, 354, 354, 366, 366, 370, 400, 410, 426, 426, 430, 432, 437, 437, 460, 470, 490, 490, 500, 520, 595, 600, 600, 660, 680, 820, 820, 900, 950, 1012, 1012, 1012, 1030, 1030, 1030, 1035, 1035, 1035, 1100, 1100, 1296, 1600, 1600, 1600, 1620, 2100, 2100, 2100, 2100, 2310, 2650, 2800, 2800, 2880, 3600, 3600, 3600, 3660, 4970, 5256, 5400, 5400, 5400, 5420, 5420, 5420, 5800, 5800, 6156, 6468, 6800, 6800, 6800, 8316, 9900, 10120, 10120, 10120, 11342, 11220, 11220, 11220, 11772, 12750, 19600, 22650, 23460, 23460, 25410, 30030, 31862, 32580, 35970, 35970, 36600, 36600, 36600, 38612, 39270, 43890, 44310, 53130, 66990, 71610, 73170, 85470, 89300, 94710, 99330, 108570, 455600, 455600, 455600, 699600, 1463400, 1463400, 1463400\}.$$

This rather complicated function was obtained by successively evaluating simpler functions by computer to see where they began to differ too much from 1, and adding in compensating simple functions to reduce the rate of growth.

Since there are 222 positive terms and 226 negative terms in e,

$$|e(x)| \leqslant 226 \quad \text{for all } x, \tag{16}$$

for when we remove the square brackets in e the function is identically zero by construction. Upon examining $e(x)$, we find that

$$e(x) = 1, \quad \text{for} \quad 1 \leqslant x < 219 \tag{17}$$

and

$$|e(x)-1| \leqslant k \quad \text{for} \quad x < n, \tag{18}$$

where k and n are as given in the following table:

k	1	2	3	4	5	6	7	8	9	10	11
n	345	568	584	804	1237	1359	1391	1393	1416	1416	1417

k	12	13	14	15	16	17	18	19	20
n	5010	5011	5881	5882	16097	16100	16100	16103	26740

k	21	22	23	24	25	26	27	28	29
n	26750	26752	26754	26759	31397	46110	46110	46112	63611

k	30	31	32	33	34	35	36	37	38
n	67158	67159	67189	69258	69259	69263	82800	82800	82813

k	39	40	41	42	43	44		45
n	85869	87542	87547	97006	97007	106591	up to	125000.

Let N be the set of n's in the above table. It follows that

$$|M(x)+4| \leqslant Q\left(\frac{x}{219}\right) + \sum_{n\in N} Q\left(\frac{x}{n}\right) + (226-45)\, Q\left(\frac{x}{12500}\right). \tag{19}$$

Using (8) and (10) in (19) we obtain

$$|M(x)+4| \leqslant 0.01247x, \quad \text{for} \quad x > 10^8$$

or

$$|M(x)| < \tfrac{1}{80}x, \quad \text{for} \quad x > 10^8.$$

This completes the proof of (4).

It is likely that with a better function $e(x)$ one could prove

$$|M(x)| < .01x, \quad \text{for} \quad x \geqslant 1137.$$

This is certainly true for $1137 \leqslant x \leqslant 10^8$.

4. Application. S. Selberg [5] has shown that, if $g(x)$ is defined by

$$g(x) = \sum_{n\leqslant x} \frac{\mu(n)}{n},$$

then $g(x)$ changes sign infinitely often. We show here that, on the other hand, $g_1(x)$, defined by

$$g_1(x) = \sum_{14\leqslant n\leqslant x} \frac{\mu(n)}{n} = g(x) - \sum_{n\leqslant 13} \frac{\mu(n)}{n},$$

is always positive, or, what is the same thing, that $g(x)$ has its minimum at $x = 13$.

We note that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 \tag{20}$$

(see e.g. R. Ayoub [1], page 114) and observe that

$$\sum_{n\leqslant 13} \frac{\mu(n)}{n} = -0.0773559\ldots \tag{21}$$

To show that

$$\left|\sum_{n\leqslant x} \frac{\mu(n)}{n}\right| < 0.07 \quad \text{for} \quad 200 < n \leqslant 10^8$$

we can use $|M(x)| < \frac{1}{2}\sqrt{x}$. Then, for $900 \leqslant x \leqslant 10^8$ we have

$$\sum_{d\leqslant x} \frac{\mu(d)}{d} = \sum_{d\leqslant 900} \frac{\mu(d)}{d} + \sum_{900<d\leqslant x-1} \frac{M(d)}{d(d+1)} - \frac{M(900)}{901} + \frac{M(x)}{x}.$$

Since

$$\sum_{d\leqslant 900}\frac{\mu(d)}{d} = 0.00328\ldots \quad \text{and} \quad M(900)=1$$

we obtain

$$(22)\qquad \left|\sum_{d\leqslant x}\frac{\mu(d)}{d}\right| < 0.00217\ldots+\frac{1}{2}\sum_{900<d\leqslant x-1}\frac{1}{d^{3/2}}+\frac{1}{2}\cdot\frac{1}{x^{1/2}} < 0.036,$$

$$\text{for } 900\leqslant x\leqslant 10^8.$$

One readily checks $g(x)$ for $1\leqslant x\leqslant 900$, and we have that, for $1\leqslant x\leqslant 10^8$, g assumes its minimum only at $x=13$. So it remains only to check for $x>10^8$.

Since

$$\sum_{d\leqslant x}\mu(d)\left[\frac{x}{d}\right]=1,$$

we have

$$(23)\qquad \sum_{d\leqslant x}\frac{\mu(d)}{d}=\frac{1}{x}+\frac{1}{x}\sum_{d\leqslant x}\mu(d)\left\{\frac{x}{d}\right\}.$$

Now,

$$\sum_{d\leqslant x}\mu(d)\left\{\frac{x}{d}\right\}=\sum_{x/2<d\leqslant x}\mu(d)\left(\frac{x}{d}-1\right)+\sum_{x/3<d\leqslant x/2}\mu(d)\left(\frac{x}{d}-2\right)+\ldots+$$

$$+\sum_{x/k<d\leqslant x/(k-1)}\mu(d)\left(\frac{x}{d}-(k-1)\right)+\sum_{d\leqslant x/k}\mu(d)\left\{\frac{x}{d}\right\}$$

$$=x\sum_{x/k<d\leqslant x}\frac{\mu(d)}{d}-\sum_{1\leqslant t\leqslant k-1}M\left(\frac{x}{t}\right)+(k-1)M\left(\frac{x}{k}\right)+\sum_{d\leqslant x/k}\mu(d)\left\{\frac{x}{d}\right\}.$$

Therefore, using (23), it follows that

$$\sum_{d\leqslant x/k}\frac{\mu(d)}{d}=\frac{1}{x}-\frac{1}{x}\sum_{1\leqslant t\leqslant k-1}M\left(\frac{x}{t}\right)+\frac{k-1}{x}M\left(\frac{x}{k}\right)+\frac{1}{x}\sum_{d\leqslant x/k}\mu(d)\left\{\frac{x}{d}\right\},\quad x\geqslant k.$$

Hence,

$$(24)\qquad \sum_{d\leqslant x}\frac{\mu(d)}{d}$$

$$=\frac{1}{kx}-\frac{1}{kx}\sum_{1\leqslant t\leqslant k-1}M\left(k\frac{x}{t}\right)+\frac{k-1}{kx}M(x)+\frac{1}{kx}\sum_{d\leqslant x}\mu(d)\left\{\frac{kx}{d}\right\},\quad \text{for}\quad x\geqslant 1.$$

Using (4) in (24), we obtain

$$\left|\sum_{d\leqslant x}\frac{\mu(d)}{d}\right|\leqslant\frac{1}{kx}+\frac{1}{80}\sum_{1\leqslant t\leqslant k-1}\frac{1}{t}+\frac{1}{80}\cdot\frac{k-1}{k}+\frac{1}{kx}\left|\sum_{d\leqslant x}\mu(d)\left\{\frac{kx}{d}\right\}\right|.$$

Let

$$A(x)=\sum_{\substack{d\leqslant x\\ \mu(d)=1}}1,\quad B(x)=\sum_{\substack{d\leqslant x\\ \mu(d)=-1}}1,\quad \text{and}\quad C(x)=\max\big(A(x),B(x)\big).$$

Then clearly

$$\left|\sum_{d\leqslant x}\mu(d)\{g(x,d)\}\right|\leqslant C(x)\quad \text{for any } g.$$

Now

$$C(x)=\tfrac{1}{2}\big(A(x)+B(x)\big)+\tfrac{1}{2}|A(x)-B(x)|=\tfrac{1}{2}Q(x)+\tfrac{1}{2}M(x).$$

Thus

$$(25)\qquad \left|\sum_{d\leqslant x}\mu(d)\{g(x,d)\}\right|\leqslant\tfrac{1}{2}Q(x)+\tfrac{1}{2}M(x).$$

From (25), using (8), we have

$$\left|\sum_{d\leqslant x}\frac{\mu(d)}{d}\right|\leqslant\frac{1}{kx}+\frac{1}{80}\sum_{1\leqslant t\leqslant k-1}\frac{1}{t}+\frac{1}{80}\cdot\frac{k-1}{k}+\frac{0.305}{k}+\frac{1}{160k},$$

$$\text{for}\quad x>60000.$$

Choosing k to be 20, we have

$$(26)\qquad \left|\sum_{d\leqslant x}\frac{\mu(d)}{d}\right|\leqslant 0.073,\quad \text{for}\quad x>60000.$$

This suffices to complete the proof.

We have shown that, if

$$g(x)=\sum_{n\leqslant x}\frac{\mu(n)}{n},$$

then $g(x)$ assumes its minimum at $x=13$. If we define $g_r(x)$ by

$$g_r(x)=\sum_{n\leqslant x}\frac{\mu(n)}{n^r},$$

then it is rather easy to show that, at least for integer $r\geqslant 2$, $g_r(x)$ assumes its minimum at $x=5$. For

$$\sum_{n\leqslant x}\frac{\mu(n)}{n^r}=1-\frac{1}{2^r}-\frac{1}{3^r}-\frac{1}{5^r}+\frac{1}{6^r}-\frac{1}{7^r}+\frac{1}{10^r}-\frac{1}{11^r}-\frac{1}{13^r}+\frac{1}{14^r}+\ldots$$

It is easy to see that the minimum cannot occur for $5 < x < 13$. For $r \geqslant 4$, we shall show that

$$\frac{1}{6^r}+\frac{1}{10^r} > \frac{1}{7^r}+\sum_{d=11}^{\infty}\frac{1}{d^r}, \tag{27}$$

so that the sum beyond $x = 5$ is always positive, and hence the minimum occurs at $x = 5$.

Since

$$\sum_{d=11}^{\infty}\frac{1}{d^r} < \int_{10}^{\infty}\frac{1}{u^r}\,du = \frac{10}{r-1}\cdot\frac{1}{10^r},$$

we have

$$\frac{1}{7^r}-\frac{1}{10^r}+\sum_{d=11}^{\infty}\frac{1}{d^r} < \frac{11-r}{r-1}\cdot\frac{1}{10^r}+\frac{1}{7^r} \leqslant \frac{2\frac{1}{3}}{10^r}+\frac{1}{7^r} = \frac{2\frac{1}{3}}{7^r}\left(\frac{7}{10}\right)^r+\frac{1}{7^r}$$

$$\leqslant \frac{2\frac{1}{3}}{7^r}\left(\frac{7}{10}\right)^4+\frac{1}{7^r} < \frac{1.6}{7^r} = \frac{1.6}{6^r}\left(\frac{6}{7}\right)^r \leqslant \frac{1.6}{6^r}\left(\frac{6}{7}\right)^4 < \frac{1}{6^r}.$$

So the minimum occurs at $x = 5$ for all $r \geqslant 4$ (not just integer r). One can use (4) to obtain

$$\left|\sum_{d\leqslant x}\frac{\mu(d)}{d^r}-\frac{1}{\zeta(r)}\right| \leqslant \frac{1}{80}\left(2+\frac{1}{r-1}\right)\frac{1}{x^{r-1}}, \tag{28}$$

for $x \geqslant 1119$ and $r > 1$.

Using this to examine $r = 2$ and $r = 3$ we again find that the minimum occurs at $x = 5$. Indeed, it seems that there is an r_0 between 1 and 2, namely the solution of

$$\frac{1}{6^r}+\frac{1}{10^r} = \frac{1}{7^r}+\frac{1}{11^r}+\frac{1}{13^r},$$

such that, for $1 \leqslant r < r_0$ the minimum occurs at $x = 13$, for r_0 there are twin minima at $x = 13$ and $x = 5$, and for $r > r_0$ the minimum occurs at $x = 5$.

References

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