A new estimate for the sum \( M(x) = \sum_{n \leq x} \mu(n) \)

by

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1. Introduction. Let \( \mu(n) \) be the ordinary Möbius function, so that if \( n = 1 \), \( \mu(n) = 1 \), and if \( n = p_1^{a_1} \cdots p_k^{a_k} \), then

\[
\mu(n) = \begin{cases} 
0, & \text{if any } a_i > 1, 1 \leq i \leq k, \\
(-1)^k, & \text{otherwise,}
\end{cases}
\]

and define \( M(x) \) by

\[
M(x) = \sum_{n \leq x} \mu(n).
\]

R. D. von Sterneck [6] showed that

\[
|M(x)| \leq \frac{1}{8} x + 8
\]

and R. Hackel [2] improved this to

\[
|M(x)| \leq \frac{1}{32} x + 155 \quad \text{for all } x.
\]

Our object here will be to obtain the result

\[
|M(x)| \leq \frac{x}{30} \quad \text{for } x \geq 1119
\]

(if we wished a result valid for all \( x \), we could say

\[
|M(x)| \leq \frac{x}{80} + 5, \quad \text{for all } x.
\]

We observe that it is well known (see e.g. R. Ayoub [1], p. 111) that the result \( M(x) = o(x) \) is equivalent to the prime number theorem.

G. Neubauer [4] has shown that

\[
|M(x)| \leq \frac{1}{30} \sqrt{x} \quad \text{for } 200 < x \leq 10^8
\]

(and that this fails to be true for a number of larger \( x \)). Using this we can prove (4) for \( 1119 < x \leq 10^8 \). For since \( \frac{1}{30} \sqrt{x} < x/80 \) for \( x > 1600 \), (4) follows for \( 1600 < x \leq 10^8 \), and one obtains by simple checking that (4) also holds for \( 1119 < x \leq 1600 \); thus, (4) remains to be verified for \( x > 10^8 \).
2. The method. We outline the method to be used, which is a refinement of that of von Sterneck. Consider the function

$$f(x) = [x] - \left[ \frac{x}{2} \right] - \left[ \frac{x}{3} \right] - \left[ \frac{x}{5} \right] + \left[ \frac{x}{30} \right].$$

Since for $x \geq 1$

$$\sum_{d \leq x} \mu(d) \left[ \frac{x}{md} \right] = 1,$$

we have

$$\sum_{d \leq x} \mu(d) \left[ \frac{x}{md} \right] + \sum_{\gamma_m < x \leq c} \mu(d) \left[ \frac{x}{md} \right] = 1 + 0 = 1,$$

for $x \geq m$, and thus

$$\sum_{d \leq x} \mu(d) \left[ \frac{x}{md} \right] = 1,$$

for $x \geq 30$.

Since $f(x) = 1$ for $1 \leq x < 6$ and 0 or 1 for $x \geq 6$, we have $f(x/d) = 1$ for $d > x/6$, so that

$$\sum_{d \leq x} \mu(d) \left( 1 - \left[ \frac{x}{d} \right] \right) \leq \sum_{d \leq x} \mid \mu(d) \mid = Q\left( \frac{x}{6} \right).$$

Thus,

$$|M(x) + 1| \leq Q\left( \frac{x}{6} \right),$$

for $x \geq 30$.

Moser and the author observed in [3] that

$$Q(x) - \frac{x}{\pi} x < \frac{1}{2} \sqrt{x},$$

for $x \geq 8$, which follows readily using the methods therein and the result

$$|M(x)| \leq \frac{1}{2} x$$

for $x > 200$,

a combination of the Hacke and Neubauer results. It follows that

$$0.606x < Q(x) < 0.615x,$$

for $x > 5000$,

and one readily checks that (10) holds for $x \geq 475$. Similarly,

$$Q(x) < 0.635x,$$

for $x \geq 75$.

Using (10) in (7) we obtain

$$|M(x)| + 1 \leq 0.103x,$$

for $x \geq 2950$.

and, by (6), for $x > 200$. If we further observe that $f(x) = 1$ for $7 \leq x \leq 10$, we have

$$|M(x)| + 1 \leq Q\left( \frac{x}{6} \right) - Q\left( \frac{x}{7} \right) + Q\left( \frac{x}{10} \right) \leq 0.079x,$$

for $x > 200$.

Using the function

$$f_1(x) = \left[ x - \frac{x}{2} \right] - \left[ \frac{x}{3} \right] - \left[ \frac{x}{5} \right] + \left[ \frac{x}{15} \right] - \left[ \frac{x}{30} \right],$$

and similar refinements to that used in deriving (13), one can obtain

$$|M(x)| + 2 \leq 0.04x,$$

for $x > 200$,

which is the same as (9). It seems difficult to get a fairly simple function like $f_1(x)$ which will substantially improve (14).

If we examine the characteristics of a "good" function $f$, we see that what we would like is a function which takes the value 1 for $1 \leq x \leq n$ for fairly large $n$, and then does not differ too widely from 1 thereafter. We shall employ the techniques of E. Waage ([7] and [8]) to obtain such a function.

3. Main result. In line with Waage, we define

$$w_k(x) = \left[ \frac{x}{k} \right] - \left[ \frac{x}{k+1} \right] - \left[ \frac{x}{k(k+1)} \right],$$

and use the symbol

$$\left( n_1, n_2, \ldots, n_m, l_1, l_2, \ldots, l_l \right)$$

to stand for the function

$$\left[ \frac{x}{n_1} \right] + \left[ \frac{x}{n_2} \right] + \cdots + \left[ \frac{x}{n_m} \right] - \left[ \frac{x}{l_1} \right] - \left[ \frac{x}{l_2} \right] - \cdots - \left[ \frac{x}{l_l} \right].$$

Let

$$U_4(x) = \left[ \left[ \left[ \left[ x \right] \right] \right] \right] = (1; 2, 2),$$

$$U_4(x) = u_4(x) + w_4(x) = (1; 2, 3, 6),$$

$$U_4(x) = U_4(x) - u_4(x) = (1; 30; 2, 3, 5),$$

(this is our $f(x)$),

$$U_4(x) = U_4(x) + u_4(x) = (1, 6, 30; 2, 3, 5, 7, 42).$$
Define $U(x), U_f(x), u(x)$, and $u'(x)$, respectively, as follows:

$U(x) = U_f(x) - u(x)$

$U_f(x) = U_f(x) + u(x)$

$u(x) = U_f(x) + u(x)$

$u'(x) = u(x) + u(x)$

$u''(x) = u(x) + u(x)$

Let $R_1, R_2, R_3$ and $R_4$ be respectively the sets

$\{1, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$

$\{18, 70, 90, 99, 114, 134, 142, 146, 163, 177, 183, 213\}$

$\{113, 131, 139, 154, 170, 173, 191\}$

Then the function $e(x)$ which we shall use is defined by the formula

$e(x) = u(x) - \sum_{\Delta x} U(p) \left(\frac{x}{\Delta x}\right) + u'(x) \left(\frac{x}{\Delta x}\right) + u''(x) \left(\frac{x}{\Delta x}\right) + u(x) \left(\frac{x}{\Delta x}\right) + U_f(x) \left(\frac{x}{42}\right)$

$- u(x) \left(\frac{x}{70}\right) - U_f(x) \left(\frac{x}{101}\right) - U_f(x) \left(\frac{x}{137}\right) - U_f(x) \left(\frac{x}{163}\right) - U_f(x) \left(\frac{x}{167}\right)$

$+ 2U_f(x) \left(\frac{x}{30}\right) + \sum_{\Delta x} u(x) \left(\frac{x}{\Delta x}\right) - \sum_{\Delta x} u(x) \left(\frac{x}{\Delta x}\right) - u(x) \left(\frac{x}{1400}\right)$

$+ u(x) \left(\frac{x}{18}\right) + u(x) \left(\frac{x}{30}\right) + u(x) \left(\frac{x}{30}\right) + u(x) \left(\frac{x}{30}\right) - u(x) \left(\frac{x}{30}\right)$

$+ 6u(x) \left(\frac{x}{60}\right) - 3u(x) \left(\frac{x}{115}\right) + u(x) \left(\frac{x}{21}\right) - 3u(x) \left(\frac{x}{46}\right) + 3u(x) \left(\frac{x}{92}\right)$

$- 3u(x) \left(\frac{x}{103}\right) - 2u(x) \left(\frac{x}{20}\right) + u(x) \left(\frac{x}{23}\right) - 2u(x) \left(\frac{x}{23}\right) - u(x) \left(\frac{x}{14}\right) - 2u(x) \left(\frac{x}{10}\right)$

New estimate for the sum $M(x) = \sum_{\Delta x} u(x)$

$= \frac{u(x)}{12} + \frac{u(x)}{11} + 3u(x) - \frac{u(x)}{2} + u(x) - \frac{u(x)}{10} - u(x) - 3u(x) + u(x) - \frac{u(x)}{10}$

$+ \frac{u(x)}{3} + \frac{u(x)}{5} - \frac{u(x)}{10} - \frac{u(x)}{2} - 2u(x) - \frac{u(x)}{10}$

$- \frac{u(x)}{5} + u(x) + 3u(x) - \frac{u(x)}{10} - \frac{u(x)}{10} - u(x) - 3u(x) + u(x) - \frac{u(x)}{10}$

$+ u(x) + u(x) + u(x) - \frac{u(x)}{2} - \frac{u(x)}{2} - \frac{u(x)}{2}$

$- \frac{u(x)}{3} - \frac{u(x)}{2} - \frac{u(x)}{2} - \frac{u(x)}{2} - \frac{u(x)}{2}$

$- u(x) - u(x) + u(x) + u(x) - u(x) + u(x) - u(x) + u(x) + u(x)$

$= \sum_{\Delta x} u(x) + \sum_{\Delta x} u(x) - \sum_{\Delta x} u(x)$

where

and


This rather complicated function was obtained by successively evaluating simpler functions by computer to see where they began to differ too much from 1, and adding in compensating simple functions to reduce the rate of growth.

Since there are 222 positive terms and 226 negative terms in \( e \),

\[ |e(x)| < 226 \quad \text{for all } x, \]

for which we remove the square brackets in \( e \) the function is identically zero by construction. Upon examining \( e(x) \), we find that

\[ e(x) = 1, \quad \text{for } 1 \leq x < 219 \]

and

\[ |e(x)| - 1 < k \quad \text{for } x < n, \]

where \( k \) and \( n \) are as given in the following table:

| \( n \) | 345 568 584 604 1237 1359 1381 1393 1416 1416 1417 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( k \) | 1 2 3 4 5 6 7 8 9 10 11 |
| 12 13 14 15 16 17 18 19 20 |
| 501 501 588 588 1697 16100 16100 16103 2674 |
| 21 22 23 24 25 26 27 28 29 |
| 26750 26752 26754 26759 31397 46110 46110 46112 63611 |
| 30 31 32 33 34 35 36 37 38 |
| 67158 67159 67189 69258 69259 69263 82800 82800 82800 |
| 39 40 41 42 43 44 45 |
| 85869 85872 85747 97006 97007 106591 up to 125000 |

New estimate for the sum \( M(x) = \sum_{n \leq x} \mu(n) \)

Let \( N \) be the set of \( n \)'s in the above table. It follows that

\[ |M(x) + 4| \leq Q \left( \frac{x}{219} \right) + \sum_{n \in N} Q \left( \frac{x}{n} \right) + (226 - 45)Q \left( \frac{x}{12500} \right). \]

Using (8) and (10) in (19) we obtain

\[ |M(x) + 4| \leq 0.01247x, \quad \text{for } x > 10^6 \]

or

\[ |M(x)| < 0.01x, \quad \text{for } x > 10^6. \]

This completes the proof of (4).

It is likely that with a better function \( e(x) \) one could prove

\[ |M(x)| < 0.1x, \quad \text{for } x > 1137. \]

This is certainly true for \( 1137 < x \leq 10^6 \).

4. Application. S. Selberg [5] has shown that, if \( g(x) \) is defined by

\[ g(x) = \sum_{n \leq x} \frac{\mu(n)}{n}, \]

then \( g(x) \) changes sign infinitely often. We show here that, on the other hand, \( g_1(x) \), defined by

\[ g_1(x) = \sum_{n \leq x} \frac{\mu(n)}{n} = g(x) - \sum_{n < 10^6} \frac{\mu(n)}{n}, \]

is always positive, or, what is the same thing, that \( g(x) \) has its minimum at \( x = 13 \).

We note that

\[ \sum_{n < 10^6} \frac{\mu(n)}{n} = 0 \]

(see e.g. R. Ayoub [1], page 114) and observe that

\[ \sum_{n < 10^6} \frac{\mu(n)}{n} = -0.073559\ldots \]

To show that

\[ \sum_{n < 10^6} \frac{\mu(n)}{n} < 0.07 \quad \text{for } 200 < n \leq 10^6 \]

we can use \( |M(x)| < \frac{1}{44}x \). Then, for \( 900 < x \leq 10^6 \) we have

\[ \sum_{d \leq x} \frac{\mu(d)}{d} = \sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d} + \sum_{d \leq 900} \frac{M(d)}{d(d+1)} - \frac{M(900)}{901} < \frac{M(x)}{x}. \]
Since
\[ \sum_{d \mid n} \frac{\mu(d)}{d} = 0.00338 \ldots \quad \text{and} \quad M(900) = 1 \]
we obtain
\[ (22) \quad \left| \sum_{d \mid n} \frac{\mu(d)}{d} \right| < 0.00217 \ldots + \frac{1}{2} \sum_{\nu \mid n, \nu \neq d} \frac{1}{\nu^2} + \frac{1}{2} \frac{1}{n^2} < 0.036, \]
for \( 900 < n \leq 10^6 \).

One readily checks \( g(x) \) for \( 1 \leq x < 900 \), and we have that, for \( 1 \leq x \leq 10^6 \),
\( g \) assumes its minimum only at \( x = 13 \). So it remains only to check for \( x > 10^6 \).

Since
\[ \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor = 1, \]
we have
\[ (23) \quad \sum_{d \mid n} \frac{\mu(d)}{d} = \frac{1}{x} + \frac{1}{x} \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor. \]

Now,
\[ \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor = \sum_{d \mid n} \mu(d) \left( \left\lfloor \frac{x}{d} \right\rfloor - 1 \right) + \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor + \ldots + \]
\[ + \sum_{d \mid n, \nu \mid n, \nu \neq d} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor - (k-1) \right) + \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor \]
\[ = x - \sum_{d \mid n, \nu \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor - \sum_{d \mid n, \nu \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor + \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor \]
\[ = x - \sum_{d \mid n, \nu \mid n} \mu(d) \left( \left\lfloor \frac{x}{d} \right\rfloor + \left(\nu - 1\right) \right) + \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor \]
\[ = x - \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor - \sum_{d \mid n \text{ odd}} M \left( \left\lfloor \frac{x}{d} \right\rfloor + \left(\nu - 1\right) \right) + \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor \]
\[ = x - \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor - \sum_{d \mid n, \nu \mid n, \nu \neq d} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor. \]

Therefore, using (23), it follows that
\[ \sum_{d \mid n} \frac{\mu(d)}{d} = \frac{1}{x} + \frac{1}{x} \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor + \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor, \quad x \geq k. \]

Hence,
\[ (24) \quad \sum_{d \mid n} \frac{\mu(d)}{d} \]
\[ = \frac{1}{kx} - \frac{1}{kx} \sum_{d \mid n, \nu \mid n, \nu \neq d} \sum_{d \mid n, \nu \mid n, \nu \neq d} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor, \quad \text{for} \quad x \geq 1. \]

Using (4) in (24), we obtain
\[ \left| \sum_{d \mid n} \frac{\mu(d)}{d} \right| \leq \frac{1}{kx} + \frac{1}{80} \sum_{l \mid n, l \neq d} \frac{1}{l} + \frac{k-1}{k} \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor. \]

Let
\[ A(x) = \sum_{d \mid n} 1, \quad B(x) = \sum_{d \mid n} 1, \quad \text{and} \quad C(x) = \max(A(x), B(x)). \]

Then clearly
\[ \left| \sum_{d \mid n} \mu(d) \left\lfloor g(x, d) \right\rfloor \right| \leq O(x) \quad \text{for any} \quad g. \]

Now
\[ C(x) = \frac{1}{2} A(x) + B(x) + A(x) B(x) = \frac{1}{2} Q(x) + \frac{1}{2} M(x). \]

Thus
\[ (25) \quad \left| \sum_{d \mid n} \mu(d) \left\lfloor g(x, d) \right\rfloor \right| \leq \frac{1}{4} Q(x) + \frac{1}{2} M(x). \]

From (25), using (8), we have
\[ \left| \sum_{d \mid n} \frac{\mu(d)}{d} \right| \leq \frac{1}{kx} + \frac{1}{80} \sum_{l \mid n, l \neq d} \frac{1}{l} + \frac{k-1}{k} \sum_{d \mid n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor + \frac{0.305}{k} + \frac{1}{1600}. \]

Choosing \( k \) to be 20, we have
\[ (26) \quad \left| \sum_{d \mid n} \frac{\mu(d)}{d} \right| \leq 0.073, \quad \text{for} \quad x > 60000. \]

This suffices to complete the proof.

We have shown that, if
\[ g(x) = \sum_{n \leq x} \frac{\mu(n)}{n}, \]
then \( g(x) \) assumes its minimum at \( x = 13 \). If we define \( g_r(x) \) by
\[ g_r(x) = \sum_{n \leq x} \frac{\mu(n)}{n}, \]
then it is rather easy to show that, at least for integer \( r \geq 2 \), \( g_r(x) \) assumes its minimum at \( x = 5 \). For
\[ \sum_{n \leq x} \frac{\mu(n)}{n} = 1 - \frac{1}{2^r} - \frac{1}{3^r} - \frac{1}{6^r} - \frac{1}{7^r} - \frac{1}{10^r} - \frac{1}{11^r} - \frac{1}{13^r} + \frac{1}{17^r} + \ldots \]
It is easy to see that the minimum cannot occur for $5 < x < 13$. For $r \geq 4$, we shall show that

$$\frac{1}{6} + \frac{1}{10^r} > \frac{1}{7} + \sum_{d \equiv 1}^{\infty} \frac{1}{d^r},$$

so that the sum beyond $x = 5$ is always positive, and hence the minimum occurs at $x = 5$.

Since

$$\sum_{d \equiv 1}^{\infty} \frac{1}{d^r} \leq \int_{\infty}^{\infty} \frac{1}{u^r} du = \frac{10}{r-1} \cdot \frac{1}{10^r},$$

we have

$$\frac{1}{7} - \frac{1}{10^r} + \sum_{d \equiv 1}^{\infty} \frac{1}{d^r} \leq \frac{11-r}{r-1} \cdot \frac{1}{10^r} + \frac{1}{7^r} \leq \frac{2}{3} \cdot \frac{10^r}{10^r} + \frac{1}{7^r} = \frac{2}{3} \cdot \left(\frac{7}{10}\right)^r + \frac{1}{7^r}$$

$$\leq \frac{2}{3} \cdot \frac{10^r}{7^r} + \frac{1}{7^r} = \frac{1.6}{7^r} < \frac{1.6}{6^r} \leq \frac{6^r}{6^r} \leq \frac{6^r}{7^r} < \frac{1}{6^r}.$$  

So the minimum occurs at $x = 5$ for all $r \geq 4$ (not just integer $r$). One can use (4) to obtain

$$\left| \sum_{d \equiv 1}^{\infty} \frac{\mu(d)}{d^r} - \frac{1}{\zeta(r)} \right| \leq \frac{1}{86} \left( \frac{1}{2} + \frac{1}{r-1} \right) \frac{1}{3^{r-1}}$$

for $x \geq 1119$ and $r > 1$.

Using this to examine $r = 2$ and $r = 3$ we again find that the minimum occurs at $x = 5$. Indeed, it seems that there is an $r_0$ between 1 and 2, namely the solution of

$$\frac{1}{6^r} + \frac{1}{10^r} = \frac{1}{7^r} + \frac{1}{11^r} + \frac{1}{13^r},$$

such that, for $1 \leq r < r_0$ the minimum occurs at $x = 13$, for $r_0$ there are twin minima at $x = 13$ and $x = 5$, and for $r > r_0$ the minimum occurs at $x = 5$.

References


[4] B. Selberg, *Über die Sumsse $\sum_{n \leq x} \frac{\mu(n)}{n}$*, Des Kongelige Norske Videnskaps Selskabs Forhandlinger (Oslo), 28 (1955), pp. 37-41.

