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A covering class of residues with odd moduli

by

JAMES H. JORDAN (Pullman, Washington)

1. Introduction. In 1952 P. Erdös [2] introduced the following concept:

DEFINITION. A finite set of ordered pairs of integers $\{(a_i, m_i)\}$ with all m_i distinct and larger than 1 is called a *covering* class of residues if each integer, n, satisfies at least one of the congruential equation $n \equiv a_i \pmod{m_i}$.

The problems posed by Erdös are:

QUESTION 1. *Does there exist a covering class of residues with all $m_i > n$ for each positive integer n?*

and,

QUESTION 2. *Does there exist a covering class of residues with all m_i odd?*

Erdös [3] offered a reward of \$ 50 for a proof of the answer to question 1 and a \$ 25 reward for the proof of a negative answer to question 2. J. Selfridge [5] offered a \$ 250 reward for a positive answer to question 2 and the example.

Erdös exhibited [1] a covering class of residues with all $m_i > 2$. J. D. Swift [1] and the author [4] exhibited covering classes of residues with all $m_i > 3$. J. Selfridge [5] announced that he has a covering class of residues with all $m_i > 7$.

Recently the author [4] generalized these concepts with the following:

DEFINITION. A finite set of ordered pairs of Gaussian Integers $\{(\alpha_j, \gamma_j)\}$ with all $|\gamma_j| > 1$ and $\gamma_k \neq \gamma_j \varepsilon$, $k \neq j$, where $\varepsilon = \pm 1$ or $\pm i$, is called a *covering class of residues in G* if every Gaussian Integer β satisfies at least one of the congruential equations

$$\beta \equiv \alpha_j \pmod{\gamma_j}.$$

The analogous problems were posed

QUESTION 3. *Does there exist a covering class of residues in G with all $|\gamma_j| > \sqrt{n}$ for each positive integer n?*

and

QUESTION 4. *Does there exist a covering class of residues in G with all $N(\gamma_j)$ odd?*

The author has offered rewards identical with those offered by Erdös and Selfridge for proofs of the answers to Questions 3 and 4.

Unfortunately this article does not answer any of the four questions but does answer an analougous question in the algebraic integers of the form $a+b\sqrt{2i}$ with a and b real integers.

Let us adopt the notation that small case Greek letters will denote element of $Z(\sqrt{-2})$ while small case Latin letters will denote real integers.

Consider the following:

DEFINITION. A finite set of ordered pairs of elements of $Z(\sqrt{-2})$, $\{(\alpha_j, \gamma_j)\}$ with all $|\gamma_j| > 1$ and $\gamma_k \neq \gamma_j \varepsilon$, where $\varepsilon = \pm 1$, is called a *covering class of residues in* $Z(\sqrt{-2})$ if each element β of $Z(\sqrt{-2})$ satisfies at least one of the congruential equations

$$\beta \equiv \alpha_j \ (\mathrm{mod}\ \gamma_j).$$

Analougous questions in this system would be:

QUESTION 5. *Does there exist a covering class of residues in* $Z(\sqrt{-2})$ *with all* $N(\gamma_j) = \gamma_j\bar{\gamma}_j = a^2+2b^2 > n$ *for every positive integer* n?

and

QUESTION 6. *Does there exist a covering class of residues in* $Z(\sqrt{-2})$ *with all* $N(\gamma_j)$ *odd?*

It is the purpose of this paper to give a positive answer to question 6 by exhibiting a covering class of residues in $Z(\sqrt{-2})$ with all norms of the moduli odd.

2. The example. Let us adopt the following notation:

(i) $\alpha = 1+\sqrt{2i}$; $\quad \beta = 1-\sqrt{2i}$; $\quad \gamma = 3+\sqrt{2i}$;

(ii) $n^* = \frac{1}{2}(3^{n-1}-1)$; $\quad m' = \frac{1}{2}(5\cdot 3^{m-1}-1)$;

(iii) For $\{\varrho_j\}_1^s$ pairwise relatively prime let $\langle(\delta_1, \varrho_1); (\delta_2, \varrho_2); \ldots; (\delta_s, \varrho_s)\rangle$ denote (δ, ϱ) where δ is the unique simultaneous solution of $\{\eta \equiv \delta_j \ (\mathrm{mod}\ \varrho_j)\}_1^s$ and $\varrho = \prod_1^s \varrho_j$.

(iv) Let

$$A = \{(n^*, \alpha^n)\}_1^6;$$
$$B = \{(n^*, \beta^n)\}_1^6;$$
$$C = \{\langle(n', \alpha^n); (m', \beta^m)\rangle\}_{1,1}^{6,6};$$

$$D = \{(0, \gamma)\};$$
$$E = \{\langle(n, \gamma); ((n+1)^*, \alpha^n)\rangle\}_1^5 = \{e_n\}_1^5;$$
$$F = \{\langle(n+5, \gamma); ((n+1)^*, \beta^n)\rangle\}_1^5 = \{f_n\}_1^5;$$
$$G = \{\langle(n-1, \gamma); (m', \alpha^m); ((n+1)^*, \beta^n)\rangle: 1 \leqslant m < n \leqslant 6\} = \{g_{m,n}\};$$
$$H = \{\langle(n+4, \gamma); (m', \beta^m); ((n+1)^*, \alpha^n)\rangle: 1 \leqslant m < n \leqslant 6\} = \{h_{m,n}\}.$$

Now we are ready for the

THEOREM. *The set* $T = \bigcup_A^H X$ *is a covering class of residues in* $Z(\sqrt{-2})$.

Proof. It suffices to show that a complete residue system modulo the least common multiple of the 89 individual moduli is covered. The least common multiple is $\alpha^6\beta^6\gamma = 2187+729\sqrt{2i}$. A convenient complete residue system modulo $\alpha^6\beta^6\gamma$ on which to focus our attention is: $R = \{x+y\sqrt{2i}: 0 \leqslant x < 2187, 0 \leqslant y < 729\}$. This set contains 5,845, 851 elements. Rather than attempting to pick an arbitrary element and establish that it is covered, we will explain what effect each component of the cover has on this number of elements.

I. 2,918,916 elements satisfy exactly one of the congruences of component A.

II. 2,918,916 elements satisfy exactly one of the congruences of B. 1,457,456 of these also satisfy exactly one of the congruences of component A, the other 1,461,460 do not satisfy any of the congruences of component A.

III. 1,457,456 elements satisfy exactly one congruence of C and none of these satisfy any of the congruences of A or B.

At this stage only 8,019 elements fail to be covered by $A \cup B \cup C$. These can be accounted for as follows:

for $0 \leqslant a \leqslant 10$

(i) 3^{6-n} of these are covered by

$$t_{a,n} = \langle(a, \gamma); (7^*, \alpha^6); (n', \beta^n)\rangle, \quad n = 1, 2, 3, 4, 5, 6,$$

for each a for a total of 4,004.

and

(ii) 3^{6-n} of these are covered by

$$r_{a,n} = \langle(a, \gamma); (7^*, \beta^6); (n', \alpha^n)\rangle, \quad n = 1, 2, 3, 4, 5, 6,$$

for each a for a total of 4,004.

and

(iii) one of these satisfies

$$s_a = \langle(a, \gamma); (7^*, \alpha^6); (7^*, \beta^6)\rangle \quad \text{for each } a \text{ for a total of } 11.$$

Now any number that is congruent to 7^* (mod α^6) must also be congruent to $(n+1)^*$ (mod α^n) for $1 \leqslant n \leqslant 6$. It is also the case that any number congruent to $m'(\mathrm{mod}\, a^m)$ must also be congruent to $(n+1)^*\,(\mathrm{mod}\, a^n)$ for $1 \leqslant n < m$.

To complete the proof we will merely mention which r, t, and s are engulfed by which congruences of the D, E, F, G, and H components:

(1) D engulfs $s_0;\ t_{0,1};\ \ldots;\ t_{0,6};\ r_{0,1};\ \ldots;\ r_{0,6};$

(2) e_n engulfs $s_n;\ t_{n,1};\ \ldots;\ t_{n,6};\ r_{n,n+1}\ \ldots;\ r_{n,6},\qquad n = 1, 2, 3, 4, 5;$

(3) f_{n-5} engulfs $s_n;\ r_{n,1};\ \ldots\ r_{n,6};\ t_{n,n-4};\ \ldots;\ t_{n,6},\qquad n = 6, 7, 8, 9, 10,$

(4) $g_{m,n}$ engulfs $r_{n-1,m},\quad 1 \leqslant m < n \leqslant 6,$

(5) $h_{m,n}$ engulfs $t_{n+4,m},\quad 1 \leqslant m < n \leqslant 6.$

Notice that all cases are engulfed by these. Hence $\bigcup_A^H X$ is a covering class of residues in $Z(\sqrt{-2})$ and the norms being divisors of the odd integer 5,845,851 are necessarily odd.

We have not used the divisors $\alpha^j\beta^j\gamma$ for $j = 1, 2, 3, 4, 5, 6$ nor $\alpha^6\gamma$ or $\beta^6\gamma$ of $\alpha^6\beta^6\gamma$ as possible moduli so perhaps a simpler covering class of residues in $Z(\sqrt{-2})$ can be found.

3. Remarks. Although this does not shed any light on questions 2 and 4, the method of constructing the example may be able to be used in attempts at settling these questions.

The basic ingredient of $Z(\sqrt{-2})$ that allowed this example to be constructed was the existence of two primes whose norms were 3. These did an excellent job of covering a large portion of $Z(\sqrt{-2})$ and only the one additional prime $3+\sqrt{2i}$ was needed to complete the cover.

The method of this paper indicates that we can obtain infinitely many essentially different covers of $Z(\sqrt{-2})$ that have odd norms.

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Sur un théorème de Rényi. II

par

H. Delange (Paris)

1. Introduction. Soient $\omega(n)$ le nombre des diviseurs premiers de l'entier positif n et $\Omega(n)$ le nombre total des facteurs dans la décomposition de n en facteurs premiers.

A. Rényi a montré (1) que, pour chaque entier $q \geqslant 0$, l'ensemble des n pour lesquels on a $\Omega(n) - \omega(n) = q$ possède une densité d_q, la suite des nombres d_q étant déterminée par le fait que, pour $|z| < 2$,

$$\sum_{q=0}^{+\infty} d_q z^q = \frac{6}{\pi^2}\prod \frac{1-z/(p+1)}{1-z/p},$$

où p parcourt la suite des nombres premiers.

Dans un article précédent (2) de même titre que celui-ci, nous avons montré que le fait que la fonction $\zeta(s)$ de Riemann n'a aucun zéro de partie réelle 1 entraîne le résultat suivant, qui précise celui de Rényi:

Si $\nu_q(x)$ est le nombre des $n \leqslant x$ pour lesquels on a

$$\Omega(n) - \omega(n) = q \qquad (q \text{ entier } \geqslant 0),$$

on a pour x infini:

$$\nu_q(x) = d_q x + o[x^{1/2}(\log\log x)^q].$$

Nous nous proposons ici de montrer que ceci peut aussi s'établir élémentairement à partir du théorème des nombres premiers sous la forme $\pi(x) \sim x/\log x$, ou plus précisèment à partir du fait équivalent que la fonction de Möbius satisfait à

$$\sum_{n\leqslant x} \mu(n) = o[x]. \tag{1}$$

On peut même établir le théorème plus général suivant:

(1) *On the density of certain sequences of integers*, Publications de l'Institut de Mathématiques de l'Académie Serbe des Sciences 8 (1955), pp. 157-162.

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