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On sets characterizing number-theoretical functions

by

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1. A complex-valued function $f(n)$ $(n = 1, 2, \ldots)$ defined on the set of natural numbers is called *additive* if for all pairs m, n of relatively prime natural numbers,

$$f(nm) = f(n)+f(m). \tag{1}$$

An additive function $f(n)$ is called *totally additive* if (1) holds for all pairs m, n of natural numbers.

We use a terminology according to which a number-theoretic function $f(n)$ is said to *vanish on the set* $\mathscr{A}$ of natural numbers if $f(n) = 0$ for all n belonging to the set $\mathscr{A}$. We call a number-theoretic function $f(n)$ *singular* if $f(n) = 0$ for all natural numbers n.

It is evident that a totally additive number-theoretic function is uniquely determined by its values on prime numbers, because

$$f(p_1^{a_1}\ldots p_r^{a_r}) = \sum_{j=1}^{r} a_j f(p_j). \tag{2}$$

From relation (2) it also follows that if $\mathscr{P}$ is a set of prime numbers and we prescribe arbitrary values a_p for $p \in \mathscr{P}$, then there exists (at least one) totally additive number-theoretic function $f(n)$ such that $f(p) = a_p$ for $p \in \mathscr{P}$.

It is easy to prove that a set of natural numbers $\mathscr{A} = \{a_1, a_2, \ldots\}$ has the last property if and only if a_i, a_j are relatively prime for all $i \neq j$. Thus the structural survey of these sets is not difficult. But we cannot say this with regard to the first property.

DEFINITION 1. We call a set $\mathscr{A}$ of natural numbers a *set of uniqueness* (concerning totally additive functions) if the unique totally additive function which vanishes on $\mathscr{A}$ is a *singular* one.

It is easy to find an example, different from the prime numbers for a *set of uniqueness*. For example, if l, k $(0 < l < k)$ are fixed relatively prime integers, then the set $\mathscr{A}$ containing the prime divisors of k and the arithmetical progression $l+jk$ $(j = 0, 1, \ldots)$ is a *set of uniqueness*.

The proof is almost trivial. Another example is the union of the set of the primes p in the arithmetical progression $p \equiv -1\,(4)$ and of the set of the numbers n^2+1 $(n = 1, 2, \ldots)$.

It seems very difficult to decide whether the set $\mathscr{P}_1$, consisting of $p+1$ where p runs over all primes, is a set of uniqueness or not. Using simple numerical calculations we can prove that if $f(n)$ is a totally additive function which vanishes for all $p+1$, then $f(p) = 0$ for $p \leqslant 50$. See the following table:

$$0 = f(3+1) = 2f(2) = 0,$$
$$0 = f(5+1) = f(3)+f(2) = f(3) = 0,$$
$$0 = f(19+1) = f(4)+f(5) = f(5) = 0,$$
$$0 = f(13+1) = f(7)+f(2) = f(7) = 0,$$
$$0 = f(43+1) = f(4)+f(11) = f(11) = 0,$$
$$0 = f(103+1) = f(8)+f(13) = f(13) = 0,$$
$$0 = f(101+1) = f(6)+f(17) = f(17) = 0,$$
$$0 = f(37+1) = f(2)+f(19) = f(19) = 0,$$
$$0 = f(137+1) = f(6)+f(23) = f(23) = 0,$$
$$0 = f(173+1) = f(6)+f(29) = f(29) = 0,$$
$$0 = f(61+1) = f(2)+f(31) = f(31) = 0,$$
$$0 = f(73+1) = f(2)+f(37) = f(37) = 0,$$
$$0 = f(163+1) = f(4)+f(41) = f(41) = 0,$$
$$0 = f(171+1) = f(4)+f(43) = f(43) = 0,$$
$$0 = f(281+1) = f(6)+f(47) = f(47) = 0.$$

We formulate our problem as Hypothesis 1-5.

H_1. Hypothesis 1. *The set $\mathscr{P}_1$ is a set of uniqueness.*

H_1 would be a simple consequence of

H_2. Hypothesis 2. *For every prime q there exists a prime p such that*

$$p+1 = kq, \tag{3}$$

where k is a suitable integer no prime divisors of which are greater than q.

Assertion in H_1 follows from H_2 by induction. It is evident that we must prove that $f(q) = 0$ for every prime q. This assertion is true for $q = 2$. Now let $q > 2$ be a prime and suppose that $f(q') = 0$ for every prime q' which is smaller than q. Then using (3) in H_2 we have $f(k) = 0$ and so $0 = f(p+1) = f(k)+f(q) = f(q)$.

In the following we shall prove that H_2 is true for all sufficiently large q if all the non-trivial zeros of Dirichlet's L-functions are on the critical line. It seems very likely that the Riemann-Piltz conjecture implies H_2 for all q, but the proof of this assertion requires extensive numerical computations.

The following well-known problem H_3 is deeper than H_2.

Let $p(k, l)$ denote the least prime in the arithmetical progression $p \equiv l (\operatorname{mod} k)$.

H_3. Hypothesis 3. *$p(k, l) \leqslant k^2$ for every $(k, l) = 1$.*

The following conjecture H_4 is deeper than H_1.

H_4. Hypothesis 4. *If $f(n)$ is a real-valued totally additive number-theoretic function increasing monotonically on $\mathscr{P}_1$, i.e.*

$$f(p+1) \geqslant f(q+1) \quad \text{if} \quad p > q \tag{4}$$

for all pairs of primes p, q, then $f(n)$ is a constant multiple of $\log n$.

Indeed, supposing that H_1 is false, there exists a *non-singular* totally additive function $g(n)$ such that $g(n) = 0$ for all elements n of $\mathscr{P}_1$.

The assertion in H_4 follows from

H_5. Hypothesis 5. *For all pairs a, b of relatively prime natural numbers the equation*

$$ap - bq = 1 \tag{5}$$

can be solved in primes p, q.

Indeed, from H_5 it follows that for every natural n the equation $n(p+1) = (n+1)(q+1)$ is solvable in primes p, q. Then $p > q$, and we have $f(n) \leqslant f(n+1)$, $n = 1, 2, \ldots$ for the function $f(n)$ defined in H_4. Now by the theorem of P. Erdös [1], stating that if a number-theoretic function is additive and monotonic, then $f(n) = c\log n$, H_4 follows.

Let $\sigma(n)$ denote the sum of all positive divisors of n. If for every prime q we can find a solution of the equation

$$\sigma(n) = q\sigma(m)$$

in square-free numbers n, m, then H_1 follows.

But we are unable to prove even the easier

H_6. Hypothesis 6. *For every prime q there exist natural numbers n, m such that*

$$\sigma(n) = q\sigma(m).$$

Definition 2. We call a set $\mathscr{A}$ of natural numbers a *set of quasi-uniqueness* if there exists a suitable set $\mathscr{B}$ of natural numbers containing finitely many elements such that the union of $\mathscr{A}$ and $\mathscr{B}$ is a *set of uniqueness.*

2. Theorem 1. *If for every sufficiently large prime q Dirichlet's L-functions* $\bmod q$ *are non-vanishing on the halfplane* $\operatorname{res} > \frac{1}{2}$, *then the set* $\mathcal{P}_1 = \{p+1\}$ *is a set of quasi-uniqueness.*

For the proof we need the following well-known results, which we formulate as Lemmas 1 and 2.

Lemma 1. *Let $X \geqslant 2$ and suppose that all Dirichlet's L-functions* $\bmod q$ *are non-vanishing on the halfplane* $\operatorname{res} > \frac{1}{2}$. *Then for every l relatively prime to m we have*

$$\pi(x, m, l) = \frac{\operatorname{li} x}{\varphi(m)} + O(x^{1/2}\log x), \tag{6}$$

where the constant in O is an absolute one.

For the proof see Prachar's book [2], p. 251, Theorem 5.1.

Lemma 2. *For every even k, $2 \leqslant k < x$, the number of solutions of*

$$p \leqslant x, \quad p+1 = kq \tag{7}$$

in primes p, q does not exceed

$$c\frac{x}{\varphi(k)\log^2(x/k)},$$

where c is an absolute constant.

The proof of this lemma follows from a standard application of Selberg's sieve method. (See Prachar [2], p. 51, Theorem 4.6.)

From these lemmas we obtain Theorem 1 very easily. Namely we shall prove the following stronger

Theorem 2. *If the condition of Theorem 1 is satisfied, then H_2 is true for every sufficiently large q.*

Proof. Let δ and ε be sufficiently small positive constants and let q_0 be so large that

$$\pi(x, q, -1) > (1-\varepsilon)\frac{x/\log x}{q-1} \quad \text{for} \quad q \geqslant q_0,\ x = q^{2+\delta}. \tag{8}$$

The existence of q_0 follows from Lemma 1.

Thus the number of solutions of

$$p+1 = kq, \quad k \leqslant q^{1+\delta}, \tag{9}$$

for fixed q and varying p is greater than

$$(1-\varepsilon)\frac{x/\log x}{q-1}.$$

From this we deduce that there exists a solution of (9) for which k has prime divisors which are all smaller than q.

For fixed j and q let us denote by N_{jq} the number of solutions of

$$p+1 = jqq', \quad p \leqslant x, \quad q \leqslant q' \leqslant \frac{x}{q}$$

in primes p and q'. We have to prove only that

$$(1-\varepsilon)\frac{x/\log x}{q-1} > \sum_{j \leqslant q^\delta} N_{jq}.$$

From Lemma 2 it follows that

$$N_{jq} < c\frac{x}{\varphi(j)\varphi(q)\log^2(x/qj)}.$$

Thus using Lemma 2 we obtain

$$\sum_{j \leqslant q^\delta} N_{jq} < c\frac{x}{\varphi(q)\log^2(x/q^{1+\delta})}\sum_{j \leqslant q^\delta}\frac{1}{\varphi(j)} \leqslant c_2\frac{\delta x\log q}{(q-1)\log^2 q} = c_2\delta(2+\delta)\frac{x}{(q-1)\log x}$$

with a suitable absolute constant $c_2 > 0$. Now let δ be so small that

$$c_2\delta(2+\delta) < 1-\varepsilon. \tag{10}$$

Hence follows our assertion in Theorem 2.

Now let $\mathcal{B}$ be the set of all primes not exceeding $q_0(\delta, \varepsilon)$. Then the union of $\mathcal{A}$ and $\mathcal{B}$ is a set of uniqueness (see the deduction of H_1 from H_2) and Theorem 1 is proved.

We remark that for the proof of Theorem 1 we do not need the full strength of the conjecture of Riemann-Piltz.

Let $L(s, \chi_D)$ denote Dirichlet's functions $\bmod D$ and let $N(\sigma, T)$ denote all the zeros $\varrho = \beta + i\gamma$ of the function $h_D(s) = \prod_{\chi = \chi_D} L(s, \chi)$ in the rectangle $\beta \geqslant \sigma$, $|\gamma| \leqslant T$.

Then using the same arguments as those of Barban, Tshudakov and Linnik in [3] we obtain the following

Lemma 3. *Let $\vartheta = \{D\}$ be an infinite sequence of natural numbers and $\varepsilon > 0$ an arbitrary constant. Suppose that the following conditions* (α), (β) *are satisfied:*

(α) $N(\sigma, T) \leqslant b_1 T^A D^{B(1-\sigma)}\log^C D$ *for* $T \geqslant 1$, $D \in \vartheta$,

(β) $h_D(\sigma+it)$ *does not vanish in the rectangle*

$$\sigma > 1-\eta(D), \quad |\gamma| \leqslant \tau,$$

where

$$\eta(D) = b_2(\log D)^{-a},\ 0 < a < 1; \quad \tau = (\log D)^M,\ M > 0;$$

and b_1, b_2, a, M, A, C, B are constants, $B \geqslant 2$.

From these assumptions it follows that

$$\pi(x, D, l) = \frac{\operatorname{li} x}{\varphi(D)}\left(1+O\left((\log x)^{-M/2}\right)\right)$$

uniformly for $x \geqslant D^{B+\varepsilon}$, $(l, D) = 1$.

Using similar arguments as in the proof of Theorem 2 we obtain

THEOREM 3. *Supposing that the assumptions* (α), (β) *in Lemma* 3 *are satisfied by* $B = 2+\delta$, $\delta < (1+1/c_2)^{1/2}-1$ (see (10)) *for every sufficiently large prime modulus* q, *we find that for every sufficiently large* q *there exists a solution of the equation*

$$p+1 = kq$$

in prime p, *so that all the prime divisors of* k *are smaller than* q. *Hence it follows that* $\{p+1\}$ *is a set of quasi-uniqueness.*

Let $\mathscr{P}^{(3)}$ denote the set of all natural numbers containing at most three prime divisors. A. I. Vinogradov in [4] proved that every sufficiently large even number is a sum of two elements from $\mathscr{P}^{(3)}$. Using his ideas we can prove that the equation

$$aP_3 - bP_3' = 1; \quad P_3, P_3' \in \mathscr{P}^{(3)}$$

is solvable for all pairs a, b of relatively prime natural numbers.

Hence we obtain

THEOREM 4. *If* $f(n)$ *is a totally additive function increasing monotonically on the set* $\{P_3+1\}$, *i.e.*

$$f(P_3+1) \geqslant f(P_3'+1), \quad \text{if} \quad P_3 \geqslant P_3'$$

for every pairs $P_3, P_3' \in \mathscr{P}^{(3)}$, *then* $f(n)$ *is a constant multiplie of* $\log n$. *Further the set* $\{P_3+1\}$ *is a set of uniqueness.*

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References

[1] P. Erdös, *On the distribution function of additive functions*, Annals of Math. (2) 47 (1946), pp. 1-20, in particular Theorem XI on p. 17.

[2] K. Prachar, *Primzahlverteilung*, Springer Verlag, 1957.

[3] M. B. Barban, Yu. V. Linnik and N. G. Tshudakov, *On prime numbers in an arithmetic progression with a prime-power difference*, Acta Arith. 9 (1964), pp. 375-390, in particular Lemmas 1, 2, 3.

[4] А. И. Виноградов, *Применение* $\zeta(s)$ *к решету Эратосфена*, Матем. сб. **41** (83) (1957), pp. 49-80.

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Теорема о нулях дзета-функции Дедекинда и расстояние между „соседними" простыми идеалами

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Классическое доказательство Хохейзеля теоремы о разности между „соседними" простыми числами (см. [6], стр. 321) опирается на знание:

а) отсутствия нулей $\zeta(\sigma+it)$ в области

$$\sigma \geqslant 1-\frac{c}{\ln^a t}; \quad t > t_0; \quad a < 1;$$

б) оценки $N(\sigma, T)$ числа нулей $\varrho = \beta+i\gamma$ функции $\zeta(\sigma+it)$ в области $\beta \leqslant \sigma$; $0 \leqslant \gamma \leqslant T$.

В настоящей работе с помощью метода И. М. Виноградова оценок тригонометрических сумм (см. [2]) мы доказываем теоремы 1 и 2:

ТЕОРЕМА 1. *Дзета-функция Дедекинда* $\zeta_K(\sigma+it)$ *произвольного поля алгебраических чисел* K *степени* n *не имеет нулей в области*

$$\sigma \geqslant 1-\frac{A}{\ln^{2/3} t(\ln\ln t)^{1/3}}; \quad t > t_0,$$

где $A > 0$ *зависит лишь от поля* K.

ТЕОРЕМА 2.

$$\zeta_K(\tfrac{1}{2}+it) \ll |t|^{n/4-c_1/n^2\ln(n+2)}$$

(c_1 — *абсолютная постоянная*).

Из теоремы 1 с помощью несложного обобщения метода Хохейзеля получаем теорему:

ТЕОРЕМА 3. *Пусть* $\pi_1(x)$ — *число простых идеалов первой степени поля* K *с нормой, не превосходящей* x. *Тогда из оценки*

$$N_K(\sigma, T) \ll T^{b(1-\sigma)}\ln^{c_2} T$$

следует, что при $\Theta > 1-1/b$

$$\pi_1(x+x^{\Theta})-\pi_1(x) \sim \frac{x^{\Theta}}{\ln x}.$$