Table des matières du tome XIII, fascicule 3

La revue est consacrée à toutes les branches de *l'Arithmétique et de la Théorie des Nombres*, ainsi qu'aux *fonctions ayant de l'importance dans ces domaines*.

Prière d'adresser les textes dactylographiés à l'un des rédacteurs de la revue ou bien à la Rédaction de

ACTA ARITHMETICA

Warszawa 1 (Pologne), ul. Śniadeckich 8.

La même adresse est valable pour toute correspondance concernant l'échange de Acta Arithmetica.

Les volumes IV et suivants de ACTA ARITHMETICA sont à obtenir chez

Ars Polona, Warszawa 5 (Pologne), Krakowskie Przedmieście 7.

Prix de ce fascicule 3.00 \$.

Les volumes I-III (reédits) sont à obtenir chez

Johnson Reprint Corp., 111 Fifth Ave., New York, N. Y.

PRINTED IN POLAND

W R O C Ł A W S K A   D R U K A R N I A   N A U K O W A

ACTA ARITHMETICA
XIII (1968)

On additive number theory*

by

K. Thanigasalam (Middlesbrought)

§ 1. Introduction. In their investigation of Waring's problem, Hardy and Littlewood considered the following question: Given positive integers k and s satisfying $k \geqslant 2$, to construct as many integers as possible which are less than a given (large) integer N and expressible in the form

$$x_1^k + x_2^k + \ldots + x_s^k$$

where $x_1, x_2, \ldots, x_s$ are positive integers.

They established the following result. If $U_s^{(k)}(N)$ denotes the number of integers not exceeding N that are expressible as the sum of s positive integral kth powers, then for every $\varepsilon > 0$ and $N > N_0(\varepsilon)$,

(1) $$U_s^{(k)}(N) > N^{a-\varepsilon}$$

where

(2) $$a = 1-(1-2/k)(1-1/k)^{s-2}.$$

The importance of this construction in the evaluation of $G(k)$ [1] in Waring's problem is easily seen from the equivalence of the following two statements.

I. Given a positive integer $k \geqslant 2$, there exists an integer $s = s(k)$ such that all sufficiently large integers are representable as the sum of s positive integral kth powers.

II. Given a positive integer $k \geqslant 2$, there exists an integer $s = s(k)$ and a positive constant $c = c(k)$ such that

$$U_s^{(k)}(N) > N-c \quad \text{for} \quad N \geqslant N_0.$$

Thus the determination of s in accordance with II would provide an independent proof of Hilbert's theorem. Linnik's [2] elementary proof

* This paper is part of author's Ph. D. thesis, submitted to the University of London.

[1] As usual $G(k)$ is defined to be the least value of s for which every large integer is the sum of s positive integral kth powers.

[2] See Khinchin [4].

of Hilbert's theorem (based on Schnirelmann's definition of the density of sequences) is on these lines.

The bound for $U_s^{(k)}(N)$ used by Vinogradov in his estimation of $G(k)$ is

(3) $$U_s^{(k)}(N) > N^{\beta-\varepsilon}$$

where

(4) $$\beta = 1-(1-1/k)^s.$$

For small values of k sharper results concerning $G(k)$ were obtained by Davenport ([3]). These were based on better lower bounds for $U_s^{(k)}(N)$.

When s is large, the bounds for $U_s^{(k)}(N)$ obtained by Davenport's method differ very little from (4). But in order to estimate $G(k)$ for large k, we need to estimate $U_s^{(k)}(N)$ with large values of s. This is the reason why Davenport's construction does not give a more powerful bound for $G(k)$ than that given by Vinogradov (for large k).

The object of this paper is to extend Davenport's method to Freiman's generalization of Waring's problem. Here the problem is to represent large integers in the form

(5) $$x_1^{k_1}+x_2^{k_2}+\ldots+x_r^{k_r}$$

where

(6) $$2 \leqslant k_1 \leqslant k_2 \leqslant \ldots \quad \text{and} \quad \sum_{i=1}^{\infty} k_i^{-1} = \infty.$$

Denote by

$$G\{k_1, k_2, \ldots\}$$

the least value of r for which all large integers are representable in the form (5); so that $G\{k, k, \ldots\}$ corresponds to $G(k)$ in Waring's problem.

In the proof of Freiman's statement by Scourfield [7], the following construction was considered. Given integers $k_1, k_2, \ldots, k_s$ satisfying

$$2 \leqslant k_1 \leqslant k_2 \leqslant \ldots \leqslant k_s,$$

and a large natural number N, denote by $U_s(k_1, k_2, \ldots, k_s; N)$ the number of integers that are less than N and expressible in the form

$$x_1^{k_1}+x_2^{k_2}+\ldots+x_s^{k_s}$$

where $x_1, x_2, \ldots, x_s$ are positive integers. The inequality used in Scourfield is

(7) $$U_s(k_1, k_2, \ldots, k_s; N) \geqslant N^{1-\Pi(k_1,\ldots,k_s)}$$

([3]) H. Davenport [1] (proof that $G(4) = 16$, and that a modified number $G^*(k)$ satisfies $G^*(4) < 14$) and [3] (proof that $G(5) < 23$ and $G(6) < 36$).

where

(8) $$\Pi(k_1, \ldots, k_s) = \prod_{i=1}^{s} (1-1/k_i).$$

This was obtained by generalizing the Hardy-Littlewood construction. If the integers $k_1, k_2, \ldots, k_s$ are small, we can obtain slightly better lower bounds for $U_s(k_1, \ldots, k_s; N)$ than (7). These when applied to the generalized Waring's problem (with small exponents) yield more precise results. Thus, we can consider the representation of integers as sums of cubes and fourth powers, cubes and fifth powers and so on. In fact the exponents in these additive representations need not all be small. Thus there are many possible problems. However, in this paper we content ourselves with two problems which may be of some general interest to number theoreticians.

We prove the following theorems.

THEOREM 1. *Every sufficiently large positive integer N is representable in the form*

(9) $$N = \sum_{s=1}^{35} x_s^{s+1}$$

where the x's are positive integers.

This is an improvement on Roth's (see [6]) result that every large integer N is representable in the form

(10) $$N = \sum_{s=1}^{50} x_s^{s+1}.$$

It is a well-known (best possible) result (of Davenport) that all large positive integers are representable as the sum of 16 fourth powers, though for certain restricted integers a lesser number of powers suffices. We show that the following result is also true.

THEOREM 2. *Every sufficiently large positive integer N is representable in the form*

(11) $$N = \sum_{s=1}^{9} x_s^4 + \sum_{i=10}^{16} x_i^5$$

where the x's are positive integers.

Remark 1. (10) was deduced from Theorem 1 of Roth [6], which states that the number of integers that are less than N and not representable in the form

(12) $$x_1^2+x_2^3+x_3^4+x_4^5 \quad \text{is} \ll N^{1-\frac{1}{10}+\varepsilon};$$

so that the construction is carried out for integers of the form

(13) $$x_5^6+x_6^7+\ldots+x_{50}^{51}.$$

Though the exposition is slightly different, the bound for $U_{46}(6, 7, \ldots$ $\ldots, 51; N)$ used by Roth is essentially [4] (8).

Remark 2. In obtaining results similar to (12), with more integral powers, and then constructing the integers which are sums of the successive powers (for example, we could first estimate the number of integers less than N which are not expressible in the form $x_1^2+x_2^3+x_3^4+$ $+x_4^5+x_5^6$, and then carry out the construction for integers of the type $x_6^7+x_7^8+\ldots$), the techniques employed in the Hardy-Littlewood method do not seem to yield a sharper result than Theorem 1. We can attribute this to the fact that both Weyl's inequality and the construction under consideration are most effective for integral powers with small exponents.

Remark 3. As regards the connection between (10) and (9), one observes the following interesting phenomenon. As we go on cutting down the number of powers required for the representation, we meet with more and more resistance since the density of the sequence x^k $(k \geqslant 2)$, increases as k decreases, so that it is difficult to dispense with smaller powers in the representation. On the other hand, we need the bounds for $U_s(k_1, \ldots, k_s; N)$ with smaller values of s and the exponents $k_1, \ldots, k_s$; so that in view of our earlier remarks, the resistance is slightly less than what one expects.

Remark 4. Though it would be very interesting to know the exact value of $G\{2, 3, \ldots\}$, we remark that with the existing techniques, Theorem 1 (viz. $G\{2, 3, \ldots\} \leqslant 35$) seems to have reached almost a stage of finality in the Hardy-Littlewood framework. Any further substantial improvement would require another new idea.

Remark 5. In the proof of Theorem 2, we consider the construction of integers of the form

$$x_1^4+x_2^4+x_3^5+x_4^5+x_5^5, \tag{14}$$

and also integers of the form

$$x_1^4+x_2^5+x_3^5+x_4^5+x_5^5. \tag{15}$$

The rest of the argument depends on that of Davenport [1]. However, for the sake of completeness and clarity, the essence of the logical structure is presented.

§ 2. Notation. $a, q, x, y, z, \ldots$ (with or without suffices) denote natural numbers. $\alpha, \beta, \gamma, \ldots$ denote real numbers. N is a large positive integer and δ a small positive number. ε is an arbitrarily small positive

(4) (8) gives the bound $U_{46}(6, 7, \ldots, 51; N) > N^{1-5/51-\varepsilon}$ and Roth uses the bound $U_{46}(6, 7, \ldots, 51; N) > N^{1-1/10+\delta}$ where δ is some positive constant.

number. The constants implied by the notation "$\ll$" depend at most on δ and ε.

$e(\alpha)$ denotes $e^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$.

Throughout a, q satisfy $a \leqslant q$ and $(a, q) = 1$ (unless otherwise specified).

For any natural number n and positive numbers X, Y satisfying

$$2 \leqslant n, \quad 1 \leqslant X \leqslant Y,$$

we define

$$\nu = n^{-1},$$

$$f_n(X, Y; \alpha) = \sum_{X \leqslant x^n \leqslant Y} e(\alpha x^n),$$

$$S_n(a, q) = \sum_{x=1}^{q} e_q(ax^n),$$

$$\tau_n(X, Y; \beta) = \nu \sum_{X \leqslant y \leqslant Y} y^{\nu-1} e(\beta y),$$

$$F_n(X, Y; a, q, \alpha) = q^{-1} S_n(a, q) \tau_n(X, Y; \alpha - a/q),$$

$$A(u, q) = \sum_a e_q(-au) \Big\{\prod_{n=2}^{5} \big(q^{-1} S_n(a, q)\big)\Big\},$$

$$A^*(u, q) = \sum_a e_q(-au) \{q^{-1} S_4(a, q)\}^6,$$

$$\mathfrak{S}(X, u) = \sum_{q \leqslant X} A(u, q), \qquad \mathfrak{S}^*(X, u) = \sum_{q \leqslant X} A^*(u, q),$$

$$\mathfrak{S}(u) = \sum_{q=1}^{\infty} A(u, q), \qquad \mathfrak{S}^*(u) = \sum_{q=1}^{\infty} A^*(u, q).$$

Let $u_1, u_2, \ldots, u_U$ be the integers that are expressible in the form

$$\sum_{s=5}^{18} x_s^{s+1} + \sum_{s=20}^{35} x_s^{s+1}, \tag{16}$$

and satisfy $u_i < N/4$, $i = 1, \ldots, U$.

Also let $v_1, v_2, \ldots, v_V$ and $\omega_1, \omega_2, \ldots, \omega_W$ be the integers expressible respectively in the forms

$$x_1^4+x_2^4+x_3^5+x_4^5+x_5^5 \quad \text{and} \quad x_1^4+x_2^5+x_3^5+x_4^5+x_5^5,$$

and satisfy

$$v_i < N/4, \quad v_i \equiv 0, 1, 2, 3, 4 \text{ or } 5 \pmod{16}, \quad i = 1, \ldots, V; \tag{17}$$

$$\omega_j < N/4, \quad \omega_j \equiv 0, 1, 2, 3, 4 \text{ or } 5 \pmod{16}, \quad j = 1, \ldots, W. \tag{18}$$

Write

(19) $$\theta(\alpha) = \prod_{n=2}^{5} f_n(N/2^n,\ N;\ \alpha),$$

(20) $$\Omega(\alpha) = \sum_{i=1}^{U} e(\alpha u_i), \quad \Omega_1^*(\alpha) = \sum_{i=1}^{V} e(\alpha v_i), \quad \Omega_2^*(\alpha) = \sum_{j=1}^{W} e(\alpha \omega_j),$$

(21) $$\Lambda(\alpha) = \theta(\alpha)\Omega(\alpha)f_{20}(1,\ N/4;\ \alpha),$$

(22) $$\Theta(a,\ q,\ \alpha) = \prod_{n=2}^{5} F_n(N/2^n,\ N;\ a,\ q,\ \alpha),$$

(23) $$\theta^*(\alpha) = \{f_4(N/16,\ N;\ \alpha)\}^6,$$

(24) $$\Theta^*(a,\ q,\ \alpha) = \{F_4(N/16,\ N;\ a,\ q,\ \alpha)\}^6,$$

so that

(25) $$\Omega(0) = U, \quad \Omega_1^*(0) = V, \quad \Omega_2^*(0) = W.$$

§ 3. Farey dissection.

(I) The' Dissection' designed in order to prove Theorem 1 is somewhat artificial. Write

(26) $$\varphi = [N^{4/5+\delta}],$$

(27) $$\tau = 14/71.$$

The unit interval

(28) $$1/\varphi < \alpha < 1+1/\varphi$$

is divided into basic and supplementary intervals as follows.

The intervals

(29) $$\alpha = a/q+\beta \quad \text{with} \quad q \leqslant N^\tau \text{ and } |\beta| \leqslant (q\varphi)^{-1}$$

will be denoted by $\mathfrak{M}_{a,q}$ and called the *basic intervals*.

The intervals

(30) $$\alpha = a/q+\beta \quad \text{with} \quad N^\tau < q < \varphi,\ |\beta| \leqslant (q\varphi)^{-1},\ \alpha \notin \text{any } \mathfrak{M}_{a,q}$$

will be denoted by $\mathfrak{m}_{a,q}$ and called the *supplementary intervals*.

This makes the basic intervals less numerous (and of course the supplementary intervals more numerous) than if we defined them with $q \leqslant N^{1/5-\delta}$. This slight deviation from the traditional division seems to be necessary to get the maximum out of the basic and supplementary intervals simultaneously.

For a given a, q we also denote the points of the interval (28) which do not belong to the corresponding $\mathfrak{M}_{a,q}$ by $\overline{\mathfrak{M}}_{a,q}$. The union of all $\mathfrak{M}_{a,q}$'s, $\mathfrak{m}_{a,q}$'s are denoted by $\mathfrak{M}$, $\mathfrak{m}$, respectively.

(II) Write

(31) $$\varphi^* = [N^{3/4+\delta}].$$

The intervals $\mathfrak{M}_{a,q}^*$ defined by

(32) $$\alpha = a/q+\beta, \quad q \leqslant N^{1/8}, \quad |\beta| \leqslant (q\varphi^*)^{-1},$$

and the union of intervals $\mathfrak{m}_{a,q}^*$'s defined by

$$\alpha = a/q+\beta, \quad N^{1/8} < q < \varphi^*, \quad |\beta| \leqslant (q\varphi^*)^{-1}, \quad \alpha \notin \text{any } \mathfrak{m}_{a,q}$$

together form the unit interval

(33) $$1/\varphi^* < \alpha < 1+1/\varphi^*.$$

Again for a given a, q the points of (33) which do not belong to the corresponding $\mathfrak{M}_{a,q}^*$ are denoted by $\overline{\mathfrak{M}}_{a,q}^*$. The union of $\mathfrak{M}_{a,q}^*$'s, $\mathfrak{m}_{a,q}^*$'s, are denoted by $\mathfrak{M}^*$, $\mathfrak{m}^*$, respectively.

It can be proved in the usual way that any two $\mathfrak{M}_{a,q}$'s and any two $\mathfrak{M}_{a,q}^*$'s are mutually exclusive.

Now write

(34) $$r(N) = \int_{1/\varphi}^{1+1/\varphi} \Lambda(\alpha)e(-N\alpha)\,d\alpha,$$

and

(35) $$r^*(N) = \int_{1/\varphi^*}^{1+1/\varphi^*} \theta^*(\alpha)\Omega_1^*(\alpha)\Omega_2^*(\alpha)e(-N\alpha)\,d\alpha$$

(cf. (20), (21) and (23)).

We observe that $r(N)$ does not exceed the number of representations of N in the form

$$N = \sum_{s=1}^{35} x_s^{s+1},$$

and that $r^*(N)$ does not exceed the number of representations of N in the form

$$N = \sum_{s=1}^{9} x_s^4 + \sum_{t=10}^{16} x_t^5.$$

Thus, in order to prove Theorems 1 and 2, it will suffice to show that

$$r(N) > 0 \quad \text{and} \quad r^*(N) > 0 \quad \text{for large } N,$$

§ 4. Preliminary results.

Lemma 1. *If $2 \leqslant k_1 \leqslant k_2$, then the number of solutions of the equation*

$$x_1^{k_1}+x_2^{k_2} = y_1^{k_1}+y_2^{k_2} \tag{36}$$

where

$$x_1, y_1 < N^{1/k_1}; \quad x_2, y_2 < N^{1/k_2}$$

is

$$\ll N^{1/k_1+1/k_2+\varepsilon}.$$

Proof. (36) can be written as

$$x_1^{k_1}-y_1^{k_1} = y_2^{k_2}-x_2^{k_2}. \tag{37}$$

Since the number of solutions of the equation

$$x_1^{k_1}-y_1^{k_1} = m, \quad \text{with} \quad x_1 \neq y_1,$$

is $\ll m^{\varepsilon}$, the number of solutions of (37) with $x_1 \neq y_1$ is $\ll N^{2/k_2+\varepsilon}$. Also the number of solutions of (36) with $x_1 = y_1$ (and hence with $x_2 = y_2$) is $\ll N^{1/k_1+1/k_2}$. The result follows since

$$\frac{1}{k_1}+\frac{1}{k_2} \geqslant \frac{2}{k_2}.$$

Corollary. *The number of integers less than N and representable in the form*

$$x_1^{k_1}+x_2^{k_2}$$

is

$$\gg N^{1/k_1+1/k_2-\varepsilon}.$$

Proof. If $r(m)$ denotes the number of representations of m in the form $x_1^{k_1}+x_2^{k_2}$, where

$$x_1 < (N/2)^{1/k_1}, \quad x_2 < (N/2)^{1/k_2}, \tag{38}$$

then $\sum_m r^2(m)$ does not exceed the number of solutions of (36) subject to (38). Hence by the lemma,

$$\sum_m r^2(m) \ll N^{1/k_1+1/k_2+\varepsilon}.$$

Also the number of integers less than N that are representable in the form

$$x_1^{k_1}+x_2^{k_2}$$

is

$$\geqslant \sum_{\substack{m\\ r(m)>0}} 1.$$

But by Cauchy's inequality

$$\sum_{\substack{m\\ r(m)>0}} 1 \geqslant \Big\{\sum_m r(m)\Big\}^2 \Big/ \Big\{\sum_m r^2(m)\Big\} \gg N^{2(1/k_1+1/k_2)}/N^{1/k_1+1/k_2+\varepsilon},$$

and the result follows.

Lemma 2. *Let $0 < \eta < 1$ and $\lambda = 1-(1-\eta)/k$, $k \geqslant 3$. Then if $t_1, t_2, \ldots, t_T$ is a sequence of positive integers satisfying*

$$t_1 < t_2 \ \ldots < t_T \leqslant P^{k\lambda}, \tag{39}$$

the number S of solutions of

$$x^k+t_i = y^k+t_j \tag{40}$$

subject to

$$P < x < 2P, \quad P < y < 2P \tag{41}$$

satisfies

$$S \ll PT+P^{1+\eta+\varepsilon}T\{P^{-2}+P^{-\eta-l-1}T\}^{1/2^l} \tag{42}$$

where $l \leqslant k-2$.

This is Theorem 1 of Davenport [3].

Lemma 3. *If $3 \leqslant k_1 \leqslant k_2 \leqslant \ldots \leqslant k_s$ and*

$$U_{s-1}(k_1, k_2, \ldots, k_{s-1}; N) > N^{a} \tag{43}$$

for all large N, where $1/k_s < a < 1$, then

$$U_s(k_1, k_2, \ldots, k_s; N) > N^{\beta-\varepsilon} \quad \textit{for} \quad N > N_0(\varepsilon) \tag{44}$$

where

$$\beta = \max_{h \leqslant k_s-2} \frac{1}{k_s}\Big\{1+\frac{(2^h-1)(k_s-1)+(h+1)}{2^h-1+a}a\Big\}. \tag{45}$$

This is deduced from Lemma 2 in the same way as Theorem 2 of Davenport [3] is deduced from Theorem 1.

Corollary. *The number β given by (45), satisfies*

$$\beta \geqslant \frac{1}{k_s}\{1+(k_s-1)a\}. \tag{46}$$

Proof.

$$\frac{(2^h-1)(k_s-1)+(h+1)}{2^h-1+a} = (k_s-1)+\Big\{\frac{(h+1)-(k_s-1)a}{2^h-1+a}\Big\}.$$

The corollary follows from this, since $a \leqslant 1$ and hence if we take $h = (k_s-2)$,

$$(h+1)-(k_s-1)a = (k_s-1)(1-a) \geqslant 0.$$

Remark 6. The full force of Lemma 3 is not used for all the exponents in the proof of Theorem 1; instead we use the corollary for large exponents. By doing so, we avoid unnecessary calculations since the slight improvements on the bounds of $U_s(k_1, k_2, \ldots, k_s; N)$ obtained by using (45) throughout is not sufficient to improve on Theorem 1. Furthermore, the inequality

$$k_1 \leqslant k_2 \leqslant \ldots \leqslant k_s$$

is irrelevant in the proof of the lemma. First we could consider sums of $(s-1)$ integral powers where the exponents are selected in any manner from the s integers $k_1, \ldots, k_s$ and then apply the Lemma to the remaining integer.

Remark 7. If, in the notation of Lemma 3, we take

$$a = 1-\prod_{=1}^{s-1}\left(1-\frac{1}{k_i}\right),$$

then by (46) we have

$$\beta \geqslant \frac{1}{k_s}\left[1+(k_s-1)\left\{1-\prod_{i=1}^{s-1}\left(1-\frac{1}{k_i}\right)\right\}\right] = 1-\prod_{i=1}^{s}\left(1-\frac{1}{k_i}\right).$$

These are precisely the bounds given by (7). Thus we see that the bounds given by (45) are slightly superior to those given by (7).

LEMMA 4. *Let*

$$a_1 = \frac{23}{132}, \quad a_2 = \frac{1109}{4190}, \quad a_3 = \frac{43622}{123111}, \quad a_4 = \frac{3217365}{7243192},$$

$$a_5 = \frac{201918499}{377437963}, \quad a_6 = \frac{10718805701}{17063905440},$$

(47) $$a' = \frac{1}{19}(7+12a_6),$$

(48) $$a^* = \frac{1}{36}(16+20a') = \left(\frac{37+20a_6}{57}\right) = \frac{42287030765}{48632130504} > \frac{185}{213}+\frac{9}{10^4}.$$

Then

$$U_2(11, 12; N) > N^{a_1-\varepsilon}, \qquad U_3(10, 11, 12; N) > N^{a_2-2\varepsilon},$$

$$U_4(9, 10, 11, 12; N) > N^{a_3-3\varepsilon}, \qquad U_5(8, 9, 10, 11, 12; N) > N^{a_4-4\varepsilon},$$

$$U_6(7, 8, 9, 10, 11, 12; N) > N^{a_5-5\varepsilon}, \qquad U_7(6, 7, 8, 9, 10, 11, 12; N) > N^{a_6-6\varepsilon},$$

$$U_{14}(6, 7, \ldots, 19; N) > N^{a'-13\varepsilon},$$

and

(49) $$U_{30}(6, 7, \ldots, 19, 21, 22, \ldots, 36; N) > N^{a^*-30\varepsilon}.$$

Proof. a_1 is obtained by using Lemma 1. a_2, a_3, a_4, a_5, a_6 are obtained from Lemma 3 by taking

$$k_s = 10,\ h = 2; \quad k_s = 9,\ h = 2; \quad k_s = 8,\ h = 3;$$
$$k_s = 7,\ h = 3; \quad k_s = 6,\ h = 3,$$

respectively.

a' is obtained as follows. From (46),

$$U_8(6, 7, \ldots, 13; N) > N^{a_7-7\varepsilon},$$

where $a_7 = \frac{1}{13}(1+12a_6)$;

$$U_9(6, 7, \ldots, 14; N) > N^{a_8-8\varepsilon},$$

where $a_8 = \frac{1}{14}(1+13a_7) = \frac{1}{14}(2+12a_6)$.

Proceeding thus inductively we get $U_{14}(6, 7, \ldots, 19; N) > N^{a'-13\varepsilon}$, where a' is defined by (47).

a^* is obtained from a' in the same way by a repeated application of (46).

COROLLARY. *We have*

(50) $$\Omega(0) \geqslant N^{1-2\tau/3+20\delta} \quad \text{(Cf. (20))}.$$

This is easily verified from (27), (48) and (49).

Remark 8. In a certain sense, Lemma 1 is the best possible result of its kind. Also Lemma 3 is more effective for small values of k_s. Thus in the construction of integers of the form

$$x_1^6+x_2^7+\ldots+x_7^{12},$$

we first apply Lemma 1 to the exponents 11, 12 and then apply Lemma 3 to the remaining smaller exponents. It is remarkable that the bounds obtained by first applying Lemma 1 to the smaller exponents 6, 7 and then applying Lemma 3 to the larger exponents, are not sharp enough to prove Theorem 1.

LEMMA 5. *Let*

$$\beta_1 = \frac{2}{5}, \quad \beta_2 = \frac{47}{85}, \quad \beta_3 = \frac{433}{604}, \quad \beta_3' = \frac{1073}{1605},$$

$$\beta_4 = \frac{7441}{8980}, \quad \beta_4' = \frac{4691}{5888}.$$

Further, let $U_s^*(k_1, k_2, \ldots, k_s; f, l, N)$ *denote the number of integers* $\equiv f(\mathrm{mod}\, l)$ *and less than* N *which are representable in the form*

$$x_1^{k_1} + x_2^{k_2} + \ldots + x_s^{k_s}.$$

Then, if

(51) $$f_1 = 0, 1 \text{ or } 2; \quad f_2 = 0, 1, 2 \text{ or } 3; \quad f_3 = 0, 1, 2, 3 \text{ or } 4 \quad \text{and} \quad f_4 = 0, 1, 2, 3, 4 \text{ or } 5,$$

we have

$$U_2^*(5, 5; f_1, 16, N) > N^{\beta_1 - 2\varepsilon},$$
$$U_3^*(5, 5, 5; f_2, 16, N) > N^{\beta_2 - 4\varepsilon},$$
$$U_4^*(4, 5, 5, 5; f_3, 16, N) > N^{\beta_3 - 6\varepsilon},$$

(52) $$U_5^*(4, 4, 5, 5, 5; f_4, 16, N) > N^{\beta_4 - 8\varepsilon};$$

$$U_4^*(5, 5, 5, 5; f_3, 16, N) > N^{\beta_3' - 6\varepsilon},$$

(53) $$U_5^*(4, 5, 5, 5, 5; f_4, 16, N) > N^{\beta_4' - 8\varepsilon}.$$

Proof. β_1 is obtained from Lemma 1, and $\beta_2, \beta_3, \beta_4$ are obtained from Lemma 3 by taking $k_s = 5$, $h = 2$; $k_s = 4$, $h = 2$; $k_s = 4$, $h = 2$ respectively, and noting that the congruence condition imposed on the integers does not affect their bounds by more than $N^{-\varepsilon}$.

Similarly β_3', β_4' are obtained from Lemma 3 by taking $k_s = 5$, $h = 3$; $k_s = 4$, $h = 2$ respectively.

A formal proof of the lemma could be given by using the arguments in the proof of Lemma 2 of Davenport [1] on noting that

$$n \equiv f(\mathrm{mod}\, 2^5) \quad \text{implies} \quad n \equiv f(\mathrm{mod}\, 16).$$

Corollary. *In the notation of* (17), (18), (20) *and* (25), *we have*

(54) $$\Omega_1^*(0) = V > N^{\beta_4 - 10\varepsilon},$$

(55) $$\Omega_2^*(0) = W > N^{\beta_4' - 10\varepsilon}.$$

These follow trivially from the lemma.

We also note for future reference that

(56) $$\frac{\beta_4 + \beta_4'}{2} > \frac{13}{16} + \frac{1}{10^4}.$$

§ 5. Lemmas for Theorem 1.

Let

(57) $$\mu_1 = \sum_{n=2}^{5} \nu = \frac{77}{60},$$

and

(58) $$\mu = \mu_1 + \frac{1}{20} = \frac{4}{3}.$$

Lemma 6. *If* $\alpha = a/q + \beta$, *where* $|\beta| \leqslant 1/2$, *we have*

$$F_n(N/2^n, N; a, q, \alpha) \ll q^{-\nu} \min(N^\nu, N^{\nu-1}|\beta|^{-1}).$$

This is essentially Lemma 5 of Davenport [1], on noting that at that stage the restriction $n \geqslant 4$ in the paper is not relevant.

Lemma 7. *If* $\alpha = a/q + \beta$, *where* $q \leqslant N^{\nu-\delta}$, $\beta \ll q^{-1}N^{\nu-1-\delta}$, *then*

$$f_n(N/2^n, N; \alpha) - F_n(N/2^n, N; a, q, \alpha) \ll q^{3/4+\varepsilon}.$$

This is essentially Lemma 8 of Davenport [1] for $n \geqslant 4$, and Lemma 7 of Davenport [2] for $n = 3$.

For $n = 2$, we can prove by the same methods that

$$f_2(N/4, N; \alpha) - F_2(N/4, N; a, q, \alpha) \ll q^{1/2+\varepsilon}.$$

Lemma 8 (Weyl's Inequality). *If* $\alpha = a/q + \beta$, *where*

$$N^{\nu-\delta} < q \leqslant N^{1-\nu+\delta} \quad \text{and} \quad \beta \ll q^{-1}N^{\nu-1-\delta},$$

then

$$f_n(N/2^n, N; \alpha) \ll N^{\nu(1-1/2^{n-1})+\delta}.$$

This is Lemma 11 of Davenport [1], and of course is obtained by partial summation using Satz 267 of Landau [5].

Lemma 9. *On* $\mathfrak{m}$, *we have*

$$f_3(N/8, N; \alpha) \ll N^{1/3-\tau/3},$$

where τ *is defined by* (27).

Proof. Every real number α belonging to the unit interval (28) can be expressed as

(59) $$\alpha = a/q + \beta \quad \text{with} \quad 0 < q \leqslant N^{2/3+\delta}, \quad |\beta| \leqslant q^{-1}N^{-2/3-\delta}.$$

The intervals for α such that q, β satisfy one of the conditions I, II, III below will be called the 'Good intervals'.

$$\text{I.} \quad N^{1/3-\delta} < q \leqslant N^{2/3+\delta}, \quad |\beta| \leqslant q^{-1}N^{-2/3-\delta};$$
$$\text{II.} \quad N^{\tau} < q \leqslant N^{1/3-\delta}, \quad |\beta| \leqslant q^{-1}N^{-2/3-\delta};$$
$$\text{III.} \quad 0 < q \leqslant N^{\tau}, \quad q^{-1}\varphi^{-1} < |\beta| \leqslant q^{-1}N^{-2/3-\delta}.$$

The points α belonging to (28) for which q, β do not satisfy any of the conditions I, II or III, will be said to form the 'Bad intervals'.

We note that the 'Bad intervals' are contained in $\mathfrak{M}$, since if α belongs to a 'Bad interval', q, β satisfy

$$0 < q \leqslant N^{\tau}, \quad |\beta| \leqslant q^{-1}\varphi^{-1}.$$

Thus, $\mathfrak{m}$ is contained in the 'Good intervals'. Hence in order to prove the lemma, it is sufficient to consider the numbers α for which q, β satisfy I, II or III.

Suppose that I is satisfied. Then by Lemma 8 (with $n=3$), we have

(60) $$f_3(N/8, N; \alpha) \ll N^{\frac{1}{3}(1-1/4)+\delta}.$$

Now let II be satisfied. Then by Lemmas 6 and 7 (with $n=3$), we have

(61) $$f_3(N/8, N; \alpha) \ll q^{3/4+\varepsilon}+q^{-1/3}N^{1/3} \ll N^{1/4+\delta}+N^{1/3-\tau/3} \ll N^{1/3-\tau/3}.$$

Finally, suppose that III is satisfied. Then, again by Lemmas 6 and 7 (with $n=3$), we have

(62) $$f_3(N/8, N; \alpha) \ll q^{3/4+\varepsilon}+q^{-1/3}N^{1/3-1}|\beta|^{-1} \ll N^{3\tau/4+\delta}+N^{1/3-1}(q|\beta|)^{-1}q^{2/3}.$$

Now
$$(q|\beta|)^{-1} < \varphi \ll N^{4/5+\delta} \quad \text{(cf. (26))},$$
and $q^{2/3} \leqslant N^{2\tau/3}$; so that
$$N^{1/3-1}(q|\beta|)^{-1}q^{2/3} \ll N^{1/3-1/5+\delta+2\tau/3} = N^{\frac{1}{3}(1-3/5+2\tau)+\delta} \ll N^{1/3-\tau/3},$$
since $\tau < 1/5$. Also trivially
$$N^{3\tau/4} \ll N^{1/3-\tau/3}.$$

Thus the lemma follows from (60), (61) and (62).

LEMMA 10. *Let*

(63) $$\theta_1(\alpha) = f_2(N/4, N; \alpha)f_4(N/16, N; \alpha)f_5(N/32, N; \alpha)f_{20}(1, N/4; \alpha).$$

Then
$$\int_0^1 |\theta_1(\alpha)|^2 d\alpha \ll N^{1+\varepsilon}.$$

Proof. The integral is precisely the number of solutions of the equation

(64) $$x_2^2+x_4^4+x_5^5+x_{20}^{20} = y_2^2+y_4^4+y_5^5+y_{20}^{20},$$

subject to

(65) $$\begin{aligned} &N^{1/2}/2 \leqslant x_2 \leqslant N^{1/2}, \quad N^{1/4}/2 \leqslant x_4 \leqslant N^{1/4},\\ &N^{1/5}/2 \leqslant x_5 \leqslant N^{1/5}, \quad 1 \leqslant x_{20} \leqslant (N/4)^{1/20},\\ &N^{1/2}/2 \leqslant y_2 \leqslant N^{1/2}, \quad N^{1/4}/2 \leqslant y_4 \leqslant N^{1/4},\\ &N^{1/5}/2 \leqslant y_5 \leqslant N^{1/5}, \quad 1 \leqslant y_{20} \leqslant (N/4)^{1/20}. \end{aligned}$$

Equation (64) can be written as
$$x_2^2-y_2^2 = y_4^4-x_4^4+y_5^5-x_5^5+y_{20}^{20}-x_{20}^{20}.$$

(i) The number of solutions with $x_2 \neq y_2$ is
$$\ll N^{2\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{20}\right)+\varepsilon} = N^{1+\varepsilon},$$
since the number of solutions of $x_2^2-y_2^2 = m$ (for a given $m \neq 0$) is $\ll m^{\varepsilon}$.

(ii) Now consider the solutions with
$$x_2 = y_2; \quad x_4^4-y_4^4 = y_5^5-x_5^5+y_{20}^{20}-x_{20}^{20} \neq 0.$$
Since the number of solutions of $x_4^4-y_4^4 = m$ is $\ll m^{\varepsilon}$, we get the required number of solutions to be
$$\ll N^{\frac{1}{2}}N^{2\left(\frac{1}{5}+\frac{1}{20}\right)+\varepsilon} = N^{1+\varepsilon}.$$

(iii) The number of solutions with
$$x_2 = y_2, \quad x_4 = y_4, \quad x_5^5-y_5^5 = y_{20}^{20}-x_{20}^{20} \neq 0$$
s (by a similar argument)
$$\ll N^{\frac{1}{2}+\frac{1}{4}}\cdot N^{2\cdot\frac{1}{20}+\varepsilon} \ll N.$$

(iv) Finally, the number of solutions with
$$x_2 = y_2, \quad x_4 = y_4, \quad x_5 = y_5, \quad x_{20} = y_{20}$$
is
$$\ll N^{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{20}\right)} = N.$$

This completes the proof of Lemma 10.

LEMMA 11. *On* $\mathfrak{M}_{a,q}$, *we have*
$$\theta(\alpha)-\Theta(a, q, \alpha) \ll N^{(\mu_1-1/5)}q^{-1/3+\varepsilon},$$
where $\theta(\alpha)$, $\Theta(a, q, \alpha)$ *and* μ_1 *are defined by* (19), (22) *and* (57).

Proof. From Lemmas 6 and 7, we have on $\mathfrak{M}_{a,q}$ (cf. (29))

(66) $$f_n(N/2^n, N; \alpha)-F_n(N/2^n, N; a, q, \alpha) \ll q^{3/4+\varepsilon},$$

(67) $$F_n(N/2^n, N; a, q, \alpha) \ll q^{-\nu}N^{\nu}$$

for $n = 2, 3, 4, 5$.

Since $q \leqslant N^{\tau}$ (cf. (21)), it is an easy verification from these that

(68) $$f_n(N/2^n, N; \alpha) \ll q^{-\nu}N^{\nu} \quad \text{for} \quad n = 2, 3, 4, 5.$$

Now

$$\theta(\alpha)-\Theta(a, q, \alpha) = f_2f_3f_4f_5-F_2F_3F_4F_5$$
$$= (f_2-F_2)f_3f_4f_5+F_2(f_3-F_3)f_4f_5+$$
$$+F_2F_3(f_4-F_4)f_5+F_2F_3F_4(f_5-F_5),$$

where f_n, F_n stand for $f_n(N/2^n, N; \alpha)$ and $F_n(N/2^n, N; a, q, \alpha)$, respectively for $n = 2, 3, 4, 5$.

Also from (66), (67) and (68) we have

$$(f_2-F_2)f_3f_4f_5 \ll q^{3/4+\varepsilon}\cdot N^{(\mu_1-1/2)}q^{-\mu_1+1/2},$$
$$F_2(f_3-F_3)f_4f_5 \ll q^{3/4+\varepsilon}\cdot N^{(\mu_1-1/3)}q^{-\mu_1+1/3},$$
$$F_2F_3(f_4-F_4)f_5 \ll q^{3/4+\varepsilon}\cdot N^{(\mu_1-1/4)}q^{-\mu_1+1/4},$$
$$F_2F_3F_4(f_5-F_5) \ll q^{3/4+\varepsilon}\cdot N^{(\mu_1-1/5)}q^{-\mu_1+1/5}.$$

Again it is an easy verification from these that

$$\theta(\alpha)-\Theta(a, q, \alpha) \ll q^{3/4+\varepsilon}\cdot N^{(\mu_1-1/5)}q^{-\mu_1+1/5} = N^{(\mu_1-1/5)}q^{-1/3+\varepsilon},$$

proving the lemma.

LEMMA 12. *We have*

$$\int_{\mathfrak{M}} |\theta(\alpha)-\Theta(a, q, \alpha)|^2 d\alpha \ll N^{2\mu_1-1-2\tau/3-10\delta}.$$

Proof. By Lemma 11, the integral is

$$\ll \sum_{q\leqslant N^\tau}\sum_a \int_{\mathfrak{M}_{a,q}} \{N^{2\mu_1-2/5}q^{-2/3+2\varepsilon}\}\,d\beta$$
$$\ll \sum_{q\leqslant N^\tau} q\cdot q^{-2/3+2\varepsilon}N^{2\mu_1-2/5}\Big(\int_{\mathfrak{M}_{a,q}} d\beta\Big)$$
$$\ll \sum_{q\leqslant N^\tau} q\cdot q^{-2/3+2\varepsilon}N^{2\mu_1-2/5}q^{-1}N^{-4/5-\delta}$$
$$\ll N^{2\mu_1-1-1/5-\delta}\cdot N^{\tau/3+\delta} \ll N^{2\mu_1-1-2\tau/3-10\delta} \quad \text{(cf. (27))}.$$

LEMMA 13. *We have*

$$\int_{\mathfrak{M}} |\theta(\alpha)-\Theta(a, q, \alpha)|\,|\Omega(\alpha)|\,d\alpha \ll N^{\mu_1-1-10\delta}\Omega(0),$$

where $\Omega(\alpha)$ is defined by (20).

Proof. By Schwarz's inequality, the integral of the lemma is

$$\ll \Big\{\int_{\mathfrak{M}} |\theta(\alpha)-\Theta(a, q, \alpha)|^2 d\alpha\Big\}^{1/2}\Big\{\int_0^1 |\Omega(\alpha)|^2 d\alpha\Big\}^{1/2}.$$

Also

$$\int_0^1 |\Omega(\alpha)|^2 d\alpha = \Omega(0),$$

and by (50)

$$\{\Omega(0)\}^{-1} \ll N^{-1+2\tau/3-10\delta}.$$

Hence, by Lemma 12, we have

$$\int_{\mathfrak{M}} |\theta(\alpha)-\Theta(a, q, \alpha)|\,|\Omega(\alpha)|\,d\alpha \ll \{N^{\mu_1-1/2-\tau/3-5\delta}\}\{N^{-1/2+\tau/3-5\delta}\Omega(0)\}$$
$$\ll N^{\mu_1-1-10\delta}\Omega(0).$$

LEMMA 14. *We have*

$$\sum_{q\leqslant N^\tau}\sum_a \int_{\mathfrak{M}_{a,q}} |\Theta(a, q, \alpha)|^2 d\alpha \ll N^{2\mu_1-1-2\tau/3+7\delta}.$$

Proof. By Lemma 6, the integral is

$$\ll \sum_{q\leqslant N^\tau}\sum_a \int_{q^{-1}N^{-4/5-\delta}}^{1/2} \{q^{-2\mu_1}N^{2\mu_1-8}\beta^{-8}\}\,d\beta \ll \sum_{q\leqslant N^\tau} q\cdot q^{-2\mu_1}N^{2\mu_1-8}(q^{-1}N^{-4/5-\delta})^{-7}$$
$$\ll N^{2\mu_1-8+28/5+7\delta}\sum_{q\leqslant N^\tau} q^{(6-17/30)} \ll N^{2\mu_1-8+28/5+7\delta}\cdot N^{(7-17/30)\tau}.$$

Now

$$2\mu_1-8+28/5+7\delta+\Big(7-\frac{17}{30}\Big)\tau = 2\mu_1-1-\frac{2\tau}{3}+7\delta,$$

since $7\tau+\tau/10 = 7/5$ (cf. (27)); so that the result follows.

Remark 9. It is for Lemma 14 that the Farey dissection has been designed artificially.

LEMMA 15. *We have*

$$\sum_{q\leqslant N^\tau}\sum_a \int_{\mathfrak{M}_{a,q}} |\Theta(a, q, \alpha)\Omega(\alpha)|\,d\alpha \ll N^{\mu_1-1-5\delta}\,\Omega(0).$$

This is deduced from Lemma 14 in the same way as Lemma 13 was deduced from Lemma 12.

LEMMA 16. *We have*

$$\int_{\mathfrak{m}} |\Lambda(\alpha)|\,d\alpha \ll N^{\mu-1-5\delta}\,\Omega(0),$$

where $\Lambda(\alpha)$ and μ $(= 4/3)$, are defined by (21) *and* (57) *respectively.*

Proof. In the notation of (63), we have

(69) $$\int_{\mathfrak{m}} |\Lambda(\alpha)|\,d\alpha \leqslant \{\max_{\alpha\in\mathfrak{m}}|f_3(N/8, N; \alpha)|\}\int_0^1 |\theta_1(\alpha)\Omega(\alpha)|\,d\alpha.$$

Also by Schwarz's inequality,

$$(70)\quad \int_0^1 |\theta_1(a)\Omega(a)|\,da \leqslant \Big\{\int_0^1 |\theta_1(a)|^2 da\Big\}^{1/2}\Big\{\int_0^1 |\Omega(a)|^2 da\Big\}^{1/2} \ll N^{1/2+\varepsilon}\big(\Omega(0)\big)^{1/2} \ll N^{1/2+\varepsilon}\cdot N^{-1/2+\tau/3-10\delta}\Omega(0)$$

by Lemma 10 and (50).

Thus from Lemma 9, (69) and (70), we have

$$\int_{\mathfrak{m}} |\Lambda(a)|\,da \ll N^{1/3+\varepsilon-10\delta}\Omega(0) \ll N^{\mu-1-5\delta}\Omega(0)$$

since $\mu = 4/3$.

Lemma 17. *We have*

$$\int_0^1 \Theta(a, q, a)e(-ua) = \Big\{\prod_{n=2}^{5} q^{-1}S_n(a, q)\Big\}\, e_q(-au)R(u),$$

where for $N/2 < u < N$, $R(u)$ *satisfies*

$$N^{\mu_1-1} \ll R(u) \ll N^{\mu_1-1}.$$

This is a standard type of result proved in the usual way.

The next two lemmas correspond to Lemmas 28 and 29 of Roth [6].

Lemma 18. *The series*

$$\mathfrak{S}(u) = \sum_{q=1}^{\infty} A(u, q)$$

is absolutely convergent, and if $u \geqslant 3$, *there exist positive constants* c_1, c_2 *such that*

$$\mathfrak{S}(u) > c_1(\log\log u)^{-c_2}.$$

Lemma 19. *If* $X \geqslant 1$, *there exists a constant* c_3 *such that*

$$\sum_{q>X} |A(u, q)| \ll X^{-c_3}u^{\varepsilon}.$$

Lemma 20. *If* $N/2 \leqslant u \leqslant N$, *then*

$$\mathfrak{S}(N^{\tau}, u) \gg N^{-\delta}.$$

This follows trivially from Lemmas 18 and 19.

Lemma 21. *Let*

$$(71)\quad I_1(N) = \sum_{q\leqslant N^{\tau}}\sum_{a}\int_0^1 \Theta(a, q, a)f_{20}(1, N/4; a)\Omega(a)e(-Na)\,da.$$

Then

$$(72)\quad I_1(N) \gg N^{\mu-1-\delta}\Omega(0).$$

Proof. We have

$$I_1(N) = \sum_{1\leqslant x^{20}\leqslant N/4}\sum_{i=1}^{U}\sum_{q\leqslant N^{\tau}}\sum_{a}\int_0^1 \Theta(a, q, a)e(x^{20}a+u_i a-Na)\,da$$

where the u_i's are defined by (16).

Thus by Lemmas 17 and 20, we have

$$I_1(N) = \sum_{1\leqslant x^{20}\leqslant N/4}\sum_{i=1}^{U}\sum_{q\leqslant N^{\tau}}\sum_{a}\Big\{\prod_{n=2}^{5} q^{-1}S_n(a, q)\Big\}e_q\{-(N-u_i-x^{20})a\}\times R(N-u_i-x^{20})$$

$$= \sum_{1\leqslant x^{20}\leqslant N/4}\sum_{i=1}^{U}\sum_{q\leqslant N^{\tau}} A(N-u_i-x^{20}, q)R(N-u_i-x^{20})$$

$$\gg \sum_{1\leqslant x^{20}\leqslant N/4}\sum_{i=1}^{U} N^{-\delta}N^{\mu_1-1} \gg N^{1/20}\cdot U\cdot N^{-\delta}\cdot N^{\mu_1-1} = N^{\mu-1-\delta}\Omega(0),$$

since $\mu = \mu_1+1/20$ and $U = \Omega(0)$.

§ 6. Proof of Theorem 1. We have

$$(73)\quad r(N) = \int_{1/\varphi}^{1+1/\varphi} \Lambda(a)e(-Na) = I_1(N)-I_2(N)+I_3(N)+I_4(N),$$

where $I_1(N)$ is defined by (71) and

$$(74)\quad I_2(N) = \sum_{q\leqslant N^{\tau}}\sum_{a}\int_{\overline{\mathfrak{M}}_{a,q}} \Theta(a, q, a)f_{20}(1, N/4; a)\Omega(a)e(-Na)\,da,$$

$$(75)\quad I_3(N) = \int_{\mathfrak{M}} \{\theta(a)-\Theta(a, q, a)\}f_{20}(1, N/4; a)\Omega(a)e(-Na)\,da,$$

$$(76)\quad I_4(N) = \int_{\mathfrak{m}} \Lambda(a)e(-Na)\,da.$$

Using the trivial estimate $f_{20}(1, N/4; a) \ll N^{1/20}$, we have from Lemmas 15 and 13, respectively,

$$I_2(N) \ll N^{\mu-1-5\delta}\Omega(0),\ I_3(N) \ll N^{\mu-1-10\delta}\Omega(0);$$

so that from Lemmas 16 and 21, we get from (73)

$$r(N) \gg N^{\mu-1-\delta}\Omega(0).$$

Thus $r(N) > 0$ for large N, proving Theorem 1.

§ 7. Lemmas for Theorem 2. Using the trivial estimates

$$\Omega_1^*(a) \ll \Omega_1^*(0), \quad \Omega_2^*(a) \ll \Omega_2^*(0) \quad (\text{cf. } (20)),$$

we note that the next three lemmas correspond to Lemmas 12, 13 and 14 of Davenport [1].

LEMMA 22.

$$\int_{\mathfrak{M}^*} |\theta^*(a) - \Theta^*(a, q, a)|\, |\Omega_1^*(a)\Omega_2^*(a)|\, da \ll N^{\frac{1}{2}-\frac{1}{16}+\varepsilon}\Omega_1^*(0)\Omega_2^*(0).$$

LEMMA 23.

$$\sum_{q\leqslant N^{1/8}} \sum_a \int_{\mathfrak{M}^*a,q} |\Theta^*(a, q, a)|\, |\Omega_1^*(a)\Omega_2^*(a)|\, da \ll \Omega_1^*(0)\Omega_2^*(0).$$

LEMMA 24. *On* $\mathfrak{m}^*$,

$$f_4(N/16, N; a) \ll N^{\frac{1}{4}-\frac{1}{32}+\delta}.$$

LEMMA 25.

$$\int_{\mathfrak{m}^*} |\theta^*(a)\Omega_1^*(a)\Omega_2^*(a)|\, da \ll N^{-1-10\delta}\theta^*(0)\Omega_1^*(0)\Omega_2^*(0).$$

Proof. By Lemma 24 and Schwarz's inequality, the integral of the lemma is

$$\leqslant \{\max_{a\in\mathfrak{m}^*}|\theta^*(a)|\}\Big\{\int_0^1 |\Omega_1^*(a)\Omega_2^*(a)|\, da\Big\}$$

$$\leqslant N^{\frac{6}{4}-\frac{6}{32}+6\delta}\Big\{\int_0^1 |\Omega_1^*(a)|^2 da\Big\}^{1/2}\Big\{\int_0^1 |\Omega_2^*(a)|^2 da\Big\}^{1/2}$$

$$= N^{\frac{6}{4}-\frac{6}{32}+6\delta}\{\Omega_1^*(0)\Omega_2^*(0)\}^{1/2} \ll N^{\frac{6}{4}-\frac{6}{32}+6\delta}\Omega_1^*(0)\Omega_2^*(0)\cdot N^{-\frac{13}{16}-20\delta}$$

from (54), (55) and (56). The lemma follows since

(77) $$N^{6/4} \ll \theta^*(0) \ll N^{6/4}.$$

The following three lemmas correspond to Lemmas 16, 25 and 26 of Davenport [1].

LEMMA 26. *We have*

(78) $$\int_0^1 \Theta^*(a, q, a)e(-ua)\, da = q^{-6}\big(S_4(a, q)\big)^6 e_q(-ua)R^*(u),$$

where for $N/2 < u < N$, $R^*(u)$ *satisfies*

(79) $$N^{1/2} \ll R^*(u) \ll N^{1/2}.$$

LEMMA 27. *If* $u \equiv 1, 2, 3, 4, 5$ *or* $6 \pmod{16}$, *the series*

$$\mathfrak{S}^*(u) = \sum_{q=1}^{\infty} A^*(u, q)$$

is absolutely convergent, and $\mathfrak{S}^*(u) > c_4$, *where* c_4 *is a positive constant.*

LEMMA 28. *For* $X \geqslant 1$,

$$\sum_{q>X} |A^*(u, q)| \ll u^{\varepsilon} X^{-1/4}.$$

LEMMA 29. *If* $N/2 \leqslant u \leqslant N$, *then*

$$\mathfrak{S}^*(N^{1/8}, u) \gg N^{-\delta}.$$

This follows from Lemmas 27 and 28.

LEMMA 30. *Let*

(80) $$I_1^*(N) = \sum_{q\leqslant N^{1/8}} \sum_a \int_0^1 \Theta^*(a, q, a)\Omega_1^*(a)\Omega_2^*(a)e(-Na)\, da.$$

Then

(81) $$I_1^*(N) \gg N^{-1-\delta}\theta^*(0)\Omega_1^*(0)\Omega_2^*(0).$$

Proof. From Lemma 26, we have (cf. (17) and (18))

$$I_1^*(N) = \sum_{i=1}^{V}\sum_{j=1}^{W}\sum_{q\leqslant N^{1/8}}\sum_a \int_0^1 \Theta^*(a, q, a)e(v_i a + \omega_j a - Na)\, da$$

$$= \sum_{i=1}^{V}\sum_{j=1}^{W}\sum_{q\leqslant N^{1/8}}\sum_a q^{-6}\big(S_4(a, q)\big)^6\ e_q\{-(N - v_i - \omega_j)a\}R^*(N - v_i - \omega_j)$$

$$= \sum_{i=1}^{V}\sum_{j=1}^{W}\sum_{q\leqslant N^{1/8}} A^*(N - v_i - \omega_j, q)R^*(N - v_i - \omega_j).$$

Hence, from (79) and Lemma 29,

$$I_1^*(N) \gg VWN^{1/2-\delta} \gg N^{-1-\delta}\theta^*(0)\Omega_1^*(0)\Omega_2^*(0),$$

since

$$N^{6/4} \gg \theta^*(0), \quad V \gg \Omega_1^*(0), \quad W \ll \Omega_2^*(0).$$

§ 8. Proof of Theorem 2. We have

(82) $$r^*(N) = \int_{1/\varphi^*}^{1+1/\varphi^*} \theta^*(a)\Omega_1^*(a)\Omega_2^*(a)e(-Na)\, da$$

$$= I_1^*(N) - I_2^*(N) + I_3^*(N) + I_4^*(N),$$

where $I_1^*(N)$ is given by (81), and

(83) $$I_2^*(N) = \sum_{q \leqslant N^{1/8}} \sum_{a} \int_{\mathfrak{M}^*_{a,q}} \Theta^*(a, q, a)\,\Omega_1^*(a)\,\Omega_2^*(a)\,e(-Na)\,da,$$

(84) $$I_3^*(N) = \int_{\mathfrak{M}^*} \{\theta^*(a) - \Theta^*(a, q, a)\}\,\Omega_1^*(a)\,\Omega_2^*(a)\,e(-Na)\,da,$$

(85) $$I_4^*(N) = \int_{\mathfrak{m}^*} (\theta^* a)\,\Omega_1^*(a)\,\Omega_2^*(a)\,e(-Na)\,da.$$

It follows trivially from Lemmas 22 and 23, that

$$I_3^*(N) \ll N^{-1-10\delta}\theta^*(0)\,\Omega_1^*(0)\,\Omega_2^*(0),$$
$$I_2^*(N) \ll N^{-1-10\delta}\theta^*(0)\,\Omega_1^*(0)\,\Omega_2^*(0).$$

Also from Lemma 25,

$$I_4^*(N) \ll N^{-1-10\delta}\theta^*(0)\,\Omega_1^*(0)\,\Omega_2^*(0).$$

Hence from (81), and (82), we have

$$r^*(N) \gg N^{-1-\delta}\theta^*(0)\,\Omega_1^*(0)\,\Omega_2^*(0);$$

so that $r^*(N) > 0$ for large N.

This completes the proof of Theorem 2.

Acknowledgement. I am indebted to prof. Roth and prof. Halberstam for constant encouragement, valable advice and criticisms. I should also like to thank prof. Halberstam, prof. Estermann and the referee for checking manuscripts and making useful changes.

Finally, I like to thank Messrs T. E. Sellars and R. Wooldridge for their kind encouragement.

References

[1] H. Davenport, *On Waring's problem for fourth powers*, Annals of Math. (2) 40 (1939), pp. 731-747.

[2] — *On Waring's problem for cubes*, Acta Math. 71 (1939), pp. 123-143.

[3] — *On sums of positive integral k-th powers*, Amer. J. Math. 64 (1942), pp. 189-198.

[4] A. Y. Khinchin, *Three pearls of number theory*, Translated from the second (1948) revised Russian edition by F. Bagemihl, H. Komm and W. Seidel, Rochester, N. Y. 1952.

[5] E. Landau, *Vorlesungen über Zahlentheorie*, Vol. I, Leipzig 1927.

[6] K. F. Roth, *A problem in additive number theory*, Proc. London Math. Soc. (2) 53 (1951), pp. 381-395.

[7] E. J. Scourfield, *A generalization of Waring's problem*, J. London Math. Soc. 35 (1960), pp. 98-116.

DEPARTMENT OF MATHEMATICS
CONSTANTINE COLLEGE OF TECHNOLOGY
Middlesbrought, England

Reçu par la Rédaction le 9. 1. 1967

ACTA ARITHMETICA
XIII (1968)

Sur les nombres qui ont des propriétés additives et multiplicatives données

par

A. O. Gelfond (Moscou)

Soit $q \geqslant 2$ un entier. Considérons la représentation d'un entier dans le système de numération de base q et posons

(1) $$N = \sum_{k=0}^{\nu} C_k q^k, \quad 0 \leqslant C_k \leqslant q-1,$$

où l'on suppose aussi $C_\nu \neq 0$, $k = 0, 1, \ldots, s$, tandis que tous les autres C_s, $s \neq \nu_k$ sont nuls.

Admettons que la fonction $f(x)$ soit additive dans le système de base q, autrement dit que

(2) $$f(N) = f(N_1) + f(N_2), \quad N = N_1 + N_2, \quad N_1 < 2^\nu, \quad N_2 = 2^\nu N_3,$$

où N, N_1, N_2, N_3 sont non négatifs, par exemple

$$f_1(N) = N; \quad f_2(N) = \sum_{1}^{\nu} C_k, \quad N = \sum_{0}^{\nu} C_k q^k, \quad 0 \leqslant C_k \leqslant q-1;$$

$$f_3(N) = af_1(N) + \beta f_2(N) = a\sum_{0}^{\nu} C_k q^k + \beta \sum_{0}^{\nu} C_k.$$

Alors

(3) $$\sum_{1}^{N} f(n) = \sum_{a_0=0}^{q-1} \cdots \sum_{a_\nu=0}^{C_k-1} f\Big(\sum_{0}^{\nu} q_k q^k\Big) + \sum_{a_0=0}^{q-1} \cdots \sum_{a_{\nu_1}=0}^{C_{\nu_1}-1} f\Big(\sum_{k=0}^{\nu_1} a_k q^k + C_\nu q^\nu\Big) + \ldots,$$

ou, en vertu de l'additivité de $f(x)$,

(4) $$\sum_{n=1}^{N} f(n) = \prod_{0}^{\nu-1} [f(0) + f(q^k) + \ldots + f((q-1)q^k)][f(0) + \ldots + f((C_\nu - 1)q^\nu)] +$$
$$+ \prod_{0}^{\nu_1-1} [f(0) + \ldots + f((q-1)q^k)][f(0) + \ldots + f((C_{\nu_1} - 1)q^{\nu_1})]\, f(C_\nu q^\nu) + \ldots,$$